

The
Competitive
Firm's
Response
to
Risk

THE COMPETITIVE FIRM'S RESPONSE TO RISK

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PREFACE

Our goals in developing *The Competitive Firm's Response to Risk* were twofold. First to formulate and second to consistently apply a versatile analytic framework to understanding and evaluating the broad set of actions available to the competitive firm for responding to risk. In particular, we have focused on extending equilibrium analysis at the micro level from the certainty case to the risk setting. In doing this we use the familiar concepts of income and substitution effects from economic theory.

The needs for achieving these goals are several. One is to further enhance our analytic capacity for explaining the many types of institutional phenomena and observed decision behavior that are attributed to risk. A second is to assemble many of the significant advances in risk analysis that are found in the professional journals of economics and related areas into a single, unified text. A third is to provide a book that focuses solely on the theory of risk analysis rather than following the common practice in economic theory texts of devoting only a chapter or two to this important topic. And a fourth is to provide one comprehensive, unified approach for conceptualizing the analysis of numerous risk responses in a fashion that yields plausible theoretical results and that is amenable to empirical analysis.

To achieve analytic uniformity, we chose to utilize the expected value–variance approach to represent the probabilistic characteristics of risk facing the firm. The ramifications of this choice are considered in detail in several of the chapters. We believe this is a useful choice. It allows economic theory under risk to be developed in a plausible, comprehensive way. Moreover, as will quickly become clear, economic theory under risk needs a simple point of departure in order to clearly predict outcomes, understand behavior in complex decision situations, and formulate and test hypotheses about the effects of various risk responses. Thus we have translated and synthesized the important work from recent journals into a consistent expected value–variance framework.

The book is organized into several parts. Chapter 1 introduces the scope

of the subject matter and presents several of the key concepts used throughout the text. Part 1 (Chapters 2 to 5) introduces decision theory concepts of utility functions, risk attitudes, and risk measures, as well as approaches to ordering individuals and their risky choices. The two chapters of Part 2 provide the book's analytic base; they characterize the multitude of firm models that exist under conditions of risk and develop the equilibrium framework.

In Parts 3 and 4 the analytical framework is applied to a broad set of choices available to the firm. Included are such topics as output responses, input adjustments, hedging, diversification, insurance, and the management of information and liquidity. Some of the applications are illustrated using specific types of firms and industry settings, while others are treated in a more general fashion. Part 4 is distinguished from Part 3 by changes in the probabilistic characteristics of the firm's risk position as the effects of risk are modified by the respective choices. Because the resulting distinction between the direct and indirect effects of risk yields a more complex setting, these chapters are organized into a separate part. Finally, the Epilogue summarizes the preceding developments and considers some of the possible future directions for risk analysis.

The book is intended as a text or reference for advanced undergraduates and graduate students in economics, business, agricultural economics, resource economics, and related areas both in the United States and elsewhere. In addition, it will be useful for professional economists in these subject matter areas who wish to review a comprehensive treatment of risk analysis for the competitive firm. We have assumed that readers have a prior acquaintance with microeconomic theory at the intermediate level as well as an introductory knowledge of calculus and statistics. Indeed, the appendix of the book contains a review of all the statistical concepts used throughout the chapters. Each chapter is structured to present the material using a combination of text discussion, mathematical derivations, and graphing of key functional relationships.

As the manuscript developed, we benefited greatly from the review comments and suggestions provided by a number of colleagues. Included are Young Chan Choe, Mark Cochran, Beverly Fleisher, Randall Kramer, Lester Mander-scheid, Jack Meyer, Rulon Pope, Stanley Thompson, and David Trechter. We are grateful for their time, effort, and contribution to the review process. In addition, a large number of students who read early drafts of some chapters in classes taught by the authors made useful suggestions and raised constructive questions. We especially appreciate the important support needed to develop and complete the manuscript that was provided by our host institutions, Michigan State University and the University of Illinois at Urbana-Champaign. Finally, we greatly appreciate the major effort and patience of our typists, Kim Olson and Debbie Greer.

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INTRODUCTION TO THE COMPETITIVE FIRM'S RESPONSE TO RISK

The purpose of this book is to extend microeconomic theory to account for the effects of risk on a firm's equilibrium conditions. This extension is needed because many aspects of firm behavior cannot be explained in a world of complete certainty. Under certainty, the theory of the firm stipulates a profit maximizing goal and considers such decisions as the optimal level of output, input and product combinations, firm size, and product pricing when markets are imperfect. Under conditions of risk, the choice set still includes these types of decisions but also introduces various responses to risk such as asset diversification, alternative forms of contracting, holding liquid reserves, restricting borrowing and financial leverage, utilizing insurance, acquiring new information, and many others. Moreover, the firm's profit-maximizing goal is modified in response to various sources of risk and the decision maker's risk attitude.

Extending the theory of the firm to account for responses to risk enriches the theoretical framework, enhances the applicability of analytic methods, and further broadens our capacity to understand the economic relationships involved. But these extensions also present significant challenges. Especially important are identifying the fundamental variables, evaluating interrelationships between them, and measuring risks, risk attitudes, and other forces involved. For example, the many linkages between a firm's production, marketing, and financial activities yield a complex framework for analysis. Moreover, the types and combinations of risk responses are numerous. Therefore we will focus on the theoretical foundations of only a few of the major risk responses and show how economic theory is extended to account for their effects on the firm's equilibrium position.

In this chapter we introduce the theory of the competitive firm under risk and the distinguishing features of the theory, including relationships to firm theory under certainty, the concepts of certainty equivalents and risk premiums,

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and the expected value–variance approach to equilibrium analysis under risk. A distinction is also made between economic theory under risk and decision theory. Each of these features and the distinction between economic theory and decision theory are addressed in the following sections and related in an introductory way to later chapters that develop these issues in greater detail.

ECONOMIC THEORY AND DECISION THEORY

Decision theory is an immense subject which studies human decision-making behavior (Morris).¹ The subject is relevant when individuals face decision problems. A decision problem exists when an individual has alternative choices, each with significant consequences, and is unsure about which choice is the best. When the individual is uncertain about the consequences of his or her choice because of stochastic states of nature, the decision problem is said to be risky (Anderson et al.). If the states of nature facing the decision maker depend on the actions of an opponent, a game theory problem exists.

Decision theory focuses on selecting the preferred choice, or a set of preferred choices, for a well-defined class of decision makers. Selection is based on the risk-return characteristics of the choices and the risk attitudes of the decision makers. Levels of risk are represented by probability distributions. Applications of decision theory employ a broad spectrum of models. Halter and Dean, for example, illustrate decision problems for stocking rates in agriculture, drilling in geology, and raw material acquisition in forestry. In solving such problems analysts from diverse disciplines may combine their skills to characterize the decision situation and derive the choice set. The preferred choice usually has limited generality, however, since it applies only to the specific conditions defined in the decision problem modeled.

Examination of economic theory under conditions of risk is a subset of the general subject matter addressed by decision theory. For example, to identify the functional relationships between important economic variables in guiding resource allocations for a firm, economic constraints must be imposed on the choice set and the choice criterion must be precisely specified. Economic theory under risk often abstracts from measurement problems and the complexities of real world decision making in order to highlight the fundamental issues. Thus the theory essentially assumes an error-free specification of the decision maker's risk attitude and error-free measurement of probability distributions.

The theory must order choices uniquely in order to employ the calculus of comparative static analysis. As an example, and to preview later results, the evaluation of how holdings of risky assets change as a decision maker's risk-free wealth increases requires that an expected utility-maximizing choice already be known. Moreover, the utility function should be continuous, differentiable, and concave to the origin to ensure that a maximum exists. Economic theory under risk in most cases begins with the assumption that a decision maker's preferences are represented by a concave utility function. In more general applications of

decision theory, however, specifying the characteristics of the utility function is one aspect of the analysis.

As later chapters will show, the procedural aspects of decision theory provide useful tools for developing and empirically testing economic theory under risk. To aid in this process, Chaps. 2 to 4 introduce decision theory concepts of utility functions and risk attitudes and review some of the measurement approaches. Also included are some of the efficiency criteria used to order individuals according to their risk attitudes and to order choices according to their risk-return characteristics.

THEORY OF THE FIRM UNDER CERTAINTY

As evidenced by numerous textbooks on microeconomics, the theory of the firm under certainty follows an orderly sequence in its development. Included in this development are such topics as single input-output production functions, factor-factor relationships, product-product relationships, optimal output and pricing (if appropriate) in the short run and long run, and economies of scale and size. Income and substitution effects are specified to aid in comparative static analysis. The theory yields a set of precise, unambiguous conditions about the organization of a firm's inputs and products so as to maximize profits under certainty conditions.

To illustrate, we could show that a firm with a production function characterized by diminishing marginal productivity and operating in perfectly competitive markets will achieve profit-maximizing combinations of inputs and products by producing so that the marginal value products in all enterprises are equal to input costs. Or, when viewed from a cost-size standpoint, profit maximization under certainty will guide the firm to a level of production where marginal revenue from added output equals the marginal cost of added output. Moreover, in the long run, competition will cause this condition to occur at the lowest point on the firm's long-run average cost curve. The U-shaped characteristic of the average cost curve implies that the firm may benefit from economies of size by expanding until diseconomies of size occur. Firms operating in less than perfectly competitive markets and facing downward sloping demand curves will maximize profits at lower levels of economic efficiency relative to the perfect market case.

The specific characteristics of these equilibrium conditions may change when risk is considered. Risk may be viewed as resulting in an additional cost to the firm which must be met in achieving an optimal organization of the firm's activities. The firm will engage in a risky activity only if it is compensated for certain costs and risk costs. The effects of these added risk costs can then be evaluated and will alter the firm's choice of activities. As under certainty conditions, an increase in the cost of a firm's operations will unambiguously reduce the optimal level of output, at least in the short run, and reduce the incentive for expanding the firm's size. However, the effects of risk on the mix

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of inputs and enterprises are less clear; they depend heavily on the sources of risk and the possible risk responses. Some types of risk may not change the optimal input combinations for the firm; some inputs are actually risk reducing and thus preferred when uncertainty is introduced. Other inputs are held in reserve or underutilized to counter adverse outcomes possible under uncertainty. We will demonstrate these effects in later chapters; the important point for now is to recognize the useful role of the theory of the firm under certainty in extending the analysis to a risky world.

RISK AVERSION, RISK PREMIUMS, AND CERTAINTY EQUIVALENTS

The concepts of risk aversion, risk premiums, and certainty equivalents are central to economic theory under risk and to the equilibrium analyses used in this book. For now, we will introduce these concepts in an intuitive way; their development and analytic uses will be shown more rigorously in later chapters.

Risk aversion does not mean that individuals are unwilling to take risks. Rather, *risk aversion* means that individuals must be compensated for taking risks in the form of a premium over and above the return on a completely certain investment. Thus a risky investment or enterprise must yield an expected return high enough (compared to a risk-free investment) to compensate the risk-averse decision maker for accepting the risk. Or, more generally, we can anticipate that one investment that is riskier than another must offer a higher expected return to be preferred by risk-averse decision makers. Similarly, the more risk-averse individuals are, the higher the compensation on the risky investments they are considering must be for these investments to be preferred to riskless alternatives.

A risk-averse decision maker prefers a riskless investment whose return is equal to the expected return on a risky investment. There is, however, some level of expected return on the risky investment, larger than the return on the safe investment, at which the decision maker is indifferent between the risky and the riskless alternatives. This difference between the expected return on the risky investment and the return on the riskless investment which leaves the firm indifferent between the two choices is defined as a *risk premium*. The return on the risk-free investment, equal to the expected return on the risky investment less the risk premium, is defined as the *certainty equivalent* of the expected return on the risky investment. The certainty equivalent and the expected risky return yield the same level of well-being.

The relationships between the certainty equivalent, the expected risky return, and the risk premium are expressed as follows:

$$\text{Certainty equivalent} = \text{expected risky return} - \text{risk premium}$$

For risk-averse decision makers, the risk premium is always positive in order to provide the compensation needed for risk bearing. Thus the certainty equivalent

of a risky investment is always less than its expected return. For risk-neutral decision makers, the risk premium is zero; for risk-preferring decision makers, the risk premium is negative, indicating their willingness to pay a premium for the opportunity to bear risk or take chances.

The magnitude of the risk premium at the micro level depends jointly on the decision maker's level of risk aversion, as reflected by his or her utility function, and on the level of risk, as determined by the investment's probability distribution. In general, risk premiums increase as risk aversion increases, as the level of risk increases, or as both events occur together. As shown in the preceding section, risk premiums for risky investments represent additional costs for the risk-averse firm that affect its optimal organization. Since higher costs reduce economic well-being, actions that reduce risk and/or risk aversion reduce risk premiums and thus reduce the costs of the firm's operation. In turn, these cost reductions result in improved economic efficiency and higher utility attainment.

In later chapters we will further develop the theory and measures of risk aversion and show how the concepts of certainty equivalents and risk premiums can account for the effects of risk on the firm's equilibrium conditions. We will also analyze the effects of various risk responses on the firm's cost of risk bearing and utility attainment. To do this, however, we must introduce explicit measures of risk and return in order to carry out the analysis. These measures are treated in the next section.

EXPECTED VALUE-VARIANCE ANALYSIS

The concept of risk that we employ focuses on the randomness, or variability of outcomes, some of which are favorable to the investor and some of which may cause losses or adversity. The range of possible outcomes is expressed as a probability distribution in which the probabilities reflect the weight or likelihood of occurrence for the respective outcomes. To facilitate risk analysis we will assume that the expected value and variance of the probability distribution adequately reflect the distribution's relevant characteristics.

The *expected value-variance* (EV) approach has had widespread use in economic and financial analysis. It was originated by Markowitz to explain investors' diversification of financial assets, later extended by Tobin to include risk-free assets, and then applied in equilibrium analysis by Sharpe, Fama, and Lintner to the risk pricing of capital assets. The explicit measures of risk and return have made the EV approach well-suited to empirical analysis. In addition, rigorous demonstrations of its usefulness as an approximate method for portfolio selection (Levy and Markowitz), especially when risk is small relative to total wealth (Tsiang), have enhanced its general applicability.

The EV approach can be derived from an expected utility (EU) maximization framework. Two sufficient conditions require an investor's utility function to be quadratic, reflecting preferences only for expected values and variances of

outcome distributions, or that the investor's expectations be modeled by normal distributions that are fully specified by their expected values and variances. If one of these conditions is met, then an expected utility-maximizing choice is made from the efficient set. However, both quadratic utility and normality are highly restrictive conditions. While quadratic utility may be a useful second-order approximation for identifying a preferred choice from an EV efficient set, its restrictive assumptions preclude its usefulness as an analytic tool. Moreover, some of the theoretical models we will analyze clearly have nonnormal distributions. Thus we need a different basis for developing an expected value-variance analytic model.

The approach followed in later chapters begins by recognizing that quadratic utility and normal distributions are sufficient but not necessary for consistency between EU and EV results. For example, for choices involving alternative combinations of a risky and a riskless asset, all the decision maker's choices are members of an EV set. Moreover, each choice from the EV set can be found by maximizing a linear function of expected values and variances for a given slope subject to the constraints of the EV set. After finding the solution which can be made to correspond to any specific expected utility solution, adjustments from equilibrium can be analyzed in terms of income and substitution effects that are stochastic counterparts of the same effects found under certainty conditions. The characteristics of the income effect were derived under expected utility maximization by Cass and Stiglitz; they apply in general to all shapes and families of probability distributions. However, the substitution effect must be derived from expected utility maximization under a more restrictive set of assumptions about the investor's preferred trade-off between the expected values and variances of choices—specifically, that this trade-off is modeled by linear isoexpected utility lines for an investor with constant absolute risk aversion (i.e., an investor whose income effect is zero).

The linearity condition cited above was obtained by Freund for normal distributions and by Pratt for nonnormal distributions as a second-order approximation. Since we are not restricting the analysis to the case of normal distributions, our estimates of the substitution effect with the expected value-variance approach are also only approximations of the results that would occur using the general expected utility model. However, for the models we analyze using this framework, the analytic results are the same as those found in other studies using the general expected utility model.

Since expected utility itself is considered an approximation to the true unknown preference function of the population of investors, and since exact estimates of probability distributions are difficult to obtain, it is reasonable and acceptable to use the expected value-variance framework. It allows economic theory under risk to be developed in a plausible, comprehensive way. Moreover, economic theory under risk needs a simple point of departure in order to clearly predict outcomes, understand decision behavior in complex situations, and test hypotheses about the effects of risk.

CONCLUDING COMMENTS

In this chapter we introduced the subject matter of this book by characterizing economic theory under risk as a framework for understanding and predicting a firm's responses to unanticipated changes in its operating environment. Some of these responses involve extensions of the theory of the firm under certainty conditions; others involve institutional and market phenomena that are unique to the existence of uncertainty. In general, risks are considered to add to the costs of operation for a firm with a risk-averse decision maker and thus reduce economic efficiency.

The concepts of risk premiums and certainty equivalents serve as important analytic measures for developing the theory of the firm under risk and for conducting various types of equilibrium analyses. These concepts will be implemented using expected values and variances of probability distributions in order to fully characterize the stochastic properties of the firm's choices. This will allow comparative static analysis of how a firm's expected utility maximizing choice changes in response to changes in probability distributions, initial wealth, and risk aversion. In turn, these comparative static results can be linked to the practical responses that a firm's decision makers may consider in achieving a risk-averting goal.

ENDNOTE

1. For a detailed discussion of decision theory see Raiffa, Schlaifer, Anderson et al., Luce and Raiffa, and DeGroot.

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PART
ONE

AN INTRODUCTION TO
DECISION MAKING UNDER RISK

WHAT IS RISK, WHY IS IT A PROBLEM, AND HOW DO WE LIVE WITH IT?

In this chapter we introduce the basic foundations of risk analysis. We begin by defining risk and uncertainty using actual decision situations as illustrations and then organize the various components of these situations into a framework for effective decision analysis under risk. The meanings of risk and uncertainty are distinguished from one another, and the expected utility model (EUM) is introduced.

WHAT IS RISK? WHERE DOES IT COME FROM?

Before defining risk and uncertainty, consider the following decision situations and their implications for uncertainty analysis.

A student preparing to leave for class notices an overcast sky suggesting possible rain. She considers taking an umbrella. This would be the safe alternative but would also involve a cost. If the sky clears, the umbrella will be a nuisance. Should she take the umbrella?

A salesperson will be making a long car trip on medium- to well-worn tires. The tires may have 10,000 miles of driving left, but their worn condition increases the possibility of tire problems. The safe, but expensive, alternative is to replace the tires. The less costly alternative is to use the old ones. However, the inconvenience of replacement along with the dangers of a high-speed blowout are important considerations. Should the salesperson purchase new tires?

A farmer must decide whether to repair his old combine or buy a new one. If corn prices and yields are favorable, the new purchase will be the better choice, since investments in the old combine cannot be recovered and even if repaired it would not be as reliable as a new one. On the other hand, poor weather and/or low corn prices could create cash flow problems which would be reduced by repairing the old combine rather than purchasing a new one. The farmer's lender is concerned

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about the farmer's liquidity. Should the farmer repair the old combine or buy a new one?

Each of these situations involves decision making when the outcome is not known with certainty. Instead, one of several outcomes may occur, and it may or may not be possible to assign a probability weight to each of them. Moreover, to evaluate the possible choices properly the decision maker needs to consider (1) how much risk he or she is willing and able to carry for a particular decision and for all sources combined, (2) what sources of information are available to estimate more precisely the risks involved, and (3) the alternatives for reducing the likelihood of these risks, transferring them to other parties, and/or building his or her capacity to bear risks. All these considerations are involved in decision making under risk.

The adjective "uncertain" describes an event whose outcome is not definitely known. An uncertain event has at least two possible outcomes and usually more. If events are uncertain, we cannot say that one is more uncertain than another. All we can say is that they are both uncertain.

Perceptions of the uncertainty associated with events are unique to individual decision makers. These perceptions can be expressed in terms of probability distributions that reflect the likelihood of occurrence for various events and outcomes. An *ex ante* assessment of an event's likelihood is made by individuals based on their experience, accumulation of knowledge, and quality of judgment. Generally, activities that build experience, increase information, and improve judgment help to improve the quality of expectations and thus enhance decision making.

To illustrate, the student above might believe, based on her experience and information, that rain is impossible. From her standpoint, this removes all uncertainty. But, it still might rain. In turn, this experience might change her expectations, as could an updated weather forecast.

Suppose, in another example, that two individuals are watching a two-horse race. Individual A lacks information about the horses, their riders, and their trainers; thus he is unsure of the winner. Individual B, however, knows that the first horse's ankle is weak, his rider is overweight, and his diet has been poor. This information, along with the knowledge that the second horse is well-prepared and is being ridden by a competent light rider, "guarantees" the outcome of the race. The outcome of the horse race is a certain event for individual B and an uncertain event for individual A. The certainty or uncertainty exists in the minds of the individuals, one of whom lacks information. If individual A had the same information as individual B, he would also view the race's outcome as certain.

We will avoid the metaphysical question of whether all events have a predestined outcome; rather, we will define certainty in terms of the knowledge of the decision maker about an event's outcome. If an individual can specify an event's outcome with insignificant doubt, he faces certainty. If knowledge is insufficient to specify a unique outcome, the individual faces uncertainty.

Decision makers may also improperly assess the quality of their information. Suppose that the horse "certain" to win stumbles and falls at the starter's gate so that the second horse wins. The point is not that the outcome of the race was certain (which it was not), but that decision maker B believed it was.

The dependence on subjective information to distinguish uncertainty from certainty should not imply that uncertainty is merely a lack of available information. An individual may collect all available information and still view the event as uncertain. Thus an uncertain event is an event with more than one possible outcome. True certainty exists only when the decision maker can specify a unique outcome of an event and his or her information corresponds to real-world conditions. These conditions are rarely met.

Certainty and uncertainty are also related to the degree of accuracy of future expectations about an event's outcome. We may predict that tomorrow's temperature at 12:00 P.M. in East Lansing, Michigan, will be between 180 and -75°F . But this prediction does not admit much accuracy compared to predicting a high temperature in the 55 to 60°F range. In general, the greater the accuracy desired in predictions, the more extensive is the knowledge required.

UNCERTAIN VERSUS RISKY EVENTS

In an earlier time, Knight distinguished between risk and uncertainty based on the empirical information available for generating probabilities. If the decision maker faced a situation similar to others which had occurred in the past and information about the outcomes of previous choices could be used to estimate probability functions, the situation was risky. A unique situation with little or no empirical basis for the formation of probability distributions Knight considered uncertain.

Decision makers must, however, make probability judgments even with little or no empirical support. And once formed, the decision process is similar whether the decision maker faces Knightian risk or uncertainty. Partly as a result, few economists maintain the distinction imposed earlier by Knight but use uncertainty and risk interchangeably. We propose, however, to distinguish between risk and uncertainty.

Events are uncertain when their outcome is not known with certainty. Uncertain events are important when their outcomes alter a decision maker's material or social well-being. We define as *risky* those uncertain events whose outcomes alter the decision maker's well-being. This definition is broader than the popular concept of risk as involving possible loss or injury and implies that risky events form a subset of uncertain events. The decision maker's response to uncertain nonrisky outcomes is indifference or irrelevance. Only risky events have significance.

We said earlier that events are either uncertain or certain and cannot be graded by degree of uncertainty. This does not hold for risky events. A decision maker facing a set of risky choices does not view them as equally risky. The

decision maker learns to order them from least risky to most risky. This ordering does not depend solely on the level of risk but on the decision maker's attitudes toward risk as well.

Other definitions of risk consider variances, likelihoods of loss, safe levels of income, or specific requirements on probability distributions. These, however, are only tools with which well-defined classes of decision makers measure and order risky choices. They should not be confused with the general definition of risk.

While risky events cannot be identified by objective probabilities involving their outcomes as Knight suggested, the quality of information about probability distributions is a useful consideration. It is recognized, of course, that all informational bases are in part subjectively determined (e.g., which information is to be included in a subjective decision); still one data base may be more empirically anchored than another. For example, basing the probability density function of rainfall in a location on 100 years of data represents a different (better) informational state than using only 5 years of data. Unfortunately, the current state of uncertainty analysis is such that we are unprepared to make such distinctions using the existing theoretical tools. Future developments may eventually correct such deficiencies.

To summarize, an uncertain event has more than one possible outcome; the likelihood of the outcomes, however, can be described by probability distributions. If the outcomes of the uncertain event alter the decision maker's well-being, the event is risky.

THE EXISTENCE OF DECISION PROBLEMS

Decision problems exist when the decision maker can take actions to alter his or her well-being, although the best choice is not known with certainty. The decision maker's choice between risky events and responses to changes in risky situations are the general topics of this book. However, formulating the choice set is also important.

The major analytic tool for solving decision problems under risk is the *expected utility model*. If a unique utility function for decision makers is known, then a unique solution can be identified. In decision theory, analysts may be content to reduce the set of possible choices to a smaller number of efficient choices which include the preferred choice. In contrast, economic theory identifies a preferred choice and then evaluates adjustments in the preferred choice in response to changes in the set of choices facing the decision maker.

The components of a decision problem include (1) the states of nature, (2) the possible outcomes, (3) the probabilities of outcomes, (4) the choices, and (5) the decision rule for ordering choices. To illustrate, let s_1, \dots, s_m represent m possible states of nature with probabilities $p(s_1), \dots, p(s_m)$ of occurring. The decision maker's choices are identified as A_1, \dots, A_n . Each choice yields

Table 2.1 Listing of possible choices A_1, \dots, A_n and their outcomes O_{ij} ($i = 1, \dots, m; j = 1, \dots, n$) over the possible states of nature s_1, \dots, s_m which occur with probability $p(s_1), \dots, p(s_m)$

Nature state	Probability of state	Choices				
		A_1		A_i		A_n
s_1	$p(s_1)$	O_{11}	...	O_{1i}	...	O_{1n}
\vdots	\vdots	\vdots		\vdots		\vdots
s_m	$p(s_m)$	O_{m1}	...	O_{mi}	...	O_{mn}

a unique outcome O_{ij} ($i = 1, \dots, m; j = 1, \dots, n$) in the respective state of nature. These components of the decision problem are shown in Table 2.1.

Having identified the possible choices, the outcomes, and their likelihoods, the decision maker must order the choices according to preference. Consider a practical decision problem. Suppose the states of nature are s_1 for rain and s_2 for no rain. Let the choices for a salesperson contemplating a trip be: A_1 , buy steel-belted radial tires which are expensive but safe; A_2 , buy less expensive and less safe Polyglass tires; and A_3 , continue to drive on the old tires. Let the probabilities of s_1 and s_2 be $p(s_1)$ and $1 - p(s_1)$, respectively. This decision problem is described in Table 2.2a.

How does the decision maker compare the outcomes (O_{11}, \dots, O_{23}) for this problem? A common denominator must be established for the outcomes to make the comparison. One approach is to create a situation for selling (or buying) the outcomes in exchange for their equivalent cash value. We might ask the decision maker: How much would you pay to drive steel belts in the rain? Suppose the response is \$350. Then we subtract the actual cost of the tires, say \$250, leaving a cash value of \$100. This value represents outcome y_{11} in Table 2.2b. Using the same procedure we can obtain the dollar equivalent values of other outcomes.

In general, then, we express the outcomes of choices in money equivalents y_{ij} in Table 2.2b. Johnson refers to the process of converting outcomes to their dollar equivalents as *premaximization*. Converting outcomes to their dollar equivalents permits an ordering of the outcomes over the states of nature. However, a decision maker's responses to the questions which give rise to the ordering may depend on many factors. Examples include the decision maker's health, acceptance by peers, relationships with others, retirement security, leisure time, probability of business survival, and so on. The effects of these factors on the ordering of outcomes is the subject matter of psychologists and sociologists; it is not pursued here. Instead, we focus on ordering the choices under fairly simple, yet general, assumptions about the decision maker's attitudes toward risk and returns.

Table 2.2 Decision matrix for a tire purchase decision

Nature state	Probability of state	Choice		
		A_1 (Buy radials)	A_2 (Buy Polyglass)	A_3 (Keep old tires)
Panel a				
s_1 , rain	$p(s_1)$	O_{11}	O_{12}	O_{13}
s_2 , no rain	$1 - p(s_1)$	O_{21}	O_{22}	O_{23}
Panel b				
s_1 , rain	$p(s_1)$	y_{11}	y_{12}	y_{13}
s_2 , no rain	$1 - p(s_1)$	y_{21}	y_{22}	y_{23}
Panel c				
s_1 , rain	$p(s_1)$	$U(y_{11})$	$U(y_{12})$	$U(y_{13})$
s_2 , no rain	$1 - p(s_1)$	$U(y_{21})$	$U(y_{22})$	$U(y_{23})$

To begin, let the probability of s_1 be one, $p(s_1) = 1$. This implies that the decision maker has complete certainty about the outcomes for his or her choices A_1 , A_2 , and A_3 in Table 2.2b. The approach then is to order choices A_1 , A_2 , and A_3 based on the dollar ordering of y_{11} , y_{12} , and y_{13} . If y_{11} is the largest, then choice A_1 is preferred. It provides the greatest satisfaction to the decision maker. Next, suppose the decision maker views states of nature as being risky, with $0 < p(s_1) < 1$. Now the ordering of choices requires that all possible outcomes be considered and compared. This comparison requires a decision rule with an index for ordering choices based on attitudes toward having different dollar values.

This indexing rule will be useful, however, only if decision makers act as though they are following it. Dillon made the analogy that the mathematical solution for determining the path of a thrown baseball requires solving complex differential equations; however, those who can catch baseballs need not solve these equations, even though their catches imply that they do. The implication here is that decision makers may adopt rules of thumb to guide them through complex decision processes. These rules have been tried and tested over time and are held with confidence by the decision maker. Similarly, an analyst who wishes to model the behavioral situation seeks a modeling approach that serves as a reasonable, reliable guide in a complex decision situation. Any decision rules that an analyst develops must produce choices consistent with the decision maker's rules of thumb, even though the decision maker may not explicitly follow these rules.

SOME DECISION RULES

One decision rule often proposed when little is known about the probability of occurrence for various states of nature is the rule of the extreme risk averter. The rule essentially says: Avoid the worst that can happen. It finds the worst outcomes for each choice and indexes the choices according to the minimum value. The best of the worst outcomes then is the most preferred. Other decision rules are referred to as safety-first rules. Several versions of the safety-first model exist, most of which employ probabilistic information about the worst possible outcomes (reviewed in Robison et al.). Some rules may result in a sequential ordering of goods with safety satisfied first. The safety-first rules are simple, but they exclude considerable information. On the other hand, if probabilistic data are lacking about the entire range of outcomes, or if the decision maker is unwilling to establish probabilities for the entire range of outcomes, this approach may be valid.

Another decision rule is: Maximize expected returns. The index for this rule is the summation of the possible monetary outcomes weighted by their respective probabilities. This weighted value for equal probabilities is called an *average*, or an *expected value*. The expected value index for the j th choice $E(A_j)$ is expressed as:

$$E(A_j) = \sum_{i=1}^m p(s_i) y_{ij} \quad (2.1)$$

The ordering of the choices is based on $E(A_j)$ ($j = 1, \dots, n$), with the maximum $E(A_j)$ being preferred.

The expected value rule considers all outcomes, along with probabilistic information about their likelihood of occurring. However, this rule is criticized because it values each dollar equally and it may not adequately weight the possibilities of low-level or high-level outcomes. The expected value or expected returns rule was generally accepted at the time of Bernoulli in the 1700s. The rule said, for example, that the best gamble had the highest expected value. Bernoulli challenged the rule, however, and asked:

... let us suppose a pauper happens to acquire a lottery ticket by which he may with equal probability win either nothing or 20,000 ducats. Will he evaluate the worth of the ticket as 10,000 (the expected value of the gamble) and would he be acting foolishly, if he sold it for 9,000 ducats? (Savage, p. 97)

Bernoulli postulated that, instead of maximizing the expected monetary values of uncertain choices, people assign *moral expectation values*, later called *utilities*, to each outcome. This concept explained how the marginal worth of a dollar at a low income level could be valued differently than an additional dollar at a high income. However, it raised the question of how to assign moral expectations or utilities to each outcome.

Bernoulli's response to this question was to use the logarithmic function to convert dollar outcomes to utilities. Then he summed the expected logarithm of

outcomes for each choice and chose the one with the largest index. Samuelson and others express doubt that all decision makers' preferences can be expressed by a logarithmic function.

In the absence of a universal utility function, each person's utility function must be measured in order to predict or prescribe his or her preferred choice. The ability to measure utility functions and the appropriate methods involved are issues of continuing debate among economists and decision theorists. Nevertheless, measurement tools developed by von Neumann and Morgenstern have been tested and refined in many applications.

Von Neumann and Morgenstern, and later others (e.g., Friedman and Savage, Herstein and Milnor, Luce and Raiffa, to name a few), deduced the expected utility model in an axiomatic system. They proved that, if an individual's behavior conforms to certain postulates, an ordinal utility function can be derived to arbitrarily assign utility values to contingent incomes. The preferred investment decision maximizes the expected value of the ordinal utility function. Different axioms have been used to deduce the EUM, and, at minimum, they include:

Ordering of choices: For any two choices A_1 and A_2 the decision maker either prefers A_1 to A_2 , prefers A_2 to A_1 , or is indifferent.

Transitivity of choices: If A_1 is preferred to A_2 , and A_2 is preferred to A_3 , then A_1 must be preferred to A_3 .

Substitution of choices: If A_1 is preferred to A_2 , and A_3 is some other choice, then a risky choice $pA_1 + (1 - p)A_3$ is preferred to another risky choice $pA_2 + (1 - p)A_3$, where p is the probability of occurrence of A_1 or A_2 .

Certainty equivalent of choices: If A_1 is preferred to A_2 , and A_2 is preferred to A_3 , then some probability p exists that the decision maker is indifferent to having A_2 for certain or receiving A_1 with probability p and A_3 with probability $1 - p$. Thus A_2 is the certainty equivalent of $pA_1 + (1 - p)A_3$.

If a decision maker obeys these axioms (and several others which are more technical), a utility function can be formulated which reflects the preferences of the decision maker.

Procedures for actually estimating utility functions are presented and evaluated in many references (e.g., Anderson et al., Robison et al.). The process is briefly reviewed here. The estimation process typically occurs in a gaming situation involving repeated applications of the certainty equivalent axiom cited above. The axiom requires four informational items: values for y_1 , y_2 , y_3 , and a probability measure p . The values of y_1 , y_2 , and y_3 are dollar-valued outcomes associated with choices. The gaming approach specifies the values of three of the items and then requires the decision maker to provide a consistent value for the fourth. The process is then repeated by changing one of the specified items and eliciting a new value for the fourth item from the decision maker, and then repeated again, and so on.

To illustrate, suppose the decision maker faces a risky alternative having a maximum possible gain of \$10,000 with probability $1 - p$. The scale of

this person's utility function is fixed by arbitrarily specifying a utility value of 1.0 for the gain, $U(10,000) = 1.0$, and a utility value of .0 for the loss $U(-10,000) = 0$. Utility values for monetary values in between these limits are found as the person indicates certainty equivalents for differing likelihoods of gains and losses. This procedure treats y_1 , y_3 , and p as the prespecified values. It assumes that the utility of the certainty equivalent (y_{CE}) equals the expected utility of the risky alternative:

$$U(y_{CE}) = p(1.0) + (1 - p)(0) = p$$

To illustrate, when probability $p = .5$, the decision maker might say that a sure loss of \$2500 is the certainty equivalent to gambling on a possible loss or gain of \$10,000 at these odds. Thus the utility of minus \$2500 is .5, $U(-2500) = .5$. When $p = .7$, the decision maker might indicate \$3500 as the certainty equivalent. Then the utility of \$3500 for this person is .7, $U(\$3500) = .7$. So far, this procedure has yielded four observations on utility and monetary values for a particular decision maker. When enough observations are available, a utility function can be developed using graphical or statistical procedures in which the function is presumed to yield a valid, reliable ordering of the decision maker's risky choices.

To complete our earlier example, assume we associate with outcomes y_{11} , y_{12} , y_{21} , and y_{22} the utility values $U(y_{11})$, $U(y_{12})$, $U(y_{21})$ and $U(y_{22})$ (see Table 2.2c). The index of preference of choices can be constructed as:

$$EU(A_1) = p_1U(y_{11}) + [1 - p(s_1)]U(y_{21})$$

$$EU(A_2) = p_1U(y_{12}) + [1 - p(s_1)]U(y_{22})$$

$$EU(A_3) = p_1U(y_{13}) + [1 - p(s_1)]U(y_{23})$$

The largest of $EU(A_1)$, $EU(A_2)$, and $EU(A_3)$ is then selected.

A number of alternative approaches and applications have been used in conducting utility games, with most of the focus on the choice of the prespecified values in the certainty equivalent axiom (Anderson et al.). The entire procedure has been subjected to considerable scrutiny and criticism, with numerous sources of possible bias associated with the interviewers' methods, preferences for specific probabilities, negative preferences toward gambling, absence of realism, inexperience of the participants, confounding of preference for income with risk attitudes (Fleisher), and other sources of error (Roumasset, Binswanger, Robison). The major criticism of empirically derived utility functions is the quality of precision of the resulting preference representations. Preference for income is a subjective perception. As in the case of the subjective perception of salinity, sweetness, temperature, and noise, no representative function can perfectly capture the ordering.

These shortcomings have prompted considerable efforts to refine, extend, and generalize the methods for directly eliciting utility functions from individual

decision makers (e.g., Halter and Mason). Thus many improvements have been made in the elicitation process. In general, as both a positive and a normative tool, the expected utility model remains the premier indexing tool.

CONCLUDING COMMENTS

The expected utility model is the premier indexing rule for ordering choices under uncertainty. It is the disciplinary tool in most of the literature focusing on economic analysis under uncertainty. Schoemaker claims it has been the major paradigm in decision making since World War II. If so, the EUM has been both durable and versatile as a tool in decision theory and economic theory.

It is not, however, a perfect predictor, because of problems in accurately measuring utility functions and the probabilities of outcomes of choices. Applications in decision theory have allowed for this imprecision through the use of various efficiency criteria (King and Robison), but in the equilibrium analysis of economic theory, imperfect measurement is not permitted because a unique solution is required.

Nevertheless the EUM is used in economic theory under risk because it is the best alternative available. It allows us to build behavioral models for sensitivity analysis. Since the introduction of the EUM in 1944, it has taken considerable time to move from economic analysis under certainty to economic analysis under risk. Nonetheless, the EUM is helping to increase the accuracy with which we describe and analyze economic problems.

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ORDERING INDIVIDUALS' RISK ATTITUDES

In this chapter we further develop the concept of a decision maker's risk attitude, as reflected by the characteristics of the utility function, and explore the effects of different risk attitudes on decision choices in a risky environment. The risk attitude, along with the decision maker's perceptions (i.e., expectations) of the amounts of risk, are two of the basic behavioral components of decision theory. The expected utility model directly reflects these components by evaluating the utility values of different monetary outcomes using probabilistic weights to represent their likelihood of occurrence.

The exact characteristics of utility functions are, of course, unique to each individual. Nonetheless, we can develop methods of classifying decision makers according to the general characteristics of their utility functions. At a high level of generality, this may consist of attitudes ordered into risk-averse, risk-neutral, and risk-preferring categories. Moreover, within the class of risk-averse agents we can further order individuals according to levels of risk aversion and the response of risk aversion to changes in wealth or other objects of utility. In turn, these ordering procedures enrich our capacity to evaluate and predict the responses of decision makers to changes in the risk characteristics of their decision environment.

In developing this framework we make use of the concepts of risk premium and certainty equivalent that were introduced in Chap. 1. Now, however, we define them more rigorously for analytic use in later chapters.

SHAPES OF UTILITY FUNCTIONS

The first general distinction commonly made between the risk attitudes of individual decision makers is based on the shape of their utility functions defined with respect to wealth or another appropriate monetary outcome. The particular shape of the function indicates the decision maker's attitude toward additional wealth and can be used to draw inferences about a person's attitude toward risk either at a unique level of wealth or over a range of wealth levels. The possible shapes may be classified as concave (to the horizontal axis), linear, or convex. Concavity (convexity) reflects diminishing (increasing) marginal utility. A linear utility function reflects constant marginal utility.

In Bernoulli's day, the most popular decision rule treated the marginal utility of income as constant. Thus a *linear utility function* was postulated for maximization:

$$U(y) = ky \quad (3.1)$$

where y represents outcomes (usually a monetary value) and k is a positive constant. To illustrate, let a decision maker with the linear utility function in Fig. 3.1 choose between three possible choices. Choice 1 has two possible outcomes, y_1 and y_2 , which are equal but opposite distances from y_{CE} and occur with equal probability. Choice 2 has a certain outcome of y equal to y_{CE} . The expected utilities of these two choices, as shown in Fig. 3.1, are both $kE(y)$, since $y_{CE} = E(y)$. Thus the decision maker is indifferent between them.

Choice 3 has more narrowly dispersed outcomes, y_3 and y_4 , which also are equal and opposite distances from y_{CE} , occur with equal probability, and have an expected value of $E(y)$. Again the expected utility of the choice is $kE(y)$. Thus the decision maker's preference for choice 3 equals his or her preference for choices 1 and 2. Moreover, the choices are unaffected by changes in variance as long as the expected values are unaffected. For this case, the decision maker is considered risk-neutral.

Bernoulli disagreed with the behavioral result of the decision rule based on a linear utility function. He used the St. Petersburg gamble and other examples to show that individuals did not order choices according to their expected values.¹ He argued instead that diminishing marginal utility better described the preferences of a decision maker. In particular, he suggested that the correct and generally applicable function was the logarithmic utility function:

$$U(y) = \log y \quad (3.2)$$

This function is still popular, but not widely applicable in describing decision makers' preferences (Samuelson).

Most economists agree with Bernoulli that a concave utility function is a more reasonable assumption than a linear utility function. The concave utility function, as shown in Fig. 3.2, is interpreted to have diminishing marginal utility ($U'' < 0$) in contrast to the constant marginal utility of the linear function. The

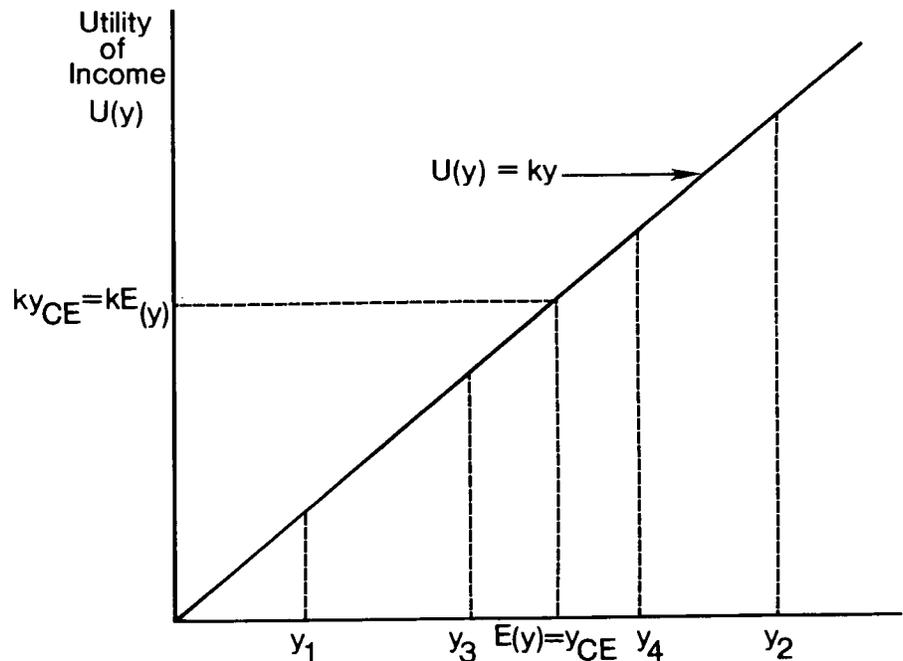


Figure 3.1 Representation of the linear utility of income function ky with constant marginal utility of income.

slopes of lines drawn tangent to the function $U(y)$ measure marginal utility. The declining slope of successively higher lines suggests diminishing marginal satisfaction for additional income. These decision makers will respond to risk in fundamentally different ways than will those with linear utility functions.

To demonstrate this difference in response, we present the decision maker with the same choices as in Fig. 3.1. Choice 1 has outcomes y_1 and y_2 , each occurring with probabilities p and $1 - p$, respectively. If $p = 1$, the expected utility of the lottery is $U(y_1)$, and if $p = 0$ the utility of the lottery is $U(y_2)$. In this example, p is $\frac{1}{2}$ and the expected value of the lottery is $E(y)$. Then the expected utility of the lottery $EU(y)$ is $\frac{1}{2}[U(y_1) + U(y_2)]$. Corresponding to this expected utility value is an income y_{CE}^* which, if received with certainty, would yield the same level of satisfaction as the lottery.

Now consider the decision maker's response to choice 2, yielding outcomes y_3 and y_4 with equal probability ($\frac{1}{2}$) and an equal distance from $E(y)$. The expected value of choice 2 is still $E(y)$, but the outcomes are much less dispersed than for choice 1. The expected utility of choice 2 is $\frac{1}{2}[U(y_3) + U(y_4)]$, which exceeds the expected utility of choice 1. Moreover, the certainty equivalent income for choice 1, y_{CE}^* , is less than that for choice 2, \hat{y}_{CE} , because of the latter's reduced dispersion of outcomes around $E(y)$.

For the decision maker with constant marginal utility, reducing the dispersion of the choice's outcomes, while leaving constant the expected values, does not alter his or her preference. Now, however, diminishing marginal utility

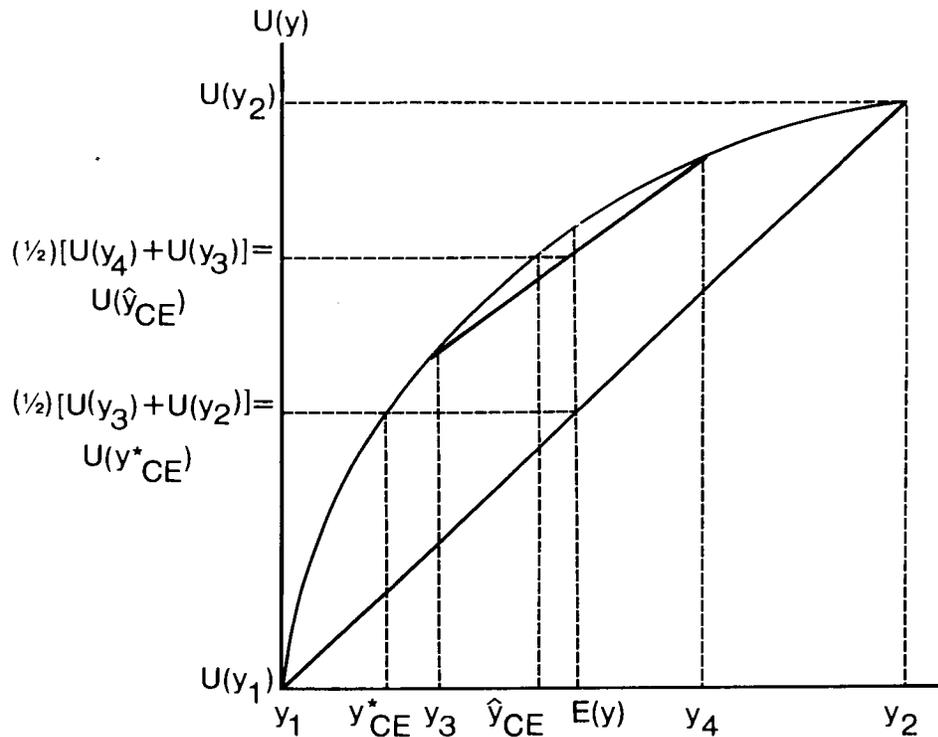


Figure 3.2 Graphical analysis of how decreasing dispersion of choice outcomes increases expected utility for risk-averse decision makers.

causes reductions in the expected utility for choices with greater dispersions and the same expected value. This result occurs because a positive increase in dispersion is more than offset in utility terms by an equal negative dispersion. More simply stated, the increase in well-being resulting from an increase in income from $E(y)$ to y_2 in Fig. 3.2 is less than the increase from y_1 to $E(y)$ because of diminishing marginal utility, even though $E(y) - y_1 = y_2 - E(y)$. Expected value-preserving spreads of probability increase symmetrically the likelihood of less favorable and more favorable events; the net impact is to reduce expected utility for a risk-averse individual with a concave utility function.

This discussion of concave utility functions could be repeated for convex functions, although the relationships would be reversed. Increasing the dispersion while leaving unchanged the expected value would increase a choice's expected utility. Thus decision makers with increasing marginal utility are called risk-preferring; that is, they prefer the choice with more widely dispersed outcomes when given two choices with the same expected value.

MORE DISCRIMINATING RISK ATTITUDE MEASURES

Concavity, linearity, and convexity of utility functions imply an ordering of individuals into three broad classes: risk-averting, risk-neutral, and risk-

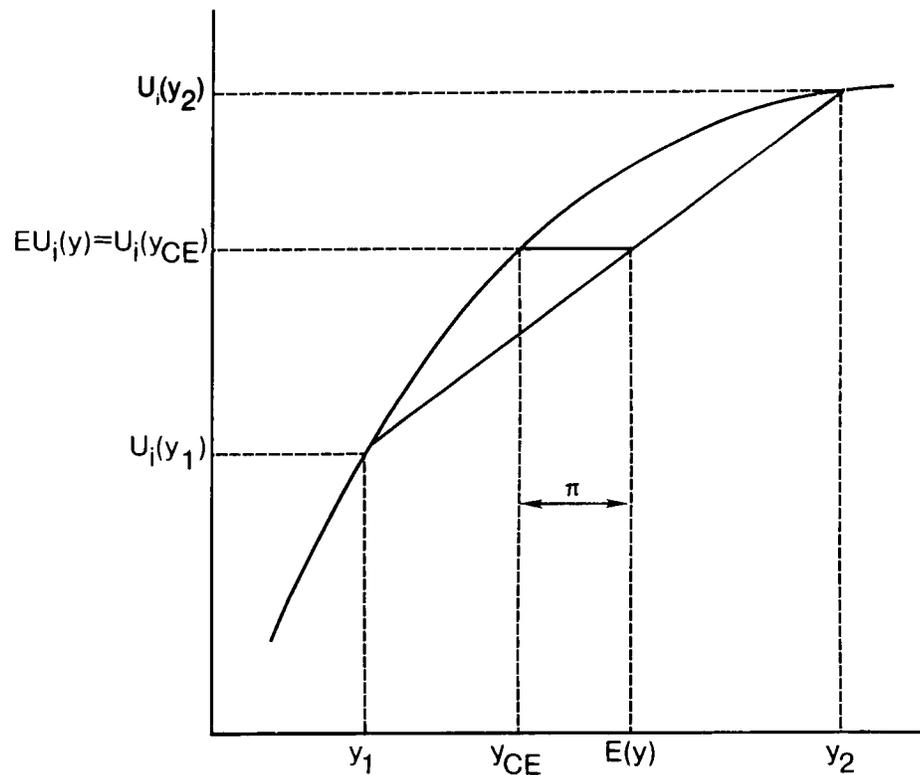


Figure 3.3 Arbitrary utility function $U_i(y)$ with indifference between the certainty equivalent income y_{CE} and the lottery with outcomes y_1 and y_2 which occur with probabilities p and $1 - p$ respectively.

preferring. These classes are indicative of the individuals' risk attitudes, but they do not give an exact measure of the degree of risk attitude. Suppose a more discriminating method is sought for ordering individuals according to their risk attitudes. We might, for example, wish to order risk-averse decision makers according to their degree of risk aversion. A common procedure is to elicit the responses of different individuals to identical sets of risky choices. A lottery with outcomes y_1 and y_2 is offered for sale to N individuals. Their maximum bids represent the certainty equivalents they would be willing to pay in exchange for the lottery. At an indifference point, the utility of the certainty equivalent y_{CE} (a maximum bid price) is equally preferred to the expected utility of the lottery.

For the i th individual with utility function $U_i(y)$, this equality is written as:

$$U_i(y_{CE}) = pU_i(y_1) + (1 - p)U_i(y_2) \quad (3.3)$$

where p = probability that y_1 will occur. We represent this indifference using the concave function $U_i(y)$ in Fig. 3.3. The expected value of the lottery is $E(y) = py_1 + (1-p)y_2$. The straight line connecting $U_i(y_1)$ and $U_i(y_2)$ indicates the expected utility for all possible values of $0 < p < 1$. The concavity of the

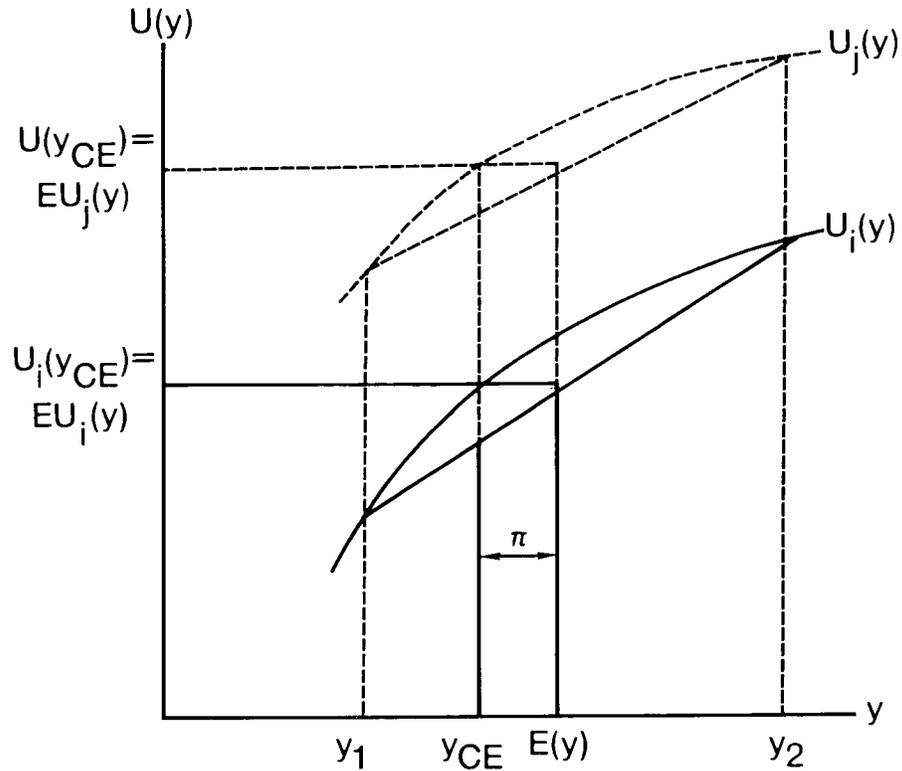


Figure 3.4 Arbitrary utility functions $U_i(y)$ and $U_j(y)$ showing indifference between y_{CE} received with certainty and the lottery with outcomes y_1 and y_2 occurring with probabilities p and $(1 - p)$, respectively.

function $U_i(y)$ suggests that the lottery's expected value must at least equal its purchase price y_{CE} . The difference between the expected value of the lottery and its certainty equivalent is the *risk premium* π that is often used to order individuals according to their risk attitudes (Pratt). The larger the risk premium, the more risk-averse the individual given the choices and the amounts of risk involved. The relationship between the shape of the utility function and the risk premium is important. Concave utility functions imply positive risk premiums, while convex functions imply negative premiums.

Consider two individuals i and j bidding certainty equivalent incomes for a lottery consisting of outcomes y_1 and y_2 with probabilities p and $1 - p$ respectively. The utility functions for the two individuals i and j are represented graphically in Fig. 3.4 as $U_i(y)$ and $U_j(y)$, where $U_i(y) = U_j(y)$ plus a positive constant. Interestingly, both bid certainty equivalent income y_{CE} in Fig. 3.4. Therefore, based on either their certainty equivalent bid price y_{CE} or the risk premium π , they have identical risk attitudes.

Consider two other individuals k and ℓ with utility functions $U_k(y)$ and $U_\ell(y)$ whose utility functions are related by the expression $U_\ell(y) = bU_k(y)$, where $b > 0$. As shown in Fig. 3.5, like individuals $U_i(y)$ and $U_k(y)$, their

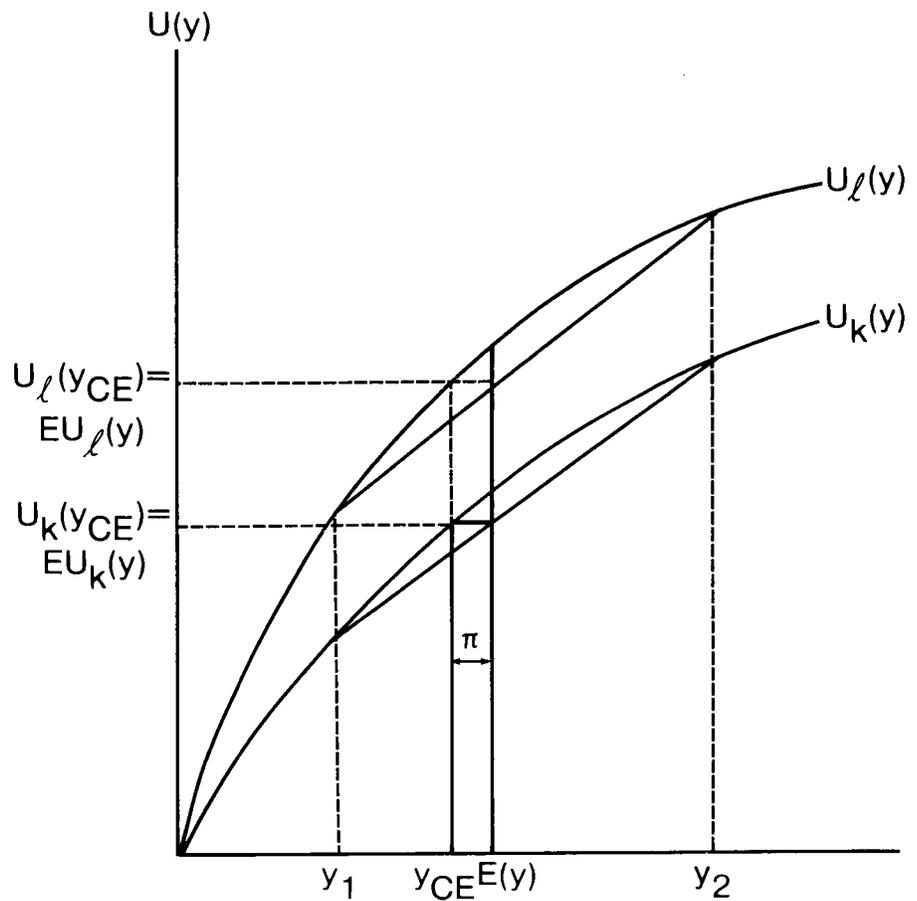


Figure 3.5 Arbitrary utility functions $U_k(y)$ and $U_l(y)$ showing indifference between y_{CE} received with certainty and the lottery with outcomes y_1 and y_2 occurring with probabilities p and $(1 - p)$ respectively.

respective certainty equivalent bid prices and risk premiums are equal.

The lesson here is that the utility function $U(y)$ is not a unique indicator of risk preferences, at least if the risk premium or certainty equivalent outcomes are to serve as risk attitude indicators. In fact, adding a constant and/or multiplying the utility function by a constant cannot change the risk premium or certainty equivalent income associated with the transformed utility function.

To show more generally the invariance of orderings caused by linear transformations of utility functions consider the utility function $U(y)$ used in the evaluation of two probability distributions $f(y)$ and $g(y)$. Suppose the expected utility of $f(y)$ exceeds the expected utility of $g(y)$. That is:

$$\int U(y)f(y) dy > \int U(y)g(y) dy$$

Choosing a different origin, say α , and changing the scale by multiplying $U(x)$ by the constant β , yields the same ordering. To show this we write:

$$\int [\alpha + \beta U(y)] f(y) dy ? \int [\alpha + \beta U(y)] g(y) dy$$

But the expected value of the constant α is just α , and the expected value of $E\beta U(y)$ is just $\beta EU(y)$. So we can write:

$$\alpha + \beta \int U(y)f(y) dy \geq \alpha + \beta \int U(y)g(y) dy$$

and, after canceling, we obtain:

$$\int U(y)f(y) dy > \int U(y)g(y) dy$$

The cardinal measure, therefore, is not the important characteristic of the utility function.

The characteristic of the utility function that does influence the risk premium is its bending rate. In Fig. 3.6 we compare two individuals with utility functions $U_i(y)$ and $U_j(y)$. As they are drawn, $U_i(y)$ bends at a greater rate than $U_j(y)$, since $U_i(y) = U_j(y)$ at y_1 and y_2 , yet is steeper at y_1 and flatter at y_2 . As a result, risk premium π_i associated with $U_i(y)$ is larger than risk premium π_j for function $U_j(y)$. This result suggests that individual i is, in general, more risk-averse than individual j but, as we shall see, this inference might be premature.

The utility functions in Fig. 3.6 are bending downward. As a function bends less in a downward or negative direction, the risk premium decreases—in Fig. 3.6 the decrease is from π_i to π_j . As the rate of bending in a negative direction approaches zero, the function $U(y)$ approaches a straight line and the risk premium π approaches zero. Thus the certainty equivalent of a risk-neutral decision maker with a linear utility function is the expected value of the lottery.

Positive bending of the function $U(y)$, on the other hand, implies a negative risk premium; that is, a decision maker will willingly pay amounts in excess of the expected value to acquire the lottery. As indicated above, these individuals are risk preferrers or risk lovers.

The direction of the bending—negative, zero, or positive—is indicated by the second derivative of $U(y)$. For $U''(y) < 0$ the bending is negative, $U''(y) = 0$ indicates no bending, and $U''(y) > 0$ implies positive bending. Thus either $U''(y)$ or the sign on the risk premium is used to classify decision makers into the risk-averse, risk-neutral, or risk-preferring category. But the magnitude of the second derivative is not a rate and therefore cannot be used for interpersonal comparisons of risk aversion because an individual's utility function is unique only up to a positive linear transformation. That is, the value of the second derivative can be arbitrarily varied by multiplying the utility function by a positive number. Hence $U''(y)$ is not suitable for ordering individuals beyond the classes of risk averters, risk neutrals, or risk preferrers.

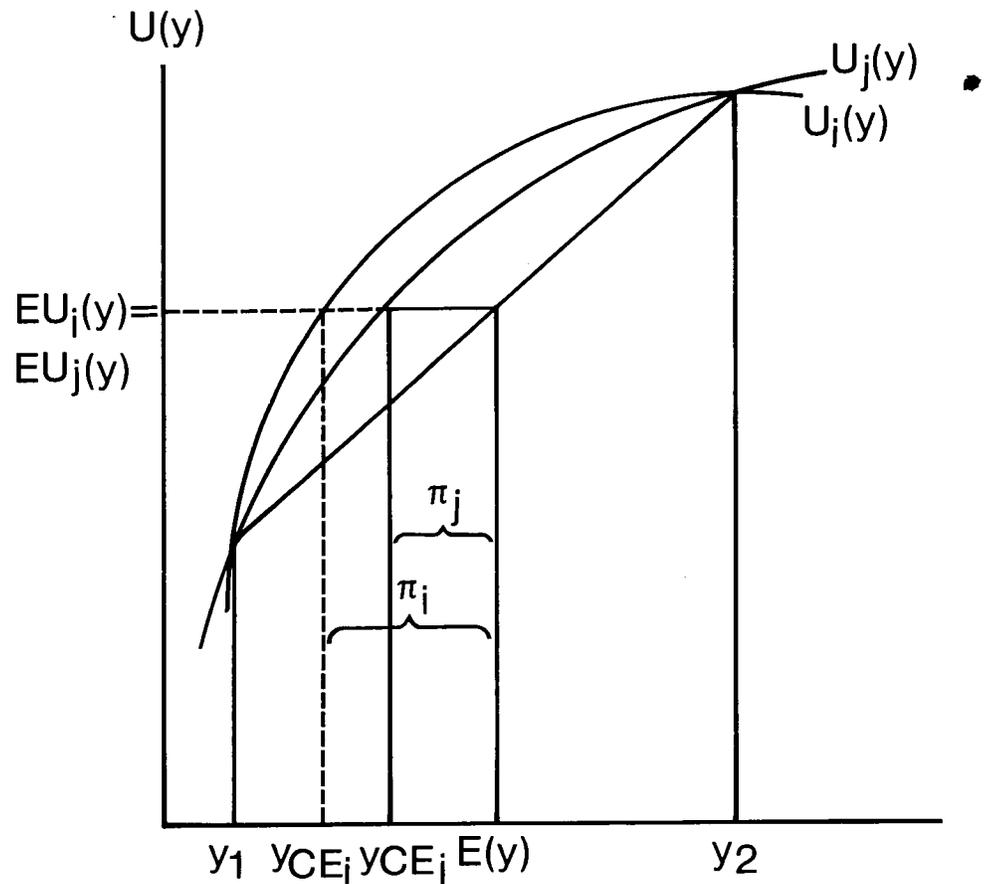


Figure 3.6 Comparison of risk attitudes of individuals i and j with utility functions $U_i(y)$ and $U_j(y)$ and certainty equivalent incomes y_{CE_i} and y_{CE_j} , respectively.

LIMITATIONS OF RISK PREMIUMS

Despite their intuitive appeal, risk premiums have limited usefulness for classifying decision makers according to their risk attitudes. They depend on probability distributions that may differ among choices and over time as well. Moreover, a risk premium is a single parameter used to represent risk attitudes described by the function $U(y)$. Thus it resembles a measure of central tendency, much like an expected value. To illustrate these limitations, consider the difficulty of classifying individuals with Friedman-Savage-type or Kahneman-Tversky-type utility functions.

Friedman and Savage argue that decision makers who display a preference for choices with both negative and positive risk premiums must have utility functions containing both concave and convex segments. Such a utility function is displayed in Fig. 3.7a. Friedman and Savage maintain that this function applies generally to all decision makers; it may reflect risk-preferring behavior for casino-type gambling and risk-averse behavior for business-type decisions.

Empirical estimates of utility functions, however, have not substantiated their claim. In fact, Kahneman and Tversky proposed a different type of preference function that is just the reverse of the Friedman-Savage function (Fig. 3.7b). Their empirical investigations have led them to conclude that people exhibit risk-preferring characteristics in loss situations and risk-averse characteristics in gain situations.

For the Friedman-Savage utility function in Fig. 3.7a, the risky prospect with possible outcomes y_1 and y_2 is equal in utility to the certain outcome of y_0 . The same is true for the utility function proposed by Kahneman and Tversky. In both cases the risk premium is zero. But in neither case are the decision makers considered risk-neutral, nor do they have identical risk attitudes. The difficulty is caused by inferring from a single parameter—the risk premium—the general characteristics of a decision maker's utility function. Thus the risk premium is considered a central tendency measure of risk aversion. It provides a risk attitude measure for a particular level of risk and over a particular range of wealth but may not accurately depict risk attitudes "in the large."

So, a risk premium cannot in general be used to order individuals according to their relative degree of risk aversion; utility functions themselves cannot be used either, since they are subject to linear transformations. The next section will introduce a unique, measurable means of ordering decision makers according to their risk attitudes. This measure reflects the bending rate of a utility function.

THE BENDING RATE AS A MEASURE OF RISK ATTITUDE

A unique measure of the direction of bending of $U(y)$ and the rate of change in slope of the function is the absolute risk aversion function. Introduced independently by Pratt and Arrow, it is defined as:

$$R(y) = \frac{-U''(y)}{U'(y)} \quad (3.4)$$

A related measure is the relative risk aversion function $R_r(y)$; it measures the elasticity of marginal utility and is defined as:

$$R_r(y) = \frac{-U''(y)y}{U'(y)} \quad (3.5)$$

Neither measure is affected by linear transformations of the utility function. They have positive values for risk averters, a zero value for risk-neutral decision makers, and negative values for risk lovers. Moreover, their uniqueness permits interpersonal comparisons at comparable wealth levels.

The absolute risk aversion function is the more commonly used risk attitude measure. Moreover, every function $U(y)$ has a corresponding function $R(y)$, and all linear transformations of $U(y)$ would map into the same function² $R(y)$. Thus representing decision makers by their absolute risk aversion function $R(y)$ is preferred to using a nonunique utility function³ $U(y)$.

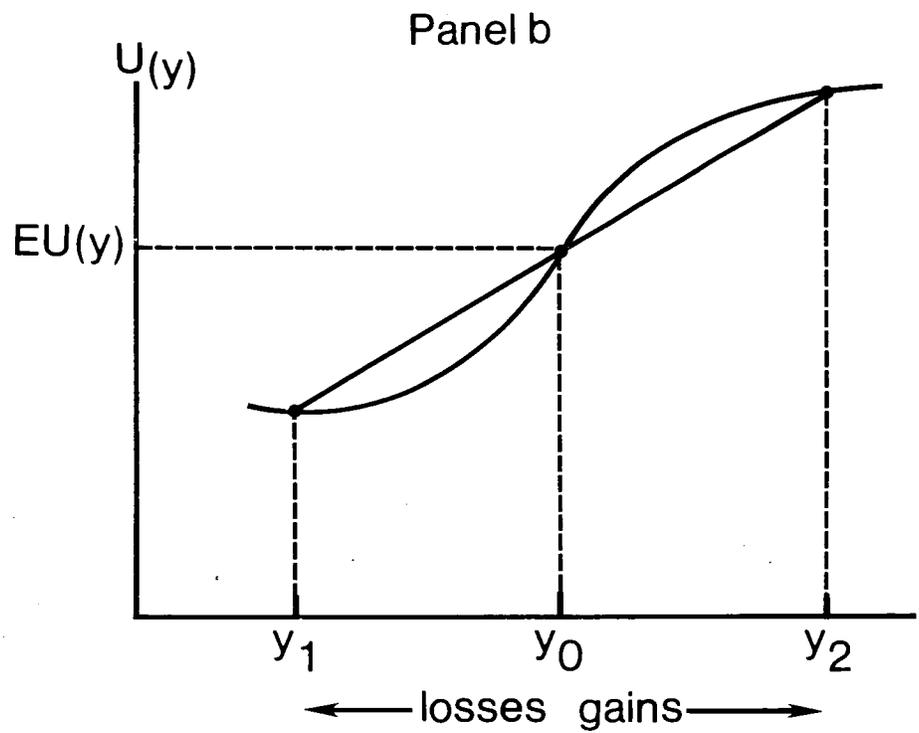
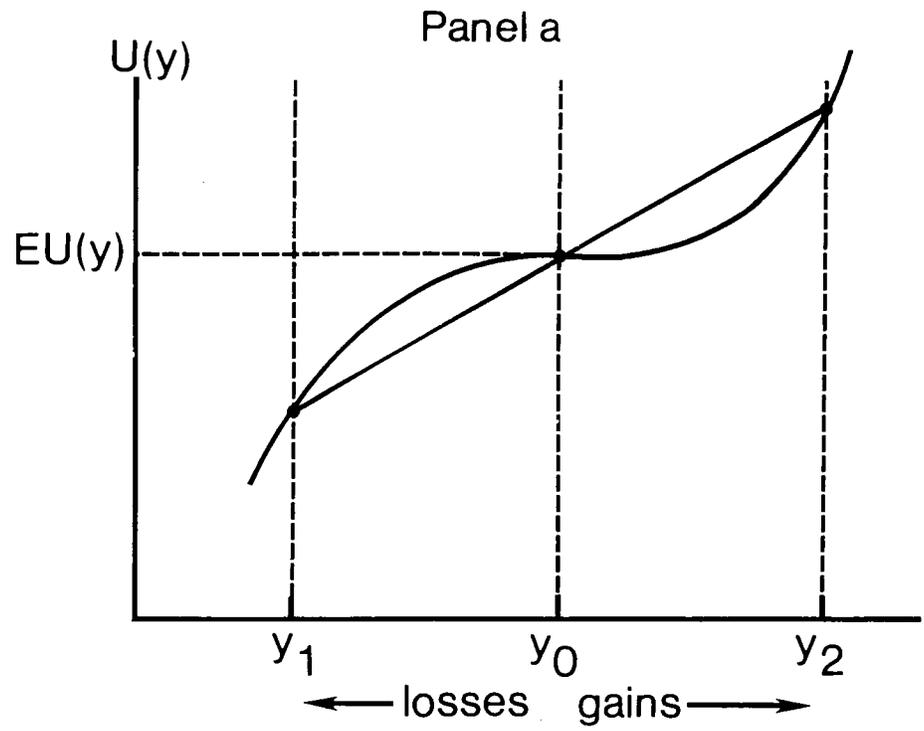


Figure 3.7 Comparison of the Friedman-Savage utility function and the Kahneman-Tversky preference function.

DARA, CARA, and IARA Risk Attitudes

The rate of bending function $R(y)$ has proven useful in classifying decision makers' risk attitudes. For all decision makers whose von Neumann-Morgenstern utility functions have derivatives $U'(y) > 0$ and $U''(y) < 0$, all values of the corresponding function $R(y)$ will be positive. Hence $R(y) > 0$ implies risk aversion. Beyond that, however, the sign of $R'(y)$ indicates how risk attitudes change as y increases.

If $R'(y) < 0$, the most usual assumption, decision makers are said to display *decreasing absolute risk aversion* (DARA). This implies that the risk premium for a lottery decreases as the decision maker moves to higher wealth levels. Similarly, $R'(y) = 0$ implies *constant absolute risk aversion* (CARA). As one might expect, the risk premium for CARA decision makers is constant regardless of changes in the decision maker's wealth. Finally, $R'(y) > 0$ implies *increasing absolute risk aversion* (IARA), suggesting that the risk premium increases for the same lottery with increases in wealth.

To illustrate DARA, CARA, and IARA utility functions, the absolute risk aversion function $R(y)$ and its derivative are calculated for three risk-averse utility functions: $U(y) = \ln y$; $U(y) = A - e^{-\lambda y}$ where $A, \lambda > 0$; and $U(y) = y - by^2$ where $b > 0$ and $(1 - 2by) > 0$. For the logarithmic utility function:

$$R(\ln y) = \frac{1}{y}$$

and

$$R'(\ln y) = \frac{-1}{y^2} < 0$$

implying a DARA risk attitude. For the negative exponential utility function:

$$R(A - e^{-\lambda y}) = \lambda$$

and

$$R'(A - e^{-\lambda y}) = 0$$

implying a CARA risk attitude. Finally, for the quadratic utility function:

$$R(y - by^2) = \frac{2b}{1 - 2by}$$

and

$$R'(y - by^2) = \frac{(2b^2)}{(1 - 2by)^2} > 0$$

implying a IARA risk attitude. Since IARA implies such a strong (and rarely observed) response to risk, one can understand why a quadratic utility function is not generally assumed except as a local approximation.

Local and Global Risk Attitude Measures

The absolute risk aversion function $R(y)$ indicates both local and global measures of risk aversion. Since $R(y)$ is a function defined over y , the measure of risk attitude can occur at any value of y . Consider a specific value for y , call it $E(y)$, and ask: For the i th and j th individual, who is more risk-averse at income $E(y)$? Another way to ask the question is: For small gambles with variance σ^2 , and expected value $E(y)$, which individual would pay the larger risk premium π to eliminate uncertainty?

To answer these questions, Pratt derived the approximate relationship:⁴

$$\pi = \frac{1}{2}R[E(y)]\sigma^2 \quad (3.6)$$

Equation (3.6) indicates that the risk premium π is equal to one-half of the product of the absolute risk aversion, measured at the expected value of the gamble $E(y)$, times the variance of the choice outcomes. Recall from Chap. 1 that the certainty equivalent income y_{CE} is found by subtracting π from $E(y)$. Using (3.6) we express y_{CE} as:

$$y_{CE} = E(y) - \frac{1}{2}R[E(y)]\sigma^2 \quad (3.7)$$

Equation (3.6) implies that, the greater the measure of risk aversion $R(y)$, the larger the required risk premium. Thus, in the small, or at a point, individuals can be ordered according to their degree of risk aversion either by their absolute risk aversion function valued at a point or by the size of the risk premium.

Ordering individuals globally or in the large creates another problem. When is one individual always more risk-averse than another? Consider two individuals i and j whose absolute risk aversion functions $R_i(y)$ and $R_j(y)$ are described in Fig. 3.8a. As the patterns of the functions $R_j(y)$ and $R_i(y)$ in Fig. 3.8a show, individual i 's risk aversion decreases as y increases, so that he or she exhibits decreasing absolute risk aversion. In contrast, individual j exhibits increasing absolute risk aversion. Both individuals face a choice with possible outcomes y_1 and y_2 and mean outcome $E(y) = \hat{y}$. From Eq. (3.6) we can determine that the i th individual is more risk-averse since $R_i(\hat{y})$ exceeds $R_j(\hat{y})$. If, however, the choice has outcomes y_3 and y_4 with mean outcome $E(\hat{y})^* = y^*$, then the j th individual is more risk-averse since $R_j(y^*)$ exceeds $R_i(y^*)$.

Now suppose the i th and j th individuals face a lottery consisting of y_2 and y_3 with mean $E(y^{**}) = y^{**}$. Which is more risk-averse? We cannot say based on the local measure of risk aversion. The individuals could be interrogated to find their respective certainty equivalents, and thus we could obtain risk premiums for the choice with outcomes y_2 and y_3 . However, we cannot infer that the individual with the larger risk premium is more risk-averse, because many utility functions with corresponding absolute risk aversion functions may have identical risk premiums. In this example, shifting the probability weights

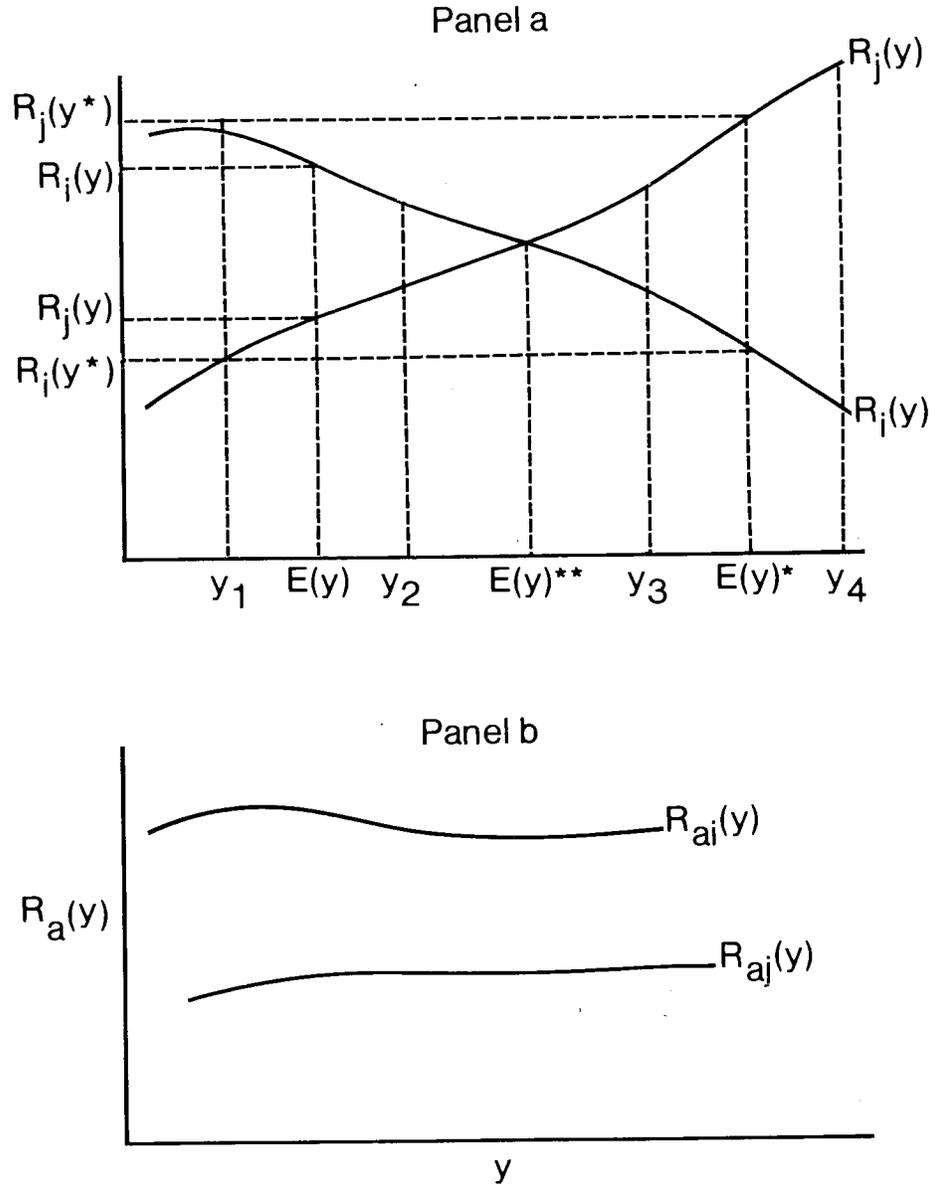


Figure 3.8 Comparison of absolute risk aversion functions $R_i(y)$ and $R_j(y)$ over outcomes y for the i th and j th individuals, respectively.

between outcomes y_2 and y_3 reverses the risk-averse orderings of the i th and j th individuals. This, however, is inconsistent with the condition that the risk attitudes are independent of probability measures.

If $R_i(y)$ were consistently greater than $R_j(y)$, as in Fig. 3.8b, then the risk premium for individual i would always exceed that for individual j , no matter what the probability distribution of choices. In this case, the i th individual would be globally more risk averse than the j th individual.

One condition which guarantees that one decision maker is more risk-averse than another is risk aversion in the large. Risk aversion in the large

requires that utility function $U^*(y)$ bends at a greater rate everywhere than does utility function $U(y)$. This is guaranteed, as Pratt showed, if $U^*(y)$ is a concave transformation of $U(y)$. The function g is a concave transformation if $g' > 0$ and $g'' < 0$, that is, if g is concave. To demonstrate how concave transformations increase the absolute risk aversion function $R(y)$, we write:

$$U^*(y) = g[U(y)] \quad (3.8)$$

To find the corresponding absolute risk aversion function $R^*(y)$, we recognize that:

$$U^{*'}(y) = g'[U(y)]U'(y) \quad (3.9)$$

and

$$U^{*''}(y) = g'[U(y)]U''(y) + [U'(y)]^2 g''[U(y)] \quad (3.10)$$

With these derivatives we can write:

$$\begin{aligned} R^*(y) &= \frac{-\{g'[U(y)]U''(y) + [U'(y)]^2 g''[U(y)]\}}{g'[U(y)]U'(y)} \\ &= R(y) + R[g(y)]U'(y) \end{aligned} \quad (3.11)$$

where $R[g(y)] = -g''[U(y)]/g'[U(y)]$. But since $R[g(y)] > 0$, as is U' :

$$R^*(y) > R(y) \quad (3.12)$$

So, increasing concavity increases the value of $R(y)$. If $R_i(y) > R_j(y)$ for all y , then decision maker i is considered globally more risk-averse than decision maker j , a condition we showed was guaranteed by one utility function being a concave transformation of another.

The consequences of decision maker i being globally more risk averse than decision maker j are:

1. For every lottery faced by individuals i and j , individual i will pay a larger risk premium (π_i) than individual j (π_j) in order to eliminate uncertainty (Pratt).
2. Equivalently, since $\pi = E(y) - y_{CE}$, where y_{CE} is the certainty equivalent income or the lottery sale price acceptable to the lottery's owner, the certainty equivalent income for individual j exceeds the certainty equivalent income for individual i .
3. Individual i 's utility function curves at a greater rate than individual j 's; that is, his marginal utility declines at a faster rate. This is true if $U_i(y) = g[U_j(y)]$ where $g' > 0$ and $g'' < 0$; that is, g is concave.
4. A lottery exists that would be acceptable to individual j but not to individual i .
5. When facing choices that combine a certain choice and a single risky asset, individual i 's preferred choice contains more of the safe asset than individual j 's.

While these statements can all be deduced from global risk aversion, it is important to maintain the distinction between local and global risk aversion. Equation (3.6) is a local measure of risk aversion at $E(y)$. The absolute risk aversion coefficient is measured for a specific income level $E(y)$. Moreover, holding σ^2 constant, the ordering of the risk premiums must be consistent with ordering of risk aversion coefficients measured at $E(y)$.

Average Risk Aversion

The global risk aversion measure is so strict that very few decision makers can be considered more or less risk-averse than others over all levels of wealth. Pratt's local approximation, on the other hand, applies only at a local level. For distributions with dispersion beyond local bounds we cannot be confident that the local measure is still useful. We suggest another risk aversion measure—an *average risk attitude measure*.

Consider first that the average of a constant is the constant. Thus for the constant absolute risk-averse utility function $-e^{-\lambda y}$, λ is a constant and an average risk attitude measure.⁵ To demonstrate how the parameter λ can be used more generally as an average risk attitude measure, consider two individuals facing the same distribution, one with absolute risk aversion function $R(y) = \lambda$ and another with a nonconstant absolute risk aversion function $R^*(y)$. Suppose that both would pay the same risk premium π to eliminate uncertainty. Then, since the average absolute risk aversion consistent with the risk premium is λ , both decision makers are said to have an average absolute risk aversion of λ for the uncertain choice.

Moreover, for any risk premium $\hat{\pi}$ associated with risk aversion function $R(y)$, we can always find a constant absolute risk aversion measure λ that leads to the same risk premium by adjusting λ until the risk premium associated with the utility function $-e^{-\lambda y}$ is $\hat{\pi}$.

Using mathematical notation, we can more precisely define average absolute risk aversion. Let the k th individual's utility function be U and her certainty equivalent for the probability distribution $g(y)$ be y_{CE_k} . Then we can write:

$$U(y_{CE_k}) = \sum U(y)g(y)$$

Also, by taking the inverse of U we could write:

$$y_{CE_k} = U^{-1} \left[\sum U(y)g(y) \right]$$

However, many utility functions may have identical certainty equivalent values for a distribution $g(y)$. Figure 3.7 illustrates graphically how two quite different utility functions (and absolute risk aversion functions) can both have the same risk premium or certainty equivalent when facing a gamble involving y_1 and y_2 with an expected value of y_0 .

One function that has the same certainty equivalent value, if λ_k is properly chosen, is:

$$U(y) = -e^{-\lambda_k y} \quad (3.13)$$

In particular, this function has a constant absolute risk aversion value that is also the average absolute risk aversion. For this function we can write:

$$-e^{-\lambda_k y_{CE_k}} = -\sum e^{-\lambda_k y} g(y)$$

and

$$y_{CE_k} = \frac{-\ln \sum e^{-\lambda_k y} g(y)}{\lambda_k}$$

The λ_k value above which equates the inverse of expected utility to the certainty equivalent y_{CE_k} is defined as the average risk aversion coefficient. The relationship between λ_k and y_{CE_k} is inverse. If $y_{CE_{k+1}} > y_{CE_k}$, then we expect the average risk aversion coefficient $\lambda_{k+1} < \lambda_k$. These results are consistent with the relationship between certainty equivalents and absolute risk aversion established for Pratt's local and global risk aversion measures.

From the above relationship, given a certainty equivalent income, the average risk aversion coefficient λ_k can always be found. This measure orders decision makers according to their risk aversion for a particular distribution. It provides a more general measure than Pratt's local measure, yet is less restrictive than his global measure. Its limitation is that it depends on the measures of probability.

Average Risk Aversion and Normal Distributions

Where the utility function is characterized by constant absolute risk aversion and $g(y)$ is normally distributed with mean $E(y)$ and variance σ^2 , the utility of the certainty equivalent income y_{CE} set equal to expected utility can be written as:

$$U(y_{CE}) = - \int (2\pi\sigma^2)^{-\frac{1}{2}} \exp \left\{ \frac{-[y - E(y)]^2}{\sigma^2} - \lambda y \right\} dy \quad (3.14)$$

Freund, and Hildreth before him, solved for y_{CE} in the above expression and found:

$$y_{CE} = E(y) - \frac{\lambda\sigma^2}{2} \quad (3.15)$$

Rearranging and remembering that $E(y) - y_{CE}$ is the risk premium yields:

$$\pi = \frac{\lambda\sigma^2}{2} \quad (3.16)$$

which is Pratt's local approximation. Because λ is constant in this case and no moments above the second exist, the expression is an exact equality and no

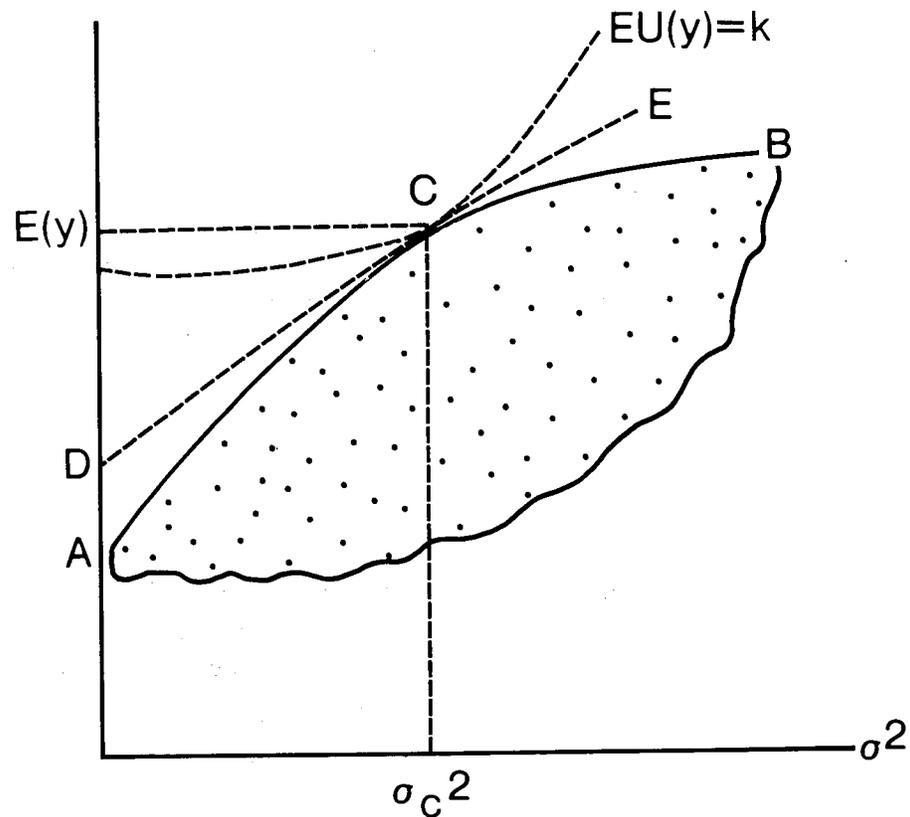


Figure 3.9 An expected value–variance efficient choice set.

EXPECTED VALUE–VARIANCE TRADE-OFFS

It is common (e.g., Binswanger, Brink and McCarl, Dillon and Scandizzo) to infer orderings of risk attitudes based on the trade-off between the expected value and variance at the equilibrium choice on EV sets. EV sets describe a risk-efficient frontier ACB , as in Fig. 3.9, from which the expected utility-maximizing choice can be found. Individuals who select choices above C are considered less risk-averse than those selecting choices actions below C . However, such an ordering may not clearly distinguish between risk-aversion measures in the small and in the large.

Consider, for example, the equation for the tangent line at C in Fig. 3.9. The intercept, D , a choice with zero variance, is a certainty equivalent outcome y_{CE} of the risky choice C with expected value $E(y)$ and variance σ_C^2 . The slope is a constant times the variance. Defining the constant slope coefficient as $\lambda/2$ yields an equation of the form:

$$E(y) = y_{CE} + \frac{\lambda}{2}\sigma^2 \tag{3.17}$$

where the intercept y_{CE} plus the slope times the variance at equilibrium equals the expected value $E(y)$ of the choice at point C . We can rearrange the equation

to obtain:

$$E(y) - y_{CE} = \frac{\lambda}{2}\sigma^2 \quad (3.18)$$

Since $E(y) - y_{CE}$ is by definition the risk premium π , we are left with Pratt's local approximation formula given in Eq. (3. 6). Twice the slope $\lambda/2$ or λ , turns out to be the local absolute risk aversion function value at $E(y)$. Since this is a local risk aversion measure, we cannot generalize about global risk attitudes based on EV slope coefficients. Only if all decision makers have constant absolute risk aversion functions can we make such global inferences about risk attitudes.

CONCLUDING COMMENTS

In this chapter we addressed the concept of ordering individuals according to their risk attitudes. We began by characterizing their attitudes into the general categories of risk-averse, risk-neutral, and risk-preferring based on the shape of their utility functions. Then we sought a more discriminating method of ordering risk attitudes within these broad categories. The bending rate of an individual's utility function serves this purpose. It can be measured by the Pratt-Arrow function of absolute risk aversion, which itself depicts a functional relationship between the level of risk aversion and wealth as the object of an individual's utility function. Following this approach an individual can be classified as having decreasing absolute risk aversion, constant absolute risk aversion, or increasing absolute risk aversion. Moreover, individuals can be compared in terms of local and global measures of their risk attitudes, although global comparisons are more difficult to achieve.

Ordering individuals according to their risk attitudes is a complicated process. Risk aversion in the strict sense is only represented by a function, but comparisons of functions are difficult. Unfortunately, the single parameters often used in comparisons, such as risk premiums, slopes on EV sets, and average risk aversion measures are not independent of the probability distributions of the choices being compared. As a result, the ordering may change when the distributions being compared are changed. Thus further work is needed to develop more general global comparisons of risk aversion.

APPENDIX 3A

Pratt has derived an approximate relationship between $R(y)$, the risk premium, and the variability of the choice. Recall that there exists a risk premium π such that the utility of the certain income $U[y - \pi]$ is equally preferable to the expected utility of any lottery $EU(y + z)$ where z is a random variable with mean zero and variance σ^2 . Thus, $E(y + z) = y$. So, we can write:

$$EU(y + z) = U(y - \pi) \quad (3A.1)$$

From Pratt we can solve for π by taking the Taylor series expansion about y of both sides of the expression above. This we write as:

$$U(y - \pi) = U(y) - \pi U'(y) + 0(\sigma^2) \quad (3A.2)$$

and

$$EU(y + z) = E \left[U(y) + zU'(y) + \frac{U''(y)}{2} z^2 + \dots \right] \quad (3A.3)$$

The term $0(\sigma^2)$ indicates a term small enough to be ignored without significantly changing the results. Taking the expectations operator inside the bracket in Eq. (3A.3) produces the result:

$$EU(y + z) = U(y) + \frac{U''(y)}{2} \sigma^2 + \dots \quad (3A.4)$$

Next we equate the two Taylor expansions and write:

$$U(y) - \pi U'(y) + 0(\sigma^2) = U(y) + \frac{U''(y)}{2} \sigma^2 + \dots \quad (3A.5)$$

Canceling out $U(y)$ from both sides and ignoring all terms beyond the second term in both expressions (since they must be small) we obtain an approximate equality relationship which we write as:

$$\pi = \frac{-U''(y)\sigma^2}{2U'(y)} \quad (3A.6)$$

Recognizing the ratio $-U''(y)/U'(y)$ as $R(y)$, we write:

$$\pi = \frac{R(y)\sigma^2}{2} \quad (3A.7)$$

ENDNOTES

1. The St. Petersburg gamble paid a prize depending on the number of coin tosses that occurred until the first heads (tails) appeared. The expected value of this gamble was infinity but no one would pay more than a finite sum to play.
2. The measure $R(y)$ is a unique measure of the bending rate and therefore a unique measure of risk associated with the utility function $U(y)$. Pratt shows this result by integrating twice the function $-U''(y)/U'(y)$. Integrating $R(y)$ gives $\log U'(x) + c$; exponentiating and integrating again gives $e^{cU(x)} + d$. But the constants of integration c and d are immaterial—that is, they do not affect risk premiums or certainty equivalent incomes. Thus the measure $R(y)$ contains all relevant information about $U(y)$, while eliminating all unnecessary information, including the constants of integration.
3. So far we have inferred that the essence of a decision maker's attitude toward risk is captured by the rate of bending in the ordinal utility function $U(y)$ or the absolute risk aversion function $R(y)$. This function alone, however, has no element of uncertainty or risk included in it. $R(y)$, for example, is simply a function defined over y . But the

manner of its derivation by finding indifference between risky alternatives makes it unclear whether the function represents simply ordinal ranking of a certain income or whether it is also a measure of risk attitude. This issue is discussed more completely by Fleisher. Whichever is true, we use the function to compare risk attitudes of individual decision makers.

4. The derivation of this measure is presented in App. 3A.
5. Define $R(y) = -U''(y)/U'(y)$. For $U(y) = -e^{-\lambda y}$, $U'(y) = \lambda e^{-\lambda y}$ and $U''(y) = -\lambda^2 e^{-\lambda y}$. Forming the ratio and canceling produces the result $R(y) = \lambda$.

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ORDERING RISKY CHOICES

In the preceding chapter, we established the concepts and methods for identifying risk attitudes and for ordering individuals according to their risk attitudes when they are faced with similar probability distributions. In this chapter, we shift the focus to the concepts and methods for ordering risky choices, based on comparisons of their probability distributions, for a given decision maker and utility function. The relationship between this chapter and the preceding one is important. The ability and accuracy of methods of ordering choices are directly related to our information about a decision maker's attitudes toward risk. The more detailed and precise the information, the more complete the ordering of risky choices. A complete ordering of risky choices requires substantial information about risk attitudes but also increases the chances of ordering errors if the attitudinal information is in error. A partial ordering of risky choices requires less comprehensive information about risk attitudes, and reduces the chances of ordering errors. Thus the ordering concept involves a trade-off between the completeness of ordering and the possibility of ordering errors.

ORDERING RULES FOR UTILITY FUNCTIONS

A decision maker's utility function is the basis for a complete ordering of choices, since it reflects all that is known about the effects of monetary outcomes on the decision maker's risk attitudes. Let $U(y)$ be the utility function of a decision maker and let $f_1(y), f_2(y), \dots, f_n(y)$ be the probability distributions describing the likelihood of outcomes for n risky choices. The decision maker must order these distributions according to his or her preferences. This is done by forming the preference (expected utility) indexes: $EU(y) f_1(y); EU(y) f_2(y), \dots, EU(y) f_n(y)$. A complete ordering of the

choices is based on the absolute differences between the index values. When an explicit answer is required, complete ordering by the utility function is warranted. The advantage, however, of such an explicit ordering is also a disadvantage: It applies only to a single decision maker. Beyond that the utility function has no application.

As noted earlier, some, such as Bernoulli, have argued that individuals have similar utility functions. The logarithmic utility function proposed by Bernoulli orders distributions according to their geometric means, the highest value being the most preferred.¹ A linear utility function, on the other hand, orders distributions based on their expected values.

There is no evidence, however, that either the logarithmic or the linear utility function accurately represents the preferences of all individuals. As a result, efforts have been made to evaluate the behavioral implications of various types of utility functions. If decision makers are risk-averse, then a quadratic utility function of the form:²

$$U(y) = y - by^2 \quad (4.1)$$

can be used where the restriction $b > 0$ must hold under risk aversion. One might also argue that a quadratic function reasonably approximates any concave utility function.³

Taking the expectation of a quadratic utility function leads to an equivalent expression of the function containing explicit parameters of probability density functions that can then be used to order risky choices. If, for example, y is stochastic with expected value $E(y)$ and variance σ^2 , then the expected value of Eq. (4.1) can be written as:

$$E(y - by^2) = E(y) - bE(y^2) \quad (4.2)$$

Since σ^2 equals $E(y^2) - (E(y))^2$, we can add and subtract $(E(y))^2$ without altering the equality and obtain:

$$\begin{aligned} E(y - by^2) &= E(y) - b \left\{ E(y^2) - [E(y)]^2 + [E(y)]^2 \right\} \\ &= E(y) - b \left\{ \sigma^2 + [E(y)]^2 \right\} \end{aligned} \quad (4.3)$$

This criterion, with $b > 0$, implies that risk is based on expected values and variances of choices. For $b > 0$, an increase in σ^2 while holding $[E(y)]^2$ constant increases the risk of choices and reduces their expected utility. Thus for choices having equal means but different variances, the choice with the smallest variance is preferred.

The quadratic function can be further restricted by limiting the value of b . Elton and Gruber, for example, argue that the y value at which negative marginal utility occurs (where $U'(y) = 1 - 2by = 0$) should be at least some specified distance from the mean of y . If the maximum value for y is, say, two standard

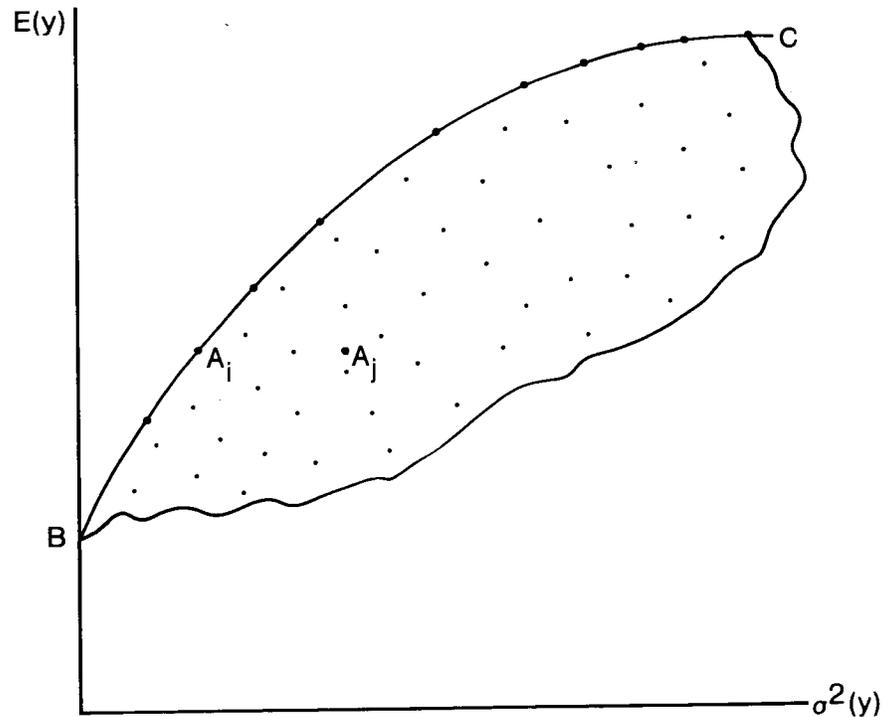


Figure 4.1 Expected value-variance efficient choice set represented by the line BC ; points interior to the line BC are inefficient.

deviations from the mean, then $y = E(y) + 2\sigma$ and $b = \{2[E(y) + 2\sigma]\}^{-1}$. With an explicit minimum value for b , a more refined criterion can be deduced.

This criterion, referred to as the expected value-variance (EV) criterion is described graphically in Fig. 4.1. Each dot in Fig. 4.1 describes the expected value and variance of a choice. Choices A_i and A_j have the same expected value, although A_i has the lower variance. Thus A_i is preferred according to the EV criterion. Again the risk attitude describes individuals whose preferences are represented by quadratic utility functions concave to the origin.

Choices along line BC are preferred to choices interior to BC for the identified class of decision makers. Sometimes the distinction used is efficient choices and inefficient choices. When choices are separated into efficient choices (e.g., points along BC) and inefficient choices (e.g., points interior to BC), then each inefficient choice is dominated by a choice in the efficient set. To illustrate, all quadratic risk-averse decision makers prefer A_i to A_j ; therefore A_j is inefficient.

STOCHASTIC DOMINANCE RULES

Imposing any functional restriction on the shape of the utility function limits its generality. As a result, more general ways to describe decision makers have

Table 4.1 Likelihood of outcomes associated with choices A_1 and A_2 with density functions $f(y)$ and $g(y)$ and cumulative functions $F(y)$ and $G(y)$, respectively.

Outcomes	Density function			Cumulative density function		
	A_1		A_2	A_1	A_2	
y_1	$f(y_1)$	=	$g(y_1)$	$F(y_1)$	=	$G(y_1)$
\vdots	\vdots		\vdots	\vdots		\vdots
y_i	$f(y_i)$	=	$g(y_i) - \alpha$	$F(y_i)$	<	$G(y_i)$
\vdots	\vdots		\vdots	\vdots		\vdots
y_k	$f(y_k)$	=	$g(y_k) + \alpha$	$F(y_k)$	=	$G(y_k)$
\vdots	\vdots		\vdots	\vdots		\vdots
y_n	$f(y_n)$	=	$g(y_n)$	$F(y_n)$	=	$G(y_n)$

been sought. The result has been more generally applicable efficiency criteria called *stochastic dominance rules* (Hadar and Russell, Hanoch and Levy).

Consider the class of decision makers who prefer more income to less, a very unrestrictive assumption. A decision maker from this class is faced with choices A_1 and A_2 whose outcomes and likelihood of outcomes are shown in Table 4.1. The outcomes, in ascending order, associated with the choices are y_1, \dots, y_n . The likelihood of outcomes for choice A_1 is described by either the density function $f(y)$ or the cumulative function $F(y)$. The likelihood of outcomes for choice A_2 is described by either the density function $g(y)$ or the cumulative function $G(y)$.

Suppose the probability functions f and g are related in the following ways: $f(y) = g(y)$ except for the i th and k th outcome. Outcome y_k is more likely if choice A_1 is selected while outcome y_i is less likely for this choice. That is, probability $\alpha < g(y_i)$ is subtracted from the likelihood of the i th outcome under A_1 and added to the likelihood of the k th occurrence. So, for choice A_1 , an event more satisfying, y_k , is more likely to occur, while a less favorable event y_i is less likely to occur. The result for all those who prefer more to less is to make choice A_1 less risky and preferred to A_2 . The effects of these probability shifts on the cumulative functions $F(y)$ and $G(y)$ are demonstrated in the last two columns of Table 4.1. The probability of outcome y , or something less (worse), is always less for choice A_1 than for choice A_2 .

This criterion, called *first-degree stochastic dominance* (FSD), is stated as follows: The choice associated with $F(y)$ is always preferred to the choice associated with $G(y)$ by all decision makers who prefer more to less if the

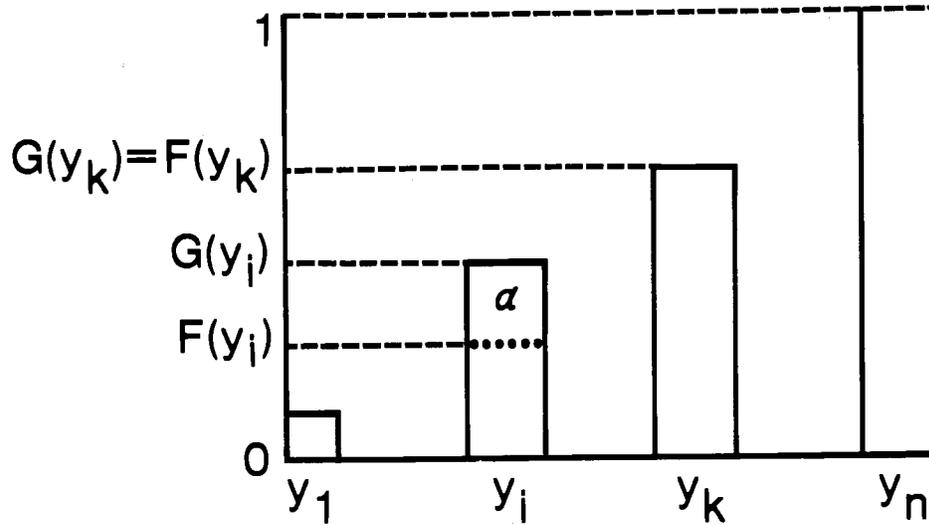


Figure 4.2 Graphical presentation of the condition on cumulative distributions $F(y)$ and $G(y)$ for $G(y)$ to be riskier or less preferred than $F(y)$.

condition $F(y) \leq G(y)$ holds for all y with strict inequality for at least one y . The condition is described graphically in Fig. 4.2. That is, the cumulative distribution function of the dominating choice always lies to the right of (below) the dominated choice.

Second-Degree Stochastic Dominance

First-degree stochastic dominance is based on the behavioral property that decision makers prefer more return to less, $U'(y) > 0$; it is the most general efficiency criterion. Its disadvantage is a limited ordering capacity. That is, the number of choices in a first-degree stochastic dominance efficient set is usually large. This is unsatisfactory because all the choices within the FSD set cannot be ranked against one another. Furthermore, if large numbers of choices are considered, the criterion becomes unworkable.

The solution is to further refine the description of decision makers' preference, and thus increase the ordering capacity. If, in addition to having utility functions exhibiting positive marginal utility [$U'(y) > 0$], decision makers are also risk-averse [$U''(y) < 0$], then *second-degree stochastic dominance* (SSD) can be used (Fishburn, Hadar and Russell, Hanoch and Levy).

Consider again a comparison between choices A_1 and A_2 having probability density functions f and g , respectively, with outcomes y_1, \dots, y_n arranged in ascending order. The distributions are constructed so that distribution f is obtained from distribution g by shifting the probability from the tails to the center of the distribution. For example, $\alpha < g(y_i)$ is shifted from the i th to the $(i + 1)$ st outcome. In contrast, $\beta < g(y_{k+1})$ is shifted back from the $(k + 1)$ st outcome to the k th outcome. The results are presented in Table 4.2.

In Table 4.2, the first probability shift α exceeds the shift β . For decision makers with diminishing marginal utility, we can unequivocally argue that the decision maker benefits by such a shift. The shift of a probability which makes the more favorable outcome y_{i+1} more likely, while reducing the likelihood of the less favorable outcome y_i , increases expected utility by the amount:

$$U(y_{i+1})\alpha - U(y_i)\alpha = \alpha \Delta U(y_i) \quad (4.4)$$

On the other hand, a shift in probability β from y_{k+1} to y_k , which is less favorable than y_{k+1} , reduces expected utility by the amount:

$$\beta U(y_{k+1}) - \beta U(y_k) = \beta \Delta U(y_k) \quad (4.5)$$

The difference between the gain of expected utility at y_i and the loss of expected utility at y_k is written as:

$$\alpha \Delta U(y_i) - \beta \Delta U(y_k) > 0 \quad (4.6)$$

It exceeds zero because $\alpha > \beta$ and because diminishing marginal utility requires the marginal utility at y_k to be less than at y_i .

Thus probability shifts which preserve the sign of the cumulative difference $\sum(G - F) \geq 0$ always imply that F is preferred to G . The cumulative distributions, along with the cumulative sum of the differences between F and G , are presented, respectively, in Fig 4.3 and 4.4. In Fig. 4.3 the cumulative distributions differ by probability amount α at y_i , where $F(y_i) < G(y_i)$, and by probability amount β , at y_k where $F(y_k) > G(y_k)$.

The cumulative sum of the difference between $F(y)$ and $G(y)$ is graphically described in Fig. 4.4. This measure is best thought of as the cumulative value of the area between the two cumulative distributions $F(y)$ and $G(y)$. And, since they differ only at points y_i and y_k , this area measure will have only two values, α and $\alpha - \beta$. In general, SSD occurs when the accumulated area of the cumulative function for choice A_1 remains less than that for A_2 as the area is summed over higher levels of monetary outcomes. This criterion allows the cumulative functions to cross, which FSD does not.

MEAN-PRESERVING SPREADS AND INCREASES IN VARIANCES

Spreading of probability from the center of a probability density function to its tails makes the transformed density function riskier than the original function for all risk-averse decision makers. This fact is the consequence of SSD. Table 4.2 described how probability can be spread so as to create an SSD-dominated distribution. Another way to spread probability leaves unaltered the expected value of the distribution.

Table 4.2 Probabilistic outcomes of choices A_1 and A_2

Outcomes	Probability density functions for		Cumulative density functions for		Sum of cumulative function differences $\sum [G(y) - F(y)]$
	A_1	A_2	A_1	A_2	
y_1	$f(y_1)$	$g(y_1)$	$F(y_1)$	$G(y_1)$	0
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
y_i	$f(y_i)$	$g(y_i) - \alpha$	$F(y_i)$	$G(y_i)$	α
y_{i+1}	$f(y_{i+1})$	$g(y_{i+1}) + \alpha$	$F(y_{i+1})$	$G(y_{i+1})$	α
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
y_k	$f(y_k)$	$g(y_k) + \beta$	$F(y_k)$	$G(y_k)$	$\alpha - \beta$
y_{k+1}	$f(y_{k+1})$	$g(y_{k+1}) - \beta$	$F(y_{k+1})$	$G(y_{k+1})$	$\alpha - \beta$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
y_n	$f(y_n)$	$g(y_n)$	$F(y_n)$	$G(y_n) = 1$	$\alpha - \beta$

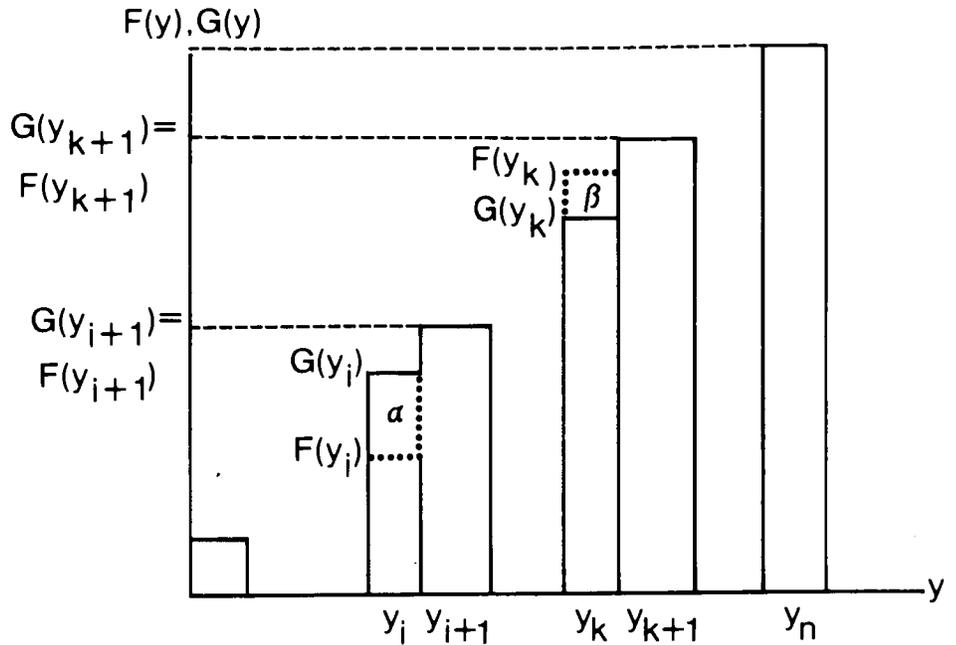


Figure 4.3 Cumulative density functions $F(y)$ and $G(y)$ for choices A_1 and A_2 .

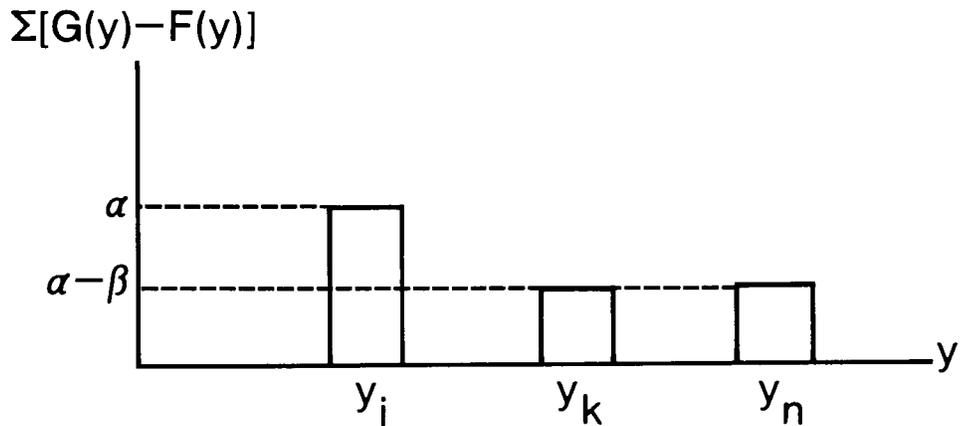


Figure 4.4 Cumulative sum of the difference $G(y) - F(y)$.

To illustrate, suppose a firm's income y is distributed with expected value $E(y)$ and variance $\sigma^2(y)$. One method of spreading probability without changing the expected value is to increase the variance of income from $\sigma^2(y)$ to $\sigma^2(\hat{y}) > \sigma^2(y)$. By doing so, the shifted distribution is made less preferred than the original for all risk-averse decision makers.

Of course, one might increase $\sigma^2(y)$ and at the same time increase $E(y)$. This spreading of the probability density distribution offsets the increase in variance with an increase in expected income, so that the preference for the original or the transformed distribution depends on the specific risk attitudes of

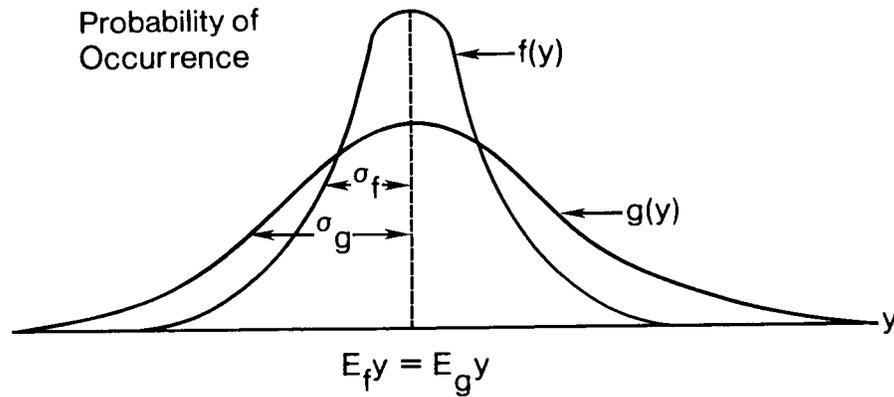


Figure 4.5 Alternative probability density functions.

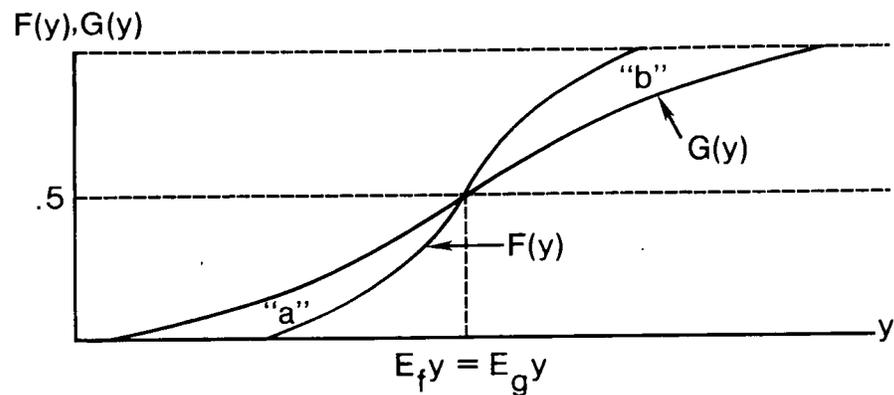


Figure 4.6 Alternative cumulative density functions.

the decision maker.

Throughout this book, we will examine the effects on an optimal choice of an increase in variance of the random factor or of the variance of income. When this occurs, increase in variance (without a change in expected income) constitutes a mean-preserving spread. Moreover, if variance is increased by spreading profitability to the tails, then the distribution with the larger variance is dominated by SSD.

Now consider a special application of SSD. Suppose the choice is between A_1 and A_2 , whose probability density functions f and g are normally distributed with equal expected values and different variances. Two such probability density functions are shown in Fig. 4.5 and their cumulative distributions in Fig. 4.6.

Since the distributions are normal and symmetric, the probability mass is divided equally at their expected values. Thus the cumulative distributions corresponding to F and G are equal and cross at their expected values, and symmetry requires that the difference in area between the distributions to the left of the expected value equal the difference in area to the right of the expected value. That is, area a in Fig. 4.6 equals area b . As a result, the cumulative

sum of the differences between G and F will always be positive, and F will be preferred to G by all risk-averse decision makers. This result leads again to the EV criterion: For normal distributions with equal expected values but different variances (standard deviations) the distribution with the smallest variance (standard deviation) is preferred. Thus two justifications for the EV set described graphically in Fig. 4.1 are: (1) risk-averse decision makers with quadratic utility functions and (2) risk-averse decision makers facing choices with normally distributed outcomes.

A third justification relied on in this book is Tobin's, which has been generalized by Meyer (1985).⁴ If choices consist of convex combinations of a single risky asset and a riskless asset, and the expected return on the risky asset exceeds the safe asset return, then all choices must be on the EV frontier. No choice is available which is not on the EV frontier. Moreover, the EV frontier describes an efficient set regardless of the probability distribution or risk attitude of the decision maker. And since the models considered here, for the most part, include only a single risky asset, we have a completely general justification for selecting the preferred choice from an EV set.

STOCHASTIC DOMINANCE WITH RESPECT TO FUNCTIONS

Each of the efficiency criteria identified above will order choices into efficient and inefficient sets for a particular class of decision makers. If one set of choices is preferred to another, this set is also less risky. Hence these efficiency criteria add definitiveness to the meaning and measurement of risk.

As practical tools, however, these efficiency criteria have a relatively low discriminatory power. For first- and second-degree stochastic dominance, the resulting efficient sets are too large (King and Robison). Moreover, the arbitrary classification of decision makers based on the derivatives of their utility functions is quite restrictive if decision makers display both risk-preferring and risk-averse attitudes.

Stochastic dominance with respect to a function is an evaluative criterion that orders choices without the restrictions of a particular utility function or specified characteristics of risk attitudes. The decision-making class is defined by upper and lower bounds on absolute risk aversion functions. Moreover, FSD and SSD are special cases of this more general efficiency criterion.

To illustrate, the class of decision makers ordered by FSD was assumed to have positive marginal utility, $U'(y) > 0$. This assumption places no bounds on the absolute risk aversion function, since $U''(y)$ can take on any value. Thus the decision-making class consistent with FSD is defined as:

$$-\infty < R(y) < \infty \quad (4.7)$$

The SSD set is more discriminating. Besides $U'(y) > 0$, it requires $U''(y) < 0$. Now the function $R(y)$ and the applicable class of decision makers are limited

to the risk-averse class with $R(y) > 0$:

$$0 < R(y) < \infty \quad (4.8)$$

Stochastic dominance with respect to functions utilizes a lower bound $R_1(y)$ and an upper bound function $R_2(y)$ on the absolute risk aversion function. Then the class of decision makers is:

$$R_1(y) \leq R(y) \leq R_2(y) \quad (4.9)$$

The necessary and sufficient conditions for ordering choices consistent with the class restriction were established by Meyer (1977). For distribution f to be preferred over g by decision makers in the class described by (4.9), the solution procedure requires a utility function $U_o(y)$ which minimizes:

$$\int [G(y) - F(y)] U'(y) dy \quad (4.10)$$

subject to:

$$R_1(y) \leq \frac{-U''(y)}{U'(y)} \leq R_2(y) \quad \text{for all } y$$

Equation (4.10) is equivalent to measuring the difference between expected utilities for distributions $G(y)$ and $F(y)$.⁵ Minimizing (4.10) involves a search for the decision maker in the defined set who is least likely to prefer F to G . If this member of the set, as defined by (4.9), actually prefers F to G , then so do all the other members of the set. Thus G is eliminated from the efficient set. If the member of the set described in (4.9) who is least likely to prefer F to G does not in fact hold this preference, then the procedure is repeated for G relative to F .

Meyer's solution to this problem requires an optimal control technique. Details of the solution are given in Meyer (1977), and an example is given in King and Robison. Applications of the technique are illustrated in Love and Robison.

CONVEX SET STOCHASTIC DOMINANCE

One characteristic of these efficiency criteria is the unanimity of the preference requirement. For example, if A_1 dominates A_2 by FSD (Table 4.1), then all decision makers who prefer more income to less must prefer A_1 to A_2 . Similarly, if A_1 dominates A_2 by SSD (Table 4.2), then all risk-averse decision makers must prefer A_1 to A_2 .

The most recent advance in the area of stochastic dominance relaxes the unanimity of the preference requirement. Convex set stochastic dominance (CSD), as developed by Fishburn and applied by Meyer (1979) and Cochran

et al., can be applied to any of the efficient sets already described. Suppose the efficient set identified by Meyer's choice among distribution algorithms consists of choices A_1, \dots, A_k, A_{k+1} . Then, according to CSD, if a convex combination of A_1, \dots, A_k can be found which dominates choice A_{k+1} , then A_{k+1} can be rejected from the efficient set. In essence, what CSD asks is: Can all the decision makers for whom the efficiency criterion applies find one member of A_1, \dots, A_k (not necessarily the same one) which is preferred to A_{k+1} ?

CONCLUDING COMMENTS

An efficiency criterion divides the decision alternatives into two mutually exclusive sets: an efficient set and an inefficient set. The efficient set contains the preferred choice of every individual whose preferences conform to the restrictions associated with the criterion. No element in the inefficient set is preferred by any of these decision makers. Thus the inefficient alternatives are no longer considered. In general, then, the stochastic dominance criteria and the expected value-variance (EV) criterion are efficiency criteria that provide a partial ordering of risky choices.

An efficiency criterion applies for a particular class of decision makers as defined by a set of restrictions on their utility functions. If these restrictions are rather general in nature, the criterion can order alternatives while requiring only minimal information about preferences. If enough alternatives are eliminated, decision makers can make a final choice from the efficient alternatives. A major problem with efficiency criteria, however, is the possible trade-off between their discriminatory power and their general applicability. Efficiency criteria that place few restrictions on preferences, and so apply for most decision makers, may not eliminate many choices from consideration. Conversely, criteria that identify small efficient sets usually require more specific information about preferences. Thus efficiency criteria help resolve some of the problems of single-valued utility functions, but have shortcomings of their own. Nonetheless, efficiency criteria are widely used in empirical analyses of risk situations.

In the remainder of this book, the preferred choice is uniquely identified from a set of possible choices. This requires that decision makers' risk attitudes be specified more precisely than required by the efficiency criteria discussed in this chapter. However, efficiency criteria will play a role. For the most part, all choices available to decision makers will be included in an EV set. This will allow us to specify the unique risk attitude which determines the optimal choice selected from an EV set as one whose desired trade-off between variance of returns and expected returns is $\lambda/2$. The consistency between this risk attitude specification and more general expected utility results will be discussed in Chap. 6.

ENDNOTES

1. The ordering equivalence of the geometric mean criterion and the expected logarithmic utility function requires that each criterion yield the same orderings of choices. Thus if there exists a positive monotonic transformation equating the expected value of the two functions, the orderings will be identical.

The geometric mean $E_g(y)$ of outcomes y_1, \dots, y_n with likelihoods of occurrence p_1, \dots, p_n respectively is:

$$E_g(y) = \prod_{i=1}^n y_i^{p_i}$$

A logarithmic function is a monotonic transformation which when applied to the above expression yields:

$$\sum_{i=1}^n p_i \log y_i = \log \left(\prod_{i=1}^n y_i^{p_i} \right)$$

Since the expected value of the log utility function is a monotonic transformation of the geometric mean, it must provide the same ordering.

2. Since utility functions are unique up to linear transformations, we can always transform quadratic functions of the form $U(y) = d + ey + fy^2$, where d, e, f are parameters, to obtain the expression above which has the single parameter b .
3. A second-order Taylor series approximation would, for example, lead to quadratic function approximation in a neighborhood.
4. A more formal statement of Meyer's generalization of the result used by Tobin is as follows. From Feller (p. 134) two cumulative density functions $F_1(\cdot)$ and $F_2(\cdot)$ differ only by location parameters a and b if:

$$F_1(y) = F_2(a + by) \quad \text{with } b > 0$$

Proposition If the set of random variables $\{y_i\}$ is described by cumulative density functions $G_i(\cdot)$ and there exists some random variable x with a finite mean and variance such that each $G_i(\cdot)$ differs from $F(\cdot)$ only by location parameters, then the ranking of y_i by expected utility can be represented by a ranking function $V(\sigma_i, \mu_i)$ where σ_i and μ_i are the standard deviation and mean of y_i (Meyer 1985).

For many of the models developed in this book, the condition stated above is satisfied; random variables do differ by location parameters, hence they can be ordered using EV methods, and no inconsistency between the EV model to be developed later and expected utility models is possible. In some later models involving the truncation of cumulative density functions to form choices, the consistency is not guaranteed except by the condition that only EV choices are available.

5. To show this let the difference in expected utility between f and g be:

$$\int U(y)f(y) dy - \int U(y)g(y) dy = \int U(y)[f(y) - g(y)] dy$$

Then applying the change-in-variable technique to integrate let $dv = f(y) - g(y)$, $v = F(y) - G(y)$, and $u = U(y)$. Then, recalling $u dv = uv]_{-\infty}^{+\infty} - \int v du$, we write:

$$\begin{aligned} \int U(y) [f(y) - g(y)] dy &= U(y) [F(y) - G(y)]_{-\infty}^{+\infty} + \int [G(y) - F(y)] U'(y) dy \\ &= \int [G(y) - F(y)] U'(y) dy \end{aligned}$$

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PART
TWO

ORGANIZING FIRM-LEVEL
RESEARCH UNDER RISK

FIRM-LEVEL MODELS UNDER RISK

The purpose of this chapter is to categorize the risk models that will be analyzed in later chapters. The possible models are numerous; only a few are considered here in order to establish the analytic approaches and to indicate the range of empirical applications.

CERTAINTY AND RISK CONDITIONS

The traditional theory of the firm is based on an assumption of perfect knowledge about the outcomes of choices. The rules for ordering choices under certainty are profit maximization subject to a cost constraint or cost minimization subject to an output constraint. These rules yield identical results. The utility function is not explicitly considered in certainty analysis because it is equivalent to profit maximization. That is, the same solution is found by maximizing profits as by maximizing utility of profits.

To demonstrate, assume the firm sells output q at a certain price p with costs of $C(q) + B$, where B is fixed cost and $C(q)$ is the variable cost. Also, assume the firm maximizes utility U of profit y :

$$\max U(y) \tag{5.1}$$

subject to:

$$y = pq - C(q) - B$$

The first-order conditions yield the optimal solution:

$$U'(y) [p - C'(q)] = 0 \tag{5.2}$$

Since $U'(y)$ is positive, it can be canceled, leaving the profit-maximizing condition in which marginal revenue equals marginal cost:

$$p = C'(q) \quad (5.3)$$

Second-order conditions, of course, require $C''(q) > 0$.

Under risk, however, the utility of profits does matter. The first-order condition for maximizing expected utility becomes:

$$E \{U'(y) [p - C'(q)]\} = 0 \quad (5.4)$$

In Eq. (5.4) E is the expectations operator which evaluates the expression by integrating continuous random variables, or by summing discrete random variables, over the range of probability density functions. Equation (5.4) may have several stochastic elements. Output price p , level of output q , the cost function $C(q)$, and fixed cost B may all experience random variation. Clearly, then, the introduction of risk produces many possible models.

The greater complexity of model organization under risk has several pay-offs. For example, numerous institutions have developed in direct response to risk. Futures markets and insurance companies are examples. Indeed, some responses by the perfectly competitive firm, in the absence of additional constraints, are described only if risk is introduced; various types of diversification and holding reserves are examples. Without stochastic models we cannot explain either the creation of some institutions or the firm's responses to risk. Under risk, however, we can begin to explain these and many other phenomena.

FIRM MODELS BY TYPE OF ACTIVITY

Consider a firm operating in a market in which it has no influence on product demand, input supply, or market prices. While these conditions seem fairly general, they still leave a wide range of responses to risk, depending on the basic activity of concern to the firm: production, marketing, or finance.

The firm's basic production process involves the acquisition of inputs x_1, \dots, x_n at prices p_1, \dots, p_n and their transformation through a production function f into product q . This relationship is expressed as:

$$q = f(x_1, \dots, x_n) \quad (5.5)$$

The firm sells the products q at output price p , holds q in inventory, or uses q to produce another product q^* . The costs per period of holding inventories r represent the return forgone by not selling q at output price p plus the cost of storage. If output q is an intermediate product, another transformation process h changes q into q^* for sale at output price p^* . For large firms, several intermediate steps may be required to produce a salable output.

The firm's marketing activities may also increase the expected utility of profits through the exchange of products between buyer and seller. Differences in the risk attitudes of market participants may motivate the exchange of goods under risk. In fact, the production models allow for trades of claims for risky outcomes that shift risk in accord with differing risk attitudes. A futures market is an example of an institutional arrangement for conducting the exchange of risky prospects.

The firm's financial activities include holding of liquid assets, borrowing and debt management, and utilizing credit (unused borrowing capacity) as a source of liquidity. These activities are critical to production and marketing. To begin production, the firm finances its inputs with internal equity capital E , debt capital D , or some combination thereof. Incurring debt involves an exchange with financial intermediaries where the firm trades some of its borrowing capacity or credit C , for debt capital D . If the return on credit reserves were zero, the firm would hold few reserves. But credit reserves are valuable because they provide the financial capacity for investing later, for making unforeseen but favorable investments, and for avoiding costly liquidations should future events prove less favorable than anticipated.

A joint analysis of these producing, marketing, and financing activities would yield a complex model. Therefore we begin by examining each activity separately. The most simple model describes the firm as acquiring inputs x_1, \dots, x_n at prices p_1, \dots, p_n and transforming them by an efficient process f to produce an output q which is sold at price p . The difference between pq and the cost of the inputs is profit y which is the object of expected utility maximization:

$$\max EU(y) \quad (5.6)$$

subject to:

$$y = pq - \sum_{i=1}^n p_i x_i - B \quad \text{and} \quad q = f(x_1, \dots, x_n)$$

Now allow the firm to hold part s of q in inventory. The expected return on inventory is \hat{p} , and the per-unit cost of holding an inventory is rsq . The expected utility of profit is now:

$$\max EU(y) \quad (5.7)$$

subject to:

$$y = (\hat{p} - r)sq + p(1 - s)q - \sum_{i=1}^n p_i x_i - B$$

$$q = f(x_1, \dots, x_n)$$

Next assume that q is an intermediate product which is sold, stored, or transformed via an efficient process h into a final output q^* which is sold at

price p^* . Let s_1 be the percentage of q stored, s_2 be the percentage used to produce p^* , and s_3 the percentage of q sold outright. The model is now:

$$\max EU(y) \quad (5.8)$$

subject to:

$$y = p^*q^* + s_1q(\hat{p} - r) + s_3qp - \sum_{i=1}^n p_i x_i - B$$

$$1 = s_1 + s_2 + s_3$$

$$q^* = h(s_2q)$$

$$q = f(x_1, \dots, x_n)$$

Next we assume that the firm's acquisition of inputs x_1, \dots, x_n is constrained by some limiting factors. These factors could include the cost and availability of credit from financial markets, diminishing marginal products, and increasing marginal risks. Financial intermediaries, for example, generally establish rules for determining borrowing limits. One type of rule might limit total credit to some multiple ζE of the firm's equity capital. Thus the cost of debt $\hat{r}D$ is subtracted from profits, and input acquisitions are limited to borrowing plus equity.

With these extensions, the basic production model is written as:

$$\max EU(y) \quad (5.9)$$

subject to:

$$y = p^*q^* + s_1q(\hat{p} - r) + s_3qp - \sum_{i=1}^n p_i x_i - \hat{r}D - B$$

$$1 = s_1 + s_2 + s_3$$

$$q^* = h(s_2q)$$

$$q = f(x_1, \dots, x_n)$$

$$\zeta E = D + C$$

$$D + E = \sum p_i x_i$$

Production models which include inventorying activities suggest that some attention be given to time. In what Hey (1981) calls "passive" risk models, decision makers allocate resources at the beginning of the period and then passively await the risky outcome. Another class of models will be considered in Part 4 of this book: The decision maker may make decisions both at the beginning of the period under risk and at the end of the period after the risky outcome has been realized. Such an intertemporal feature adds considerable realism (and complexity) to the analysis of a firm's response to risk.

CHARACTERIZING THE MARKETS

The basic market environment is a perfectly competitive market which is distinguished from its certainty counterpart. Both the certainty and risk models assume the firm is a price taker for its outputs and inputs; however, the risk model relaxes the assumption of perfect knowledge about input and output prices and production processes. Under risk, only the probability distributions of output or input prices, quantities of output, or availability of inputs may be known. If the firm has accurate knowledge of these distributions and meets the other conditions, then it is considered to operate in a perfectly competitive market.

Other market environments can be modeled as well. Consider, for example, a product whose price is not uniform among all product suppliers. This is typical of products sold infrequently, of products whose quality is not standardized or easily compared, and of products about which information is costly to obtain. In this market one supplier experiences some monopolistic control. Because information acquisition is costly to consumers, the seller knows that a price above the minimum will not result in a complete market loss as would occur in a perfectly competitive certainty market.

Consumers may also exercise monopoly control. Information is not free but is obtained at a cost, and information about a firm's price-setting policies tends to be cumulative. Eventually a high-priced firm will lose some of its customers to lower-priced firms. The potential for customer loss may counterbalance the seller's monopoly power.

Regulated markets are also important. These markets typically have firms which are granted monopoly power over the sale of a product, as in the case of public utilities. Other examples are salespeople who are granted exclusive marketing rights for a product in a particular area and labor unions which may obtain an exclusive right to provide labor services to a firm. In exchange for monopoly power, regulatory agencies impose varying degrees of price control. Still, monopoly power in one industry does not rule out substitutes. A wood stove, for example, substitutes for a gas furnace, and increased insulation substitutes for more energy use. Therefore, despite the regulated monopoly's control over its market, such as the public utilities' control, the quantity demanded may still be stochastic. Moreover, monopoly power does not eliminate stochastic input prices nor provide for control over future regulatory actions. Thus regulated firms must still make allocative decisions in the face of uncertainty.

NATURE OF THE FIRM'S INPUTS AND OUTPUTS

An important relationship exists between the activities of the firm and the types of inputs and outputs. Static production models most often assume that assets are both perfectly divisible in acquisition and use and nondurable. These assumptions are valid for some products, but not for others. Gasoline at a filling

station may be sold by the gallon, but deliveries to a farm may be available only in specific amounts. Fertilizer is conceivably available by the ounce, but bulk producers and distributors more likely deal in larger, less divisible quantities. Other assets such as tractors, combines, computers, cows, and buildings are not available in completely divisible quantities.

The services from most inputs are, however, divisible in use. A tractor is driven a second at a time, and fertilizer, if deemed desirable, can be applied an ounce at a time. This distinction between divisibility in acquisition and in use is not regularly made in the literature. Yet it plays a major role in models of investment and disinvestment.

The distinction between durables and nondurables has received more attention. Durables can be used in more than one production period, while nondurables cannot. Durables tie up resources beyond the single period of production; they impose an inventory cost on the firm, sometimes called a *fixed cost*. Durables force the decision maker to choose whether to disinvest now or later. Such decisions do not arise with nondurables.

Reversibility is another criterion for classifying assets. Risk is reduced if an investment, once made, can be undone. To reverse an investment, two conditions must be met. First, resale of the asset must be possible. Second, the difference between acquisition price and salvage value should not be so great as to discourage investment in or disinvestment from the asset.

The characteristics of divisibility versus lumpiness in acquisition and use, durability, and reversibility apply to both inputs and outputs. Unless a product is durable, hedging or storing is not a feasible alternative. Lumpiness may also be important in determining the firm's marketing strategies. While hedging or forward-contracting strategies may appear desirable for the firm, contracts may be available only in lumpy sizes. This may force the firm into making a "yes" or "no" decision rather than taking the fine-tuning marginal approach in which the level of contracting is tailored precisely to the level of expected production.

ORGANIZING MODELS BY SOURCE OF RISK AND RISK RESPONSES

The two most important categories for classifying firms under uncertainty are (1) the sources of risk and (2) the firm's risk responses. These two categories often are closely related. In some cases a direct relationship exists between the source of risk and the possible risk responses; in other cases the relationship is less evident. Establishing and measuring these relationships are important objectives of risk analysis.

We begin this section by identifying potential sources of uncertainty. In reference to the expected utility model in Eq. (5.9), the firm's sources of risk include:

1. p^* , an uncertain price for its final product

2. p , an uncertain price for its intermediate product
3. \hat{p} , an uncertain future price for its stored intermediate product
4. r , the storage cost per unit of q
5. p_i ($i = 1, \dots, n$), the price of its inputs x_1, \dots, x_n
6. \hat{r} , the price of borrowed funds used by the firm, and implicitly the return on unused borrowed funds or credit
7. The input-output relationship between q^* and q described by h
8. The input-output relationship between q and x_1, \dots, x_n described by f
9. The input-output relationship between $D + C$ and E described by the parameter ζ
10. The availability of x_1, \dots, x_n

Other sources of risk are attributed to the reliability and availability of information about the probability density functions for random events and about the utility function of the decision maker. We resolve risky decisions by maximizing single-valued utility functions over probability functions, both of which are assumed to be measured without errors. In reality, however, error-free measurements are rare. Thus the incorporation of new information is another dimension of firm-level models.

Standing in relationship to the sources of risk are the potential responses to risk. These include:

1. Adjusting input levels x_1, \dots, x_n and output q
2. Holding reserves $s_1 q$
3. Holding credit reserves C
4. Holding reserves of inputs x_1, \dots, x_n
5. Integrating vertically to produce q^*
6. Gathering information
7. Postponing decisions
8. Forward-contracting
9. Hedging
10. Diversifying enterprises, that is, integrating horizontally to produce products in addition to q , \hat{q} , and q^*
11. Acquiring risk-reducing inputs
12. Investing in production processes with a flat average cost curve
13. Buying flexible inputs
14. Buying insurance
15. Specializing
16. Adjusting financial leverage
17. Diversifying operations spatially
18. Spreading transactions over time
19. Participating in public programs designed to reduce risk
20. Utilizing share leasing of resources

Consider how these responses to risk are related to the source of risk. Suppose one of the output prices p^* , p , or \hat{p} , one of the input prices p_1, \dots, p_n ,

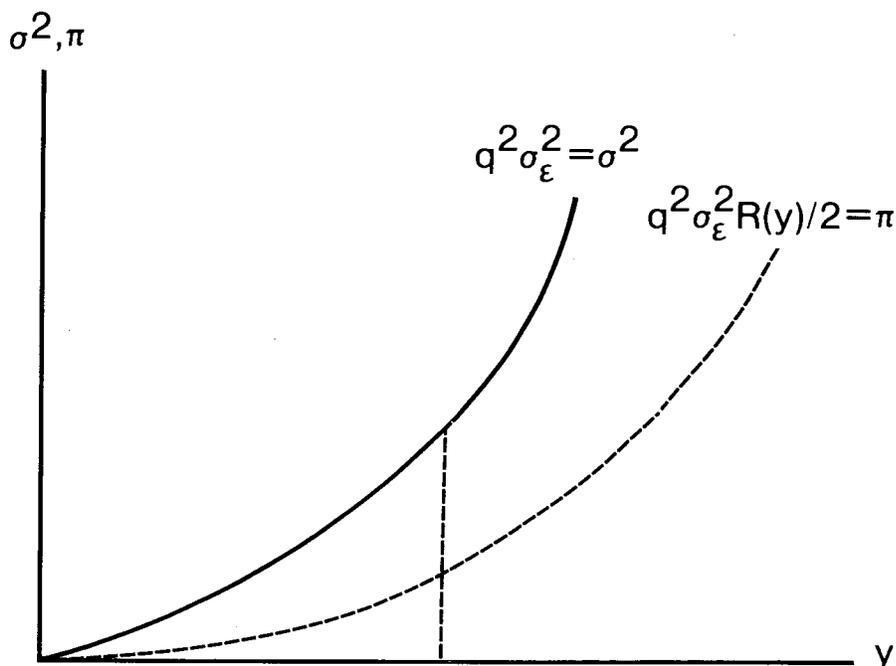


Figure 5.1 Quadratic relationship between levels of output q sold and total variance σ^2 and risk premium π .

or the cost of credit r is a random variable. The simplest, most direct way to reduce the firm's cost of risk π is to reduce the firm's holdings of q . To illustrate, consider the risk premium π which was earlier identified using Pratt's approximation:

$$\pi = \frac{R[E(y)]\sigma^2}{2} \quad (5.10)$$

where σ^2 = variance of profits and $R[E(y)]$ a measure of risk aversion at mean outcome level $E(y)$. The firm facing a variance of prices σ_ϵ^2 and purchasing or selling output q or input x_i has total variance σ^2 which is related to the level of outputs sold q by the expression $\sigma^2 = q^2\sigma_\epsilon^2$. To find Pratt's cost of risk π we multiply σ^2 by $R[E(y)]/2$ to obtain the dotted curve in Fig. 5.1.

A complication arises because expected returns often are reduced as risk is reduced. Thus economic analysis examines how firms determine the trade-off between expected returns and variance of returns. Figure 5.1 illustrates a fundamental fact about risk reduction. It occurs only as a result of altering the firm's holdings of risky assets. This response to risk links all the models we will examine and allows us to develop a similar approach to their solution.

Diversification is another means of altering the firm's risky asset holdings. Suppose the firm can produce two outputs q_1 and q_2 that are sold at stochastic prices of $p + \epsilon_1$ and $p + \epsilon_2$ where ϵ_1 and ϵ_2 have expected values of zero and variances of σ_1^2 and σ_2^2 , respectively. Let their covariance be σ_{12} or $\rho\sigma_1\sigma_2$, where ρ is a correlation coefficient. Recall (see the appendix, A Statistical Review) that

the total variance of returns from output sold σ^2 is:

$$\sigma^2 = q_1^2 \sigma_1^2 + q_2^2 \sigma_2^2 + 2\rho q_1 q_2 \sigma_1 \sigma_2 \quad (5.11)$$

Since the expected prices for q_1 and q_2 are equal, the firm's preferred combination of q_1 and q_2 depends on the combination of q_1 and q_2 which reduces σ^2 . If $\sigma_1^2 = \sigma_2^2$ and ϵ_1 and ϵ_2 were perfectly correlated, the firm could achieve no reduction in σ^2 by diversifying its production; that is, when $q = q_1 + q_2$ and $\sigma_1^2 = \sigma_2^2$ and $\rho = 1$, (5.11) can be written as:

$$\sigma^2 = [q_1^2 + (q - q_1)^2 + 2q_1(q - q_1)] \sigma_1^2 = q^2 \sigma_1^2 \quad (5.12)$$

and the total variance depends only on the total of $q_1 + q_2$ or q .

If, however, $\sigma_1^2 = \sigma_2^2$ and $\rho < 1$, total variance is reduced by diversification, as σ^2 can be written as:

$$\sigma^2 = [q^2 + 2(1 - \rho)q_1^2 - 2q_1(q - q_1)] \sigma_1^2 \quad (5.13)$$

The total variance-minimizing q_1 value can be found from the derivative of σ^2 with respect to q_1 :

$$\frac{d\sigma^2}{dq_1} = 2q_1 - q = 0 \quad \text{and} \quad q_1 = \frac{q}{2} \quad (5.14)$$

Samuelson has generalized the above result to show that, for n identically and independently distributed risky assets, the optimal solution for each is q/n . This result also holds when, instead of independence of the risky assets, we assume symmetric interdependence.

Continuing with our two risky asset examples, assume $\sigma_1^2 \neq \sigma_2^2$ and $\rho \neq 0$. Constraining $q_1 + q_2 = q$, σ^2 can then be written as:

$$\sigma^2 = q_1^2 \sigma_1^2 + (q - q_1)^2 \sigma_2^2 + 2q_1(q - q_1)\rho\sigma_1\sigma_2 \quad (5.15)$$

and the variance-minimizing value for q_1 can be found from the expression:

$$\frac{d\sigma^2}{dq_1} = q_1(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) + q(\rho\sigma_1\sigma_2 - \sigma_2^2) = 0 \quad (5.16)$$

and

$$q_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad (5.17)$$

When $\rho = 1$ and $\sigma_1^2 = \sigma_2^2$, q_1 is undefined. This result implies that $q_1 = 0$ and $q_1 = q$ are equally acceptable solutions. On the other hand, as σ_1 increases relative to σ_2 , the smaller becomes the right hand side of Eq. 5.17 and the optimal q_1 .

Whether or not diversification pays under the most general conditions described above depends on how variance increases with an increase in q . If all assets are sold as q_2 , then σ^2 increases by the quantity:

$$\frac{d\sigma^2}{dq} = 2q\rho\sigma_1\sigma_2 \quad (5.18a)$$

If the portfolio is diversified as in (5.13), the increase is:

$$\frac{d\sigma^2}{dq} = 2(q - q_1)\sigma_2^2 + 2q_1\sigma_2^2 \quad (5.18b)$$

The increase in (5.18b) is smaller than in (5.18a) precisely when $\rho\sigma_1 < \sigma_2$. Thus, as long as the adjusted standard deviation associated with $\epsilon_1(\rho\sigma_1)$ is not large relative to the standard deviation of $\epsilon_2(\sigma_2)$, diversification pays.

In other cases, gains can be obtained from specialization. Specialization, for example, may permit learning or introduce quality control so that the variance of outcomes decreases. This condition could be characterized as increasing returns to scale in risk. Mathematically, let the variance of returns from q , σ_q^2 , be expressed as σ^2/q , suggesting that σ_q^2 actually decreases with increased output levels. Total variance then is:

$$\sigma_q^2 = q\sigma^2 \quad (5.19)$$

This expression is linear rather than quadratic, so that a smaller variance is not possible, even with diversification. Still, the principle we introduced before applies. To alter risk, the firm must adjust its holdings of risky assets either by diversifying or specializing.

Other common responses to risk include integrating vertically and exchanging risky prospects for safe ones through hedges or forward contracts. Even the exchange of the firm's resources for additional information may reduce the perception of variance. Insurance schemes, selling equity shares in the business, and bringing in additional partners are other ways to alter the costs of risk to the firm. Risk reduction may not be the only goal of these actions, but in many cases it is. These methods are well-suited for the business that cannot easily diversify. A decreasing average cost curve or decreasing variance with higher levels of output may force a firm to specialize.

If the firm wishes to reduce risk, it can diversify the ownership of risk. The idea is developed as follows. Consider the cost of risk for an investment in q . It is written as:

$$\pi = R[E(y)] \frac{q^2\sigma^2}{2} \quad (5.20)$$

where $E(y)$ = expected return $q^2\sigma^2$ = variance of investment. Now divide q between n partners so that the i th owner holds an amount q/n of the risky asset and faces a risk premium of:

$$\pi_i = \frac{R[E(y)] (q/n)^2\sigma^2}{2}$$

The total risk premium of the n partners, assuming $R[E(y)]$ is the same for all partners, is:

$$n\pi_i = \frac{R[E(y)] q_2 \sigma^2}{2n} \quad (5.21)$$

Letting $n\pi_i$ be the total risk premium π^* , we can show that increasing n decreases the total cost of risk; differentiating π^* with respect to n , we obtain:

$$\frac{d\pi^*}{dn} = \frac{-R[E(y)] \sigma^2}{2n^2} = \frac{-\pi^*}{n} < 0 \quad (5.22)$$

Hence our result: Decrease the cost of risk by increasing the number of people sharing it. Sharing of the firm's risks may be arranged through hedging, share leases, participation in government programs, adding shareholders, and so on.

These two strategies may be combined. Inviting many people to participate in a risky investment reduces any one firm's commitment to the project and allows diversification into other projects which may be negatively correlated with the first.

Another general approach to risk reduction involves the holding of reserves. Holding reserves does not reduce the likelihood of risks occurring, but it does provide resources for coping with adverse events when they do occur. Holding credit reserves, cash reserves, or both, allows the firm to respond to financial emergencies or to undertake new investment opportunities. Holding excess machinery capacity allows the firm to respond to equipment failures. Holding crops and other goods in inventory permits a response to favorable changes in prices for these outputs.

But all holdings of reserves also involve a cost. By holding a credit reserve, the firm gives up possible returns obtained from using the funds in alternative investments. By holding machinery and outputs in reserve, the firm gives up earnings from other uses of these funds and experiences the risk of adverse price changes.

CONCLUDING COMMENTS

We return to our original conclusion that the principal response to risk involves changes in the holding of risky assets. This can be accomplished in numerous ways. In addition, relationships arise between the source of risk and the possible risk responses. Some risky investments are not well suited to division of ownership. Some risky assets exhibit increasing returns to scale under risk, while others do not. Some goods are not storable. The firm can control some factors, such as input purchases, and not others, such as lender-controlled credit reserves. The best approach in examining the usefulness of potential risk responses is a case-by-case approach.

Now, consider the analytic problem of examining a firm that faces a multitude of sources of risk, has available to it numerous risk responses, and operates in competitive, monopolistic, and regulated markets. In particular,

consider the case of 10 risk sources, 20 possible responses, operation in 3 different markets involving 3 different types of activities for purchases and sales of 6 different goods. The number of potential models, assuming all maximize expected utility, is: $10 \times 20 \times 3 \times 3 \times 6 = 10,800$. Obviously, the list of potential models could be increased to well over 10,800. Thus a need exists to simplify and standardize the approach for analyzing such a large number of different models. We will introduce this approach in the next chapter and apply it to a selected set of the potential models in later chapters.

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EQUILIBRIUM ANALYSIS UNDER RISK

As established in Chap. 1, a distinction between decision theory and economic theory is the former's emphasis on the selection of either a preferred choice or a set of preferred choices. In contrast, economic theory addresses changes in a preferred choice as changes occur in one or more of the underlying parameters. Under risk, these parameters may include changes in the variability of choice outcomes or changes in the expected level of choice outcomes. Thus economic theory under risk extends the firm's equilibrium analysis from the certainty case to the risk case. Using the expected value–variance framework, this extension yields a stochastic counterpart of the certainty case in which income and substitution effects resulting from shifts in EV efficient sets are distinguished by an investor's risk aversion characteristics.

In this chapter we introduce the fundamental properties of equilibrium analysis that serve as the analytic framework for model evaluations in later chapters. We begin by establishing the concept of expected value–variance analysis and show how maximizing the expected utility of a choice from an EV efficient set under certain conditions is equivalent to maximizing the certainty equivalent of the choice. Following this, we define the income and substitution effects under risk and illustrate them with an application to a savings problem.

ORIGINS OF EXPECTED VALUE–VARIANCE ANALYSIS

This chapter introduces an analytic framework that characterizes expected utility solutions in terms of their expected values E and variances V . To justify this approach, we will review the debate over the use of EV models and the more general EU models. Then we will show that the EV approach is more strongly justified as an analytic or deductive tool than as an empirical tool, although most studies using the EV approach have emphasized empirical applications.

EV Models and Decision Theory

Since its development by Markowitz as a portfolio selection tool, the EV model has been a popular method for ordering choices into efficient and inefficient sets. The EV efficient set is defined as the choices or sets of choices that provide minimum variance for alternative levels of expected returns. The efficient set is considered to contain the preferred choice for a well-defined set of investors. In contrast, the inefficient set does not contain the preferred choice.

Earlier, the conditions justifying the use of the EV model included assumption of a quadratic utility function or normally distributed outcomes. Tobin showed that, if the investor's utility function were quadratic, so that preferences were expressed only about the expected value and variance of choices, the preferred choice would be a member of the EV set (see also Chap. 4). Samuelson showed that a risk-averse expected utility maximizer's preferred choice would be from an EV set as long as the choices' outcomes were normally distributed and thus fully characterized by their expected values and variances.

Another justification for the EV approach was shown by Tobin but has generally been overlooked. He showed that expected utility-maximizing decisions are always members of an EV set when choices are represented by various combinations of a risky asset and a safe asset. The resulting choice set has no choices that are excluded from the EV set. Meyer has since shown that Tobin's condition is a special case of a more general condition requiring linear combinations of a random variable (Meyer).¹

Thus the EV approach is justified on the basis of four conditions: (1) quadratic utility, (2) normality, (3) choices involving a single random variable, and (4) choices involving linear combinations of the random variable. None of these conditions except 4 is very satisfactory relative to the characteristics of most empirical situations. Quadratic utility implies that marginal utility becomes negative beyond some level of monetary outcome and that the investor being modeled is characterized by increasing absolute risk aversion. Few variables take on values that range from negative to positive infinity as normality implies or are symmetrically distributed. And most decision situations concern choices involving more than one risky asset. These are, of course, all only sufficient conditions.

These shortcomings of conditions underlying the EV approach have made its justification in empirical analysis dependent on the ability to approximate results obtained with the more general EU model. Porter, for example, showed that EV sets of randomly constructed stock portfolios were consistent with EU models, except for portfolios having small expected values and variances. Tsiang demonstrated that various restrictions on skewness could yield a close correspondence between the EV and EU results. Levy and Markowitz showed similar effects of EV analysis as an effective approximating approach to portfolio selection. Moreover, the appropriateness of quadratic utility has been defended as a second-order Taylor series approximation to

all risk-averse utility functions. Thus the debate involving EU and EV models as decision tools has largely focused on the approximating capacity of the EV model.

The EV Model as a Deductive Tool

Decision theory models and the resulting efficient sets are valuable for their empirical content. They identify efficient choices and the resulting measures of financial performance (e.g., expected values and variances). They contribute importantly to improving the quality of decision making when the objective function is already known, and to predicting the magnitude of outcomes from various decision alternatives. But the empirical measures of expected values and variances of efficient choice distributions are not necessary in an analysis that serves to deduce economic theory under risk. The deductive or analytic results are valuable not for their measurability but for their characterization of relationships between variables and for showing the direction of change in relevant variables as changes occur in other factors that comprise the decision environment. These relationships and changes are basic to an analytic model.

Analytic models must of necessity be developed in as simple a fashion as possible, while still reflecting as rich a content as possible. Only a limited number of relationships can be analyzed simultaneously through deduction. Thus most analytic models formulated under risk contain only a single random variable. Adding more variables quickly complicates the derivations and tends to obscure the underlying relationships. In fact, stochastic dominance conditions are indeterminate for the evaluation of multiple risky assets under most circumstances (Hader and Russell).

Restricting the analytic model to be linear combinations of a random variable suggests that Tobin's rationale for EV analysis is applicable. That is, with all choices formed by combinations of a risky and a safe asset, only EV efficient choices are available. With this approach, we can seek to build an EV analytic model that yields results consistent with those obtained from the more general EU models. The point of departure for doing this graphically is to characterize any solution to the EV set in terms of maximizing a linear tangent line to the EV set where the line has slope $\lambda/2$. Since the choice of the slope is arbitrary, any solution from the EV set can be described using this approach.

In Fig. 6.1, the tangency point for the isoexpected utility line $E[U(\bar{y})]$ and the EV set AB occurs at choice C , yielding expected wealth $E(y_c)$ and variance $\sigma^2(y_c)$. To further develop this approach, let the linear tangent line in Fig. 6.1 be extended until it reaches the vertical expected return axis at y_{CE} . Since y_{CE} has zero variance, it is considered a completely certain return that is equivalent in terms of expected utility to the risky choice with expected return $E(y_c)$. Thus we can call y_{CE} the certainty equivalent to risky expected return $E(y_c)$ and refer to the line between y_{CE} and $E(y_c)$ as the certainty equivalent line. Moreover, the optimal risky solution can be obtained by maximizing the certainty equivalent subject to the restriction that the choice occurs from the EV

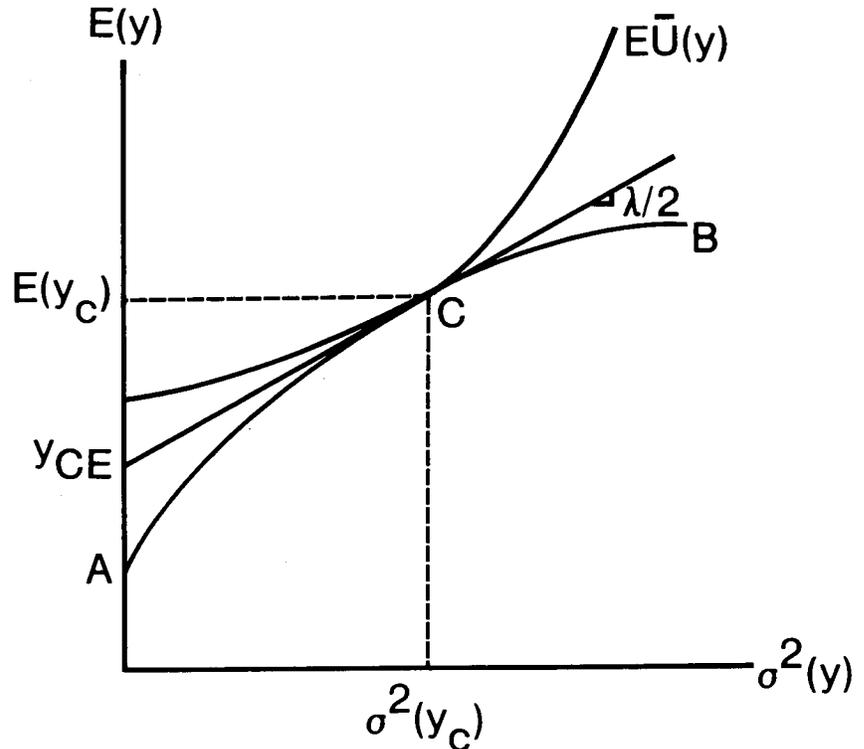


Figure 6.1 Equilibrium point C established by the tangency between an isoexpected utility line $E\bar{U}(y)$ and an EV set ACB identified by the y_{CE} linear tangent.

set and that its slope equals the equilibrium slope at the tangency between the isoexpected utility line $E\bar{U}(y)$ and the EV set, a slope designated as $\lambda/2$. This objective can be expressed as:

$$\max y_{CE} = E(y) - \frac{\lambda}{2}\sigma^2(y)$$

In this framework, the maximization of y_{CE} is equivalent to finding an optimal solution based on the equality of slope $\lambda/2$ between the EV set and the isoexpected utility lines, as illustrated by the tangency at C in Fig. 6.1. Maximizing the expected utility of a choice, where the expected value and variance are the objects of utility, is equivalent to maximizing its certainty equivalent. Moreover, it follows that if a change occurs in the shape or location of the EV set, then the optimal slope $\lambda/2$ must change to reflect a new point of tangency between an investor's isoexpected utility lines and the revised EV set. The optimal slope, then, can be described as a functional relationship:

$$\lambda = \lambda(r, \sigma_e^2, W_0, \dots)$$

where r and σ_e^2 are the mean and variance, respectively, of the return on the risky asset, W_0 is the level of allocable resources, and "... represents other variables affecting the location of the EV set.

The general problem is to formulate an analytic framework that shows how the optimal value of the control variable responds to changes in other variables

that shift and/or rotate the EV set. This in turn involves a description of how $\lambda(\dots)$ responds to changes in the EV set. This problem is approached by dividing the adjustment into two parts: an income effect and a substitution effect. The income effect is identical to the results obtained by Cass and Stiglitz for the generalized EU model. The substitution effect, to be derived later, requires a limiting assumption that may not always yield identical results for the EV and the EU models. The net results of the income and substitution effects, however, nearly always yield results for the EV model that conform to those for the EU model.

This use of the EV model is based on its strength as an analytic tool rather than as a decision theory tool. In this light, the primary analytic strengths of the EV model include: its relative ease in deriving optimal solutions and conducting equilibrium analysis; the natural relationship between the concepts of risk and variability and the statistical concept of variance; the ease with which it can be graphed using expected values and variances as the two dimensions; and finally, although not considered here, the extension of micro results to aggregative analysis. Although not justified by Tobin's single risky asset argument, the EV model permits a natural framework for analyzing multiple risky assets which is not analytically possible within the generalized EU framework. These strengths make the EV approach well-suited for modeling and analyzing many types of decision situations.

ASSET CLASS AND THE INCOME EFFECT UNDER RISK

Under conditions of certainty, an increase in an investor's wealth will increase the purchase of normal assets, leave constant the purchase of neutral assets, and reduce the purchase of inferior assets. Since all investors are considered to maximize profits in a certain world, and hence do exactly the same thing, these differences in asset purchases arising from a change in wealth are attributed to the characteristics of the assets, not to the investor. Under risk, however, this is not true. As wealth increases, investors facing the same investment choices may respond differently because of differences in their risk attitudes. Thus equilibrium analysis under risk utilizes information about these risk attitudes.

We begin the analysis by specifying a set of investors who can choose between a risky and a safe asset in formulating their portfolios. If these investors experience an increase in risk-free wealth, we must identify the conditions under which they will increase, leave constant, or decrease their holdings of the risky asset. The classification of risky assets as normal, neutral, or inferior is based on the generalization of a proof (contained in App. 6A) from Cass and Stiglitz that has the following result:

Theorem 6.1 If there are two assets, one risky and one safe, the total purchases of the risky asset increase, remain unchanged, or decrease with increases in initial wealth as there is decreasing, constant, or increasing absolute risk aversion.

This classification of risky assets is in turn based on the absolute risk aversion characteristics of the investor. A corollary to Theorem 6.1 is that, if the purchase of the risky asset increases, remains constant, or decreases with increases in risk-free wealth, depending on whether absolute risk aversion $R'(y)$ decreases, remains constant, or increases, then so does the variance of the choice. Together, the theorem and corollary imply that a risky asset is a normal, neutral, or inferior good depending on whether $R'(y) \lesseqgtr 0$.

Under conditions of certainty, the income effect is defined as a change in demand for an asset resulting from an increase in real income with prices held constant. Under risk, however, prices of assets are often random variables and, to identify the income effect, their probability distributions are assumed to remain constant as risk-free wealth increases. Thus the income effect under risk is the change in demand for an asset resulting from an increase in risk-free wealth while holding probability distributions constant.

Within the expected value–variance framework the income effect is expressed by a parallel shift in the EV set. To show the effect graphically, let the EV set be curve ACB in Fig. 6.2, with preferred choice C occurring at the point of tangency between an isoexpected utility curve $E[\bar{U}(y)]$ and the EV set ACB . Preferred choice C has expected value $E(y_C)$ and variance $\sigma^2(y_C)$. Now assume that the investor receives an increase in risk-free wealth that causes an upward parallel shift in the EV set. The upward parallel shift reflects the addition of risk-free wealth to all previous portfolios in the amount $(1+r)\Delta W_0$, where ΔW_0 is the change in wealth and r is the safe return. The increase in risk-free wealth also extends the EV set from B to B' because more risky assets can be held than before. Graphically the new EV set is $A'C'B'$.²

Now consider how investors within the three classes of absolute risk aversion will adjust to the parallel shift in the EV set. For the decision maker with constant absolute risk aversion, $R'(y) = 0$, the risky asset is a neutral good and the investor's holdings remain unchanged. The preferred choice is C' with the same variance as before. Because the shift is parallel, the slope at C' , given by a linear tangent, is the same as the slope at C .

For the investor with decreasing absolute risk aversion, $R'(y) < 0$, the new risky asset is a normal good; its purchase increases with an increase in risk-free wealth. This investor will move from the original choice B to a location on the new EV set $A'C'B'$ that lies to the right of C' . This new choice is represented by C'' . Since the slope of the EV set declines as one moves from A' to B' , the equilibrium slope at C'' is less than the slope at C' .

A similar analysis indicates that risky assets are inferior goods for investors with increasing absolute risk aversion. In this case, the equilibrium choice lies

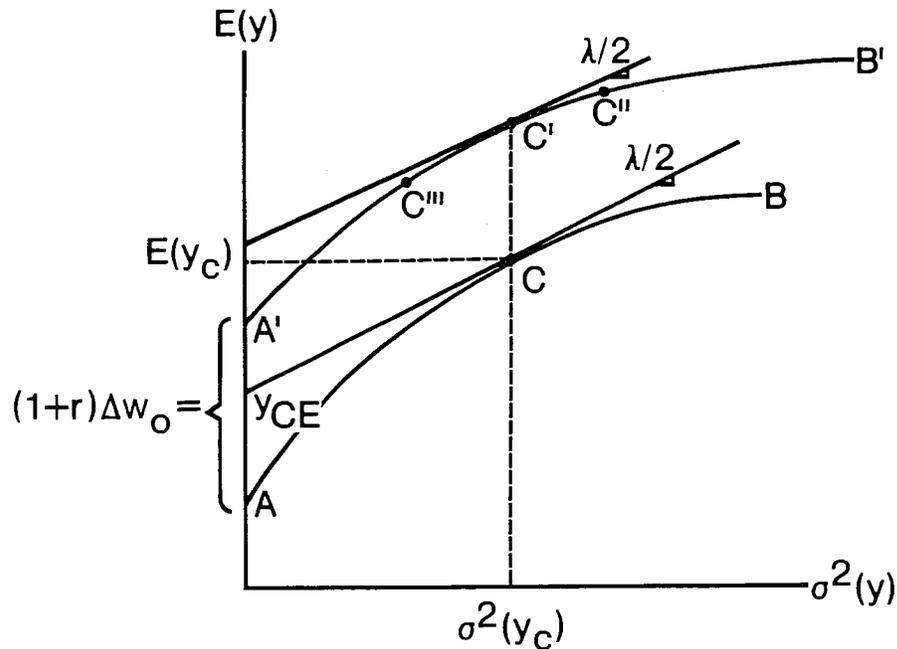


Figure 6.2 Income effects under risk measured on expected value-variance set ACB .

to the left of C' , say at C''' , with an increased slope.

To summarize the income effect, for an increase in risk-free wealth that extends the EV set and shifts it upward in a parallel fashion, the equilibrium slope decreases, remains constant, or increases for investors with decreasing, constant, or increasing absolute risk aversion, respectively. Moreover, since changes in the slope of an EV set correspond to changes in the holdings of risky assets, the purchases of risky assets decrease, remain constant, or increase for investors with increasing, constant, or decreasing absolute risk aversion, respectively.

THE SUBSTITUTION EFFECT UNDER RISK

Under conditions of certainty, the substitution effect is the change in demand for an asset resulting from a change in relative prices after compensating the decision maker for a change in real income; i.e., keeping utility fixed. Under risk, the substitution effect is the change in quantity demanded resulting from a change in the probability distribution of price after compensating the investor for a change in risk-free income.

To illustrate, consider an investor who is in equilibrium at point C on EV set ACB in Fig. 6.3. Let the probability distribution change so that the expected return on the risky asset increases. This change causes the EV set to rotate counterclockwise to a new location $AC'B'$. The substitution effect resulting

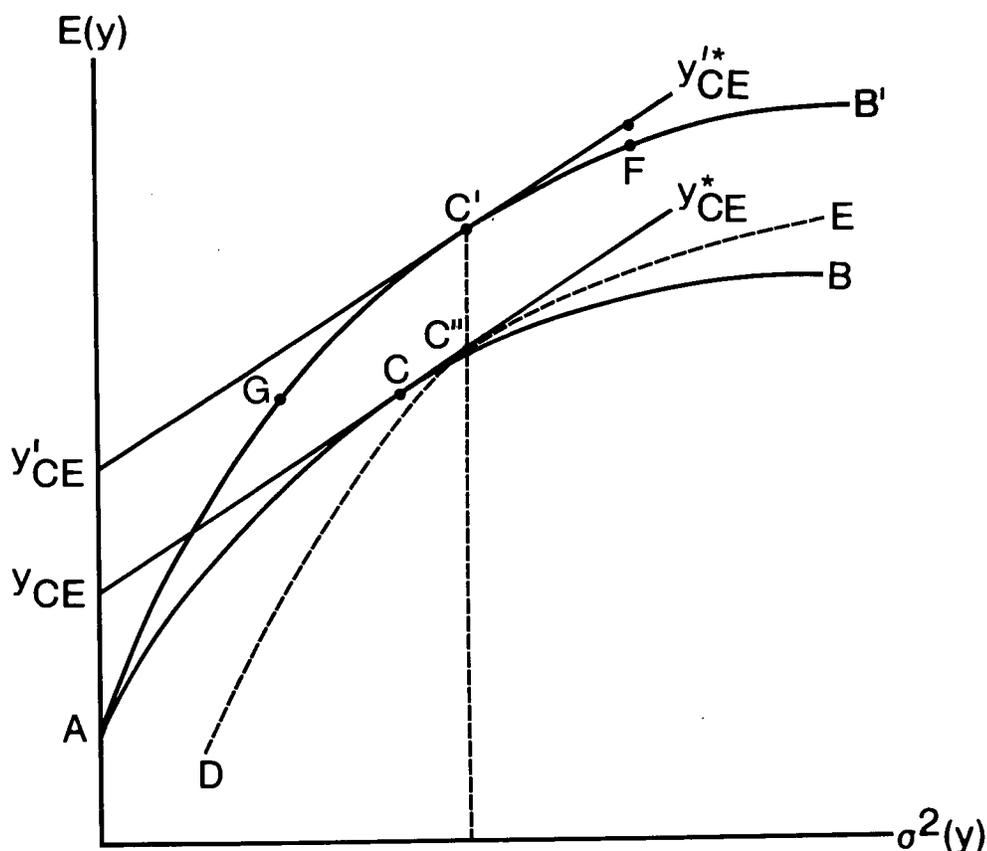


Figure 6.3 Substitution effect under risk obtained by measuring the adjustment of a constant absolute risk-averse investor.

from this change can be found in two ways. One way is to subtract the income effect from the total change in the EV set, with the remainder representing the substitution effect. The second way is to find the size of the adjustment in the equilibrium solution for those investors whose income effect is zero. As shown above, these investors display constant absolute risk aversion.

Following the second approach, which is based on an initial equilibrium at C , we first subtract real income until the investor is returned to his or her initial isoexpected utility line. An important condition is the shape of this isoexpected utility line. Freund showed that the line was linear when the risky asset was normally distributed. Moreover, Pratt's second-order approximation of the risk premium also yields a linear relationship:

$$y_{CE} = E(y) - \frac{\lambda}{2} \sigma^2(y) \quad (6.1)$$

But this relationship is the same as the linear tangent line used earlier in the chapter to obtain the initial characterization of the solution, where λ

was a function of the parameters defining the location of the EV set. Thus, to identify the substitution effect under risk, we assume that an investor with constant absolute risk aversion has an isoexpected utility line that is linear in expected value and variances. This specification is, of course, an approximation when the distributions are not normal. However, the complete generality of the income effect plus the more limiting assumptions imposed on the specification of the substitution effect yield highly reliable analytic results compared to those obtained from the more general expected utility model.³

To find the substitution effect, let the isoexpected utility line y_{CEY*CE} in Fig. 6.3 be the straight line described in Eq. (6.1) with slope $\lambda/2$. Initially, the preferred choice occurs at tangency point C between the isoexpected utility line y_{CEY*CE} and EV set ACB . Let an increase in expected return on the risky asset, or a decrease in its variance, rotate the EV set from ACB to $AC'B'$. The substitution effect is found by subtracting risk-free income from set $AC'B'$, causing it to shift downward in a parallel fashion until a new tangency at C'' occurs between the isoexpected utility line y_{CEY*CE} and the shifted EV set DE . Any change in the demand for the risky asset between choices C and C'' is due to the substitution effect.

A more convenient measure of the substitution effect is based on the observed EV set $AC'B'$ rather than the unobserved set DE . Since the variances at C'' and C' on EV sets DE and $AC'B'$, respectively, are the same, and the slopes at choices C'' and C' are equal, the change in demand for risky assets between C and C' is the same as the change in demand between C and C'' . Thus, since EV set $AC'B'$ is observed and DE is not, the substitution effect is more conveniently measured as the difference between C and C' .

Finally, we add the income effect to Fig. 6.3. For upward parallel shifts, the equilibrium slope decreases, remains constant, or increases for investors with decreasing, constant, or increasing absolute risk aversion, respectively. Adjusting from equilibrium point C' on EV set $AC'B'$, investors with decreasing absolute risk aversion choose a new equilibrium position to the right of C' with a lower slope (i.e., F). Decision makers with increasing absolute risk aversion will choose an equilibrium position to the left of C' (i.e., G) with a higher slope. Those with constant absolute risk aversion choose equilibrium position C' without a slope change, signifying no income effect.

MATHEMATICAL FORMULATION

To this point, the income and substitution effects associated with shifts in EV sets have been analyzed in terms of equilibrium slopes. Extending the adjustment theory to evaluate holdings of individual assets and specific changes in risk-free wealth and probability distributions requires a mathematical description.

The substitution effect is found by expressing the certainty equivalent model as in (6.1), holding λ constant, and observing an asset's change in response to a given change in the parameters of the EV set. Similarly, the income effect

Table 6.1 Responses to an increase in a parameter α describing the location of an EV set

	Decreasing absolute risk aversion (DARA)	Constant absolute risk aversion (CARA)	Increasing absolute risk aversion (IARA)
Certainty equivalent income increases as α increases and the EV set shifts upward	$\frac{\partial \lambda}{\partial \alpha} < 0$	$\frac{\partial \lambda}{\partial \alpha} = 0$	$\frac{\partial \lambda}{\partial \alpha} > 0$
Certainty equivalent income decreases as α increases and the EV set shifts downward	$\frac{\partial \lambda}{\partial \alpha} > 0$	$\frac{\partial \lambda}{\partial \alpha} = 0$	$\frac{\partial \lambda}{\partial \alpha} < 0$

involves the change in demand for an asset in response to a change in λ . In sum, if x is a risky asset, then the total change in the demand for x , dx , in response to a change in α is:

$$\frac{dx}{d\alpha} = \left(\frac{\partial x}{\partial \alpha} \right)_{\lambda \text{ constant}} + \frac{\partial x}{\partial \lambda} \frac{\partial \lambda}{\partial \alpha} \quad (6.2)$$

where α = any parameter (variance, expected return, risk-free wealth) defining the location of the EV set. The first term on the right in (6.2) is the substitution effect, with λ held constant. The a second term identifies the income effect. If an increase in α increases the certainty equivalent income, then $\partial \lambda / \partial \alpha < 0$ for a decision maker with decreasing absolute risk aversion. For increasing or constant absolute risk aversion, $\partial \lambda / \partial \alpha > 0$ or $\partial \lambda / \partial \alpha = 0$ respectively. These results are summarized in Table 6.1.

INCOME AND SUBSTITUTION EFFECTS APPLIED TO A SAVINGS PROBLEM

The income and substitution effects under risk are illustrated by the following example involving an investor's two-period consumption problem. The investor is assumed to have a utility function $U(c)$ for consumption of asset c . The amounts of c consumed in periods 1 and 2 are, respectively, c_1 and c_2 . However, because c_2 is consumed one period in the future, its present worth is ρc_2 , where:

$$\rho = \frac{1}{1+r} \quad (6.3)$$

and $r =$ a time preference rate.

The consumption decision is limited by allocable funds W_0 . If it is assumed that the price per unit of c_1 and c_2 is 1, then the investor can spend $W_0 - c_2$ units of W_0 in period 1 and save c_2 units for consumption in period 2. But the units saved earn a rate of return \tilde{r} , where:

$$\tilde{r} = r_2 + \epsilon \quad (6.4)$$

and ϵ is a stochastic element with expected value 0 and variance σ_ϵ^2 . Thus the expected value and variance of \tilde{r} are r_2 and σ_ϵ^2 , respectively. The investor must allocate W_0 between consumption in the current period and savings for consumption in period 2. Consumption in period 1 is known to equal $W_0 - c_2 \leq W$. However, consumption in period 2 is $(1 + \tilde{r})c_2$, which depends on the stochastic element ϵ . A favorable \tilde{r} more than compensates the investor for postponing consumption, but an unfavorable \tilde{r} may have adverse consequences. The focus here is on the responses of savings and consumption decisions to changes in risk-free wealth W_0 , expected returns r_2 , and variance σ_ϵ^2 . We shall examine these issues using the certainty equivalent consumption model.

First we construct an EV set for all combinations of expected present value (6.5) and variance (6.6) of consumption:

$$E(c_1 + \rho c_2) = W_0 - c_2 + \rho(1 + r_2)c_2 \quad (6.5)$$

and

$$\sigma^2(c_1 + \rho c_2) = \rho^2(c_2^2)\sigma_\epsilon^2 \quad (6.6)$$

The extremes of the consumption and savings decision are $c_2 = 0$ with no variance—a completely safe decision—and $c_2 = W_0$, a decision to save everything with all consumption at risk.

Expected utility of consumption with the allocable wealth constraint added can now be approximated with the certainty equivalent formula expressed as a function of c_2 :

$$(c_1 + \rho c_2)_{CE} = E(c_1 + \rho c_2) - \frac{\lambda}{2}\sigma^2(c_1 + \rho c_2) \quad (6.7)$$

where (6.5) and (6.6) define $E(c_1 + \rho c_2)$ and $\sigma^2(c_1 + \rho c_2)$.

Maximizing the certainty equivalent formula for c_2 yields:

$$\frac{d(c_1 + \rho c_2)_{CE}}{dc_2} = \rho(1 + r_2) - 1 - \lambda\rho^2 c_2 \sigma_\epsilon^2 = 0 \quad (6.8)$$

and solving for c_2 gives:

$$c_2 = \frac{\rho(1 + r_2) - 1}{\lambda\rho^2\sigma_\epsilon^2} \quad (6.9)$$

Since the function is a quadratic in c_2 , the necessary second-order conditions hold.

Now let the investor experience an increase in risk-free wealth, W_0 . To show the change in consumption, we consider the sum of the income and substitution effects for the three classes of absolute risk aversion. First, the substitution effect is zero for all investors starting with a constant-slope (λ) isoexpected utility line approximated by the certainty equivalent formula in (6.7): $dc_2/dW_0 = 0$. The income effect is:

$$\frac{dc_2}{dW_0} = \frac{-[\rho(1+r_2) - 1](\partial\lambda/\partial W_0)}{(\lambda\rho\sigma_\epsilon)^2} \quad (6.10)$$

Since W_0 shifts the EV set upward, $\partial\lambda/\partial W_0 \lesseqgtr 0$ depending on whether the investor has a DARA, CARA, or IARA risk attitude. If $r_2 > r$, the income effect is positive for DARA, negative for IARA, and zero for CARA. This implies that DARA (IARA) decision makers save more (less) with an increase in wealth and that CARA decision makers save the same amount.

The total effect of an increase in wealth on saving is the sum of the income and substitution effects. For an increase in W_0 alone, the sum is equal to the income effect alone. But, for an increase in r_2 , the sum of the income and substitution effects is:

$$\frac{dc_2}{dr_2} = \underbrace{(\lambda\rho\sigma_\epsilon^2)^{-1}}_{\text{(substitution effect)}} - \underbrace{\frac{[\rho(1+r_2) - 1](\partial\lambda/\partial r_2) - \lambda\rho}{(\lambda\rho\sigma_\epsilon)^2}}_{\text{(income effect)}} \quad (6.11)$$

Again, assuming $r_2 > r$, the sign of the income effect is determined by $\partial\lambda/\partial r_2$. Since an increase in r_2 rotates the EV set upward, $\partial\lambda/\partial r_2 \lesseqgtr 0$ for a DARA, CARA, or IARA risk attitude. Thus the income effect is positive for decreasing absolute risk aversion. Combined with a positive substitution effect, this result means that savings increase with an increase in expected return.

For increasing absolute risk aversion the response of savings to an increase in expected returns is ambiguous. Because this class of investors is less willing to assume risk as wealth increases, their negative income effect combines with a positive substitution effect to leave the total effect ambiguous.

Now consider an increase in the variability of the return from saving. Should current consumption c_1 increase because "a bird in the hand is worth two in the bush," or should saving c_2 increase to ensure minimum consumption standards in the second period? Rothschild and Stiglitz show that c_1 increases with quadratic utility. Our results are:

$$\frac{dc_2}{d\sigma_\epsilon^2} = - \left\{ \frac{[\rho(1+r_2) - 1]}{\lambda\rho^2(\sigma_\epsilon^2)^2} + \frac{[\rho(1+r_2) - 1](\partial\lambda/\partial\sigma_\epsilon^2)}{(\lambda\rho\sigma_\epsilon)^2} \right\} \quad (6.12)$$

Assume that $r_2 > r$. Increasing σ_ϵ^2 rotates the EV set downward, and for DARA and CARA decision makers $\partial\lambda/\partial\sigma_\epsilon^2 \geq 0$ and $dc_2/d\sigma_\epsilon^2 < 0$ (see Table 6.1).

This implies that c_1 increases; c_1 can also decrease for quadratic utility decision makers whose absolute risk aversion function is IARA, as Rothschild and Stiglitz claim, but this result is not guaranteed in our model.⁴

CONCLUDING COMMENTS

This chapter has introduced the theoretical structure for the remainder of the book. When choices consist of alternative amounts of a single risky asset or linear functions of a single random variable, two important rationales exist for use of EV models. First, all choices are EU efficient and, therefore, the EV set must contain choices selected by EU maximizers. Moreover, by selecting the proper slope parameter, $\lambda/2$, our EV model can be made to select the same choice from the EV set as would any particular EU maximizer. This EV justification is completely general and applies to all the models developed in the later chapters of the book.

After justifying the use of an EV choice set, an EV analytic model was developed which allowed us to separate adjustments in EV set shifts into income and substitution effects.

The important feature that distinguishes choices under risk from the certainty case is the change in an investor's absolute risk aversion function as risk-free wealth increases. This characteristic is related to changes in equilibrium slopes on EV sets. An increase in risk-free wealth causes a parallel upward shift in the EV set. Compared to the slope at the original equilibrium, the new equilibrium slope decreases, remains constant, or increases for investors with decreasing, constant, or increasing absolute risk aversion, respectively.

Having measured the income effect, the task of measuring the substitution effect remained. If investors are distinguished by their income effects, then the substitution effect is simply the response of constant absolute risk averse investors whose income effect is zero. Their isoexpected utility lines are linear. This result allows the substitution effect to be measured by holding constant the slope in the certainty equivalent formula. The total effect is then the sum of the income and substitution effects. The adjustments to shifts in the EV set using the EV model will be consistent with EU maximizers under the sufficiency condition of quadratic utility, normality of outcomes, or EV choices consisting of linear combinations of a random variable.

All of the models presented in Part 3 are models in which the choices are a linear function of the random variable. For those models, consistency in the rankings of choices using EU or EV analysis is guaranteed. In Part 4 we use the properties of the EV model used in Part 3 to analyze models that may not always be consistent with EU models. But, on the other hand, no one can claim all the sufficiency conditions for consistency between EU and EV models have been discovered. Until we understand these condi-

tions more fully, we justify the use of the EV model even when it is not consistent with EU models because it is interesting in its own right. Moreover, the proper test of the EV model is not its absolute consistency with EU models but its ability to describe and predict decision maker behavior under risk.

APPENDIX 6A

Theorem 6A.1 If there are two assets, one risky and one safe, the total purchases of the risky asset increase, remain unchanged, or decrease with initial wealth as there is decreasing, constant, or increasing absolute risk aversion.

PROOF Define W_0 as initial wealth allocated between the purchase of an input x at price p_x for use in the process $f(x)$; $W_0 - p_x x$ is invested in a safe asset earning a known rate of return r . Let the output $f(x)$ be sold at a stochastic price of $p + \epsilon$, where ϵ is distributed with mean 0 and variance σ_ϵ^2 . Profits for a particular ϵ value can be written as:

$$y = (p + \epsilon)f(x) + r(W_0 - p_x x) - p_x x \quad (6A.1)$$

and expected utility of profits as $EU(y)$. The optimal allocation between x and the safe asset equal to $W_0 - p_x x$ can be found by maximizing the expected utility of profits with respect to x :

$$EU'(y)[(p + \epsilon)f'(x) - rp_x - p_x] = 0 \quad (6A.2)$$

Since $U'(y) > 0$ by assumption, (6A.2) equals zero just in case the bracketed expression equals zero.

Now we ask how the purchase of x will change as risk-free wealth W_0 increases? To answer, we totally differentiate (6A.2) with respect to x and W_0 and obtain the result:

$$\frac{dx}{dW_0} = \frac{rEU''(y)[(p + \epsilon)f'(x) - rp_x - p_x]}{EU''(y)[(p + \epsilon)f'(x) - rp_x - p_x]^2 + EU'(y)(p + \epsilon)f''(x)} \quad (6A.3)$$

Since it is assumed that $U''(y) < 0$ and $U'(y) > 0$, we can sign the denominator if $f''(x) < 0$; this we assume. Then the denominator multiplied by the negative sign is unambiguously positive. Thus the sign of dx/dW_0 depends on the sign in the numerator. We write the numerator as: $rEU''(y)[\cdot]$ which is equal to:

$$EU''(y)[\cdot] = E \left[\frac{U''(y)}{U'(y)} \right] U'(y)[\cdot] \quad (6A.4a)$$

which can be written as:

$$EU''(y)[\cdot] = -E[R(y)]U'(y)[\cdot] \quad (6A.4b)$$

since $-U''(y)/U'(y) = R(y)$.

Next set $R^*(y)$ equal to $R(W_0r)$; that is, define $R^*(y)$ to be the value of the risk aversion function measured at a safe income level of rW_0 . This permits us to write:

$$\begin{aligned} -ER(y)U'(y)[\cdot] &= E[R^*(rW_0) - R(y)]U'(y)[\cdot] \\ &- R^*(rW_0)EU'(y)[\cdot] \end{aligned} \quad (6A.5)$$

But from (6A.2) $EU'(y)[\cdot] = 0$, which allows us to write:

$$-R(y)U'(y)[\cdot] = E[R^*(rW_0) - R(y)]U'(y)[\cdot] \leq 0 \quad (6A.6)$$

If $R'(y) < 0$, then for $[\cdot] > 0$, $R^*(rW_0) < R(y)$, and when $[\cdot] < 0$, $R^*(rW_0) > R(y)$ and (6A.6) is positive. This implies $dx/dW_0 > 0$. Conversely, if $R'(y) > 0$, then for $[\cdot] > 0$, $R^*(rW_0) > R(y)$ and when $[\cdot] < 0$, $R^*(rW_0) < R(y)$. This implies $dx/dW_0 < 0$. And finally, if $R'(y) = 0$, then $R^*(rW_0) = R(y)$ which implies $dx/dW_0 = 0$.

ENDNOTES

1. Choices from EV sets are linear functions of the random variable if the random variable appears to the power 1.
2. This description of EV choices rules out short selling, which could of course be permitted. Moreover, the description of adjustments to shifts in EV sets ignores the possibility of CARA decision makers who prefer the riskiest solution on an EV set and would prefer still riskier positions if not wealth-constrained.
3. The graphical presentation indicates a significant difference in the estimation of the substitution effect. However, when the substitution effect is measured using calculus, the substitution effect measured in a small neighborhood is precisely measured.
4. Corresponding to the quadratic utility function $U(y) = y + by^2$ is the absolute risk aversion function:

$$R(y) = \frac{-2b}{1 + 2by} > 0 \quad \text{for} \quad (1 + 2by) > 0 \quad \text{and} \quad b < 0$$

Increasing y , therefore, increases the value of $R(y)$:

$$R'(y) = \frac{(2b)^2}{1 + 2by} > 0$$

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PART
THREE

AN ANALYSIS OF FIRM-LEVEL
RESPONSE TO RISK

OPTIMAL OUTPUT UNDER PRICE RISK

In this chapter we explore the effects of risky output prices on the firm's optimal level of production. We contrast the results with optimal output under certainty conditions and show the nonneutral effects of fixed costs and income taxes. The analysis is based on an approach first used by Sandmo, which has been adapted here to utilize the expected value–variance framework.¹ As the analysis will show, the expected value–variance formulation avoids the ambiguous output response to increased risk that characterized Sandmo's results. Thus it yields results consistent with those of Ishii and others who have shown, using the expected utility approach, that the level of output under risk will decrease for a decision maker with decreasing absolute risk aversion.

MODEL ASSUMPTIONS

Consider the case of a perfectly competitive firm whose principal activity is to produce a single good with a fixed plant size. Thus no investment or disinvestment decisions are allowed. The only reference to capital in the firm's operations is a fixed cost that represents the rental price of capital. Further assume that the model is based on a single time period and that all the firm's production is sold at the prevailing market price. This leaves the firm's only response to changes in the exogenously determined prices for inputs and outputs as a change in the level of output. The model is then specified solely in terms of output q , a variable cost function $C(q)$, and a fixed cost B .

Some of the models in later chapters permit the firm some flexibility in responding to risk. For example, some of the inputs may be chosen after the risky output price is known. However, this specification is not allowed here; all input decisions occur before the actual output price is known.

Finally, we reiterate the assumptions that underlie the analysis. These assumptions include choices whose stochastic outcomes are fully specified by their means and variances and that are members of an expected value–variance efficient set. These assumptions allow the model to be formulated to yield a unique solution from among the choices comprising the expected value–variance efficient set. This solution is found by maximizing certainty equivalent income y_{CE} :

$$\max y_{CE} = E(y_i) - \frac{\lambda}{2}\sigma^2(y_i) \quad (7.1)$$

where $E(y_i)$ = expected value of i th choice, $\sigma^2(y_i)$ = variance of i th choice, and $\lambda/2$ = equilibrium slope along expected value–variance set.

THE CERTAINTY MODEL

Let the firm produce a single output, q , at variable cost $C(q)$, where $C(0) = 0$. The output price is known and equal to p . The firm also pays a fixed rental price B on capital. Profit y is then:

$$y = pq - C(q) - B \quad (7.2)$$

We take the derivative of y with respect to q and equate it to zero in order to find the profit maximizing condition:

$$p = C'(q) \quad (7.3)$$

which requires that marginal revenue equal marginal cost. Fixed cost B has no effect on the optimal level of output.

A profit-maximizing rather than a profit-minimizing condition is guaranteed only if $C''(q) > 0$. Marginal costs must increase at an increasing rate, otherwise, greater output with a constant output price would yield infinitely large profits.

Taking the total derivative of the first-order condition with respect to p and q indicates that supply q increases with output price p as long as the second-order condition on $C(q)$ is satisfied:

$$\frac{dq}{dp} = \frac{1}{C''(q)} > 0 \quad (7.4)$$

To examine the certainty model on an after-tax basis, we let T equal the applicable marginal income tax rate. After-tax profits y^* are:

$$y^* = [pq - C(q) - B](1 - T) \quad (7.5)$$

As the first-order condition indicates, the proportional tax rate has no impact on the final output since canceling $1 - T$ yields the same before-tax profit-maximizing condition:

$$p(1 - T) = C'(q)(1 - T) \quad (7.6)$$

THE RISK MODEL

We define a perfectly competitive market under risk as one in which the firm is a price taker and has identified the probability density function of all random variables. Using the certainty model as a base allows us to compare the output levels under certainty and risk, the impact of fixed costs, the slope of the supply curve, the effect of taxes, and the maximization conditions.

Consider a firm facing a risky price for the product it produces. Under risk conditions, let p be the expected output price such that:

$$E(p + \epsilon) = p \quad (7.7)$$

where the random variable ϵ has expected value 0 and variance σ_ϵ^2 . Risky profit is now:

$$y = (p + \epsilon)q - C(q) - B \quad (7.8)$$

Expected profit is:

$$E(y) = pq - C(q) - B \quad (7.9)$$

and variance of profits is:

$$\sigma^2(y) = q^2\sigma_\epsilon^2 \quad (7.10)$$

To find the firm's optimal output under risk we form the certainty equivalent of the profit expression:

$$y_{CE} = pq - C(q) - B - \frac{\lambda}{2}q^2\sigma_\epsilon^2 \quad (7.11)$$

which is maximized with respect to q . The first-order condition is:

$$p - C'(q) - \lambda q\sigma_\epsilon^2 = 0 \quad (7.12)$$

As long as marginal costs $C'(q)$ are positive, the output level satisfying the first-order conditions under risk will be less than the corresponding output under certainty. Figure 7.1 illustrates these results, with the solid and dashed lines representing marginal cost under certainty and risk conditions, respectively. The difference between the two curves is line $\lambda q\sigma_\epsilon^2$ which represents the cost of risk—the costs the firm would willingly forgo from its expected profits if the difference were received with certainty. This cost of risk reduces the firm's output relative to production under certainty. Output levels under risk and certainty differ by the amount $q_c - q_r$ on the horizontal axis.

To ensure an expected utility-maximizing solution, the second-order condition requires that H is positive, where:

$$H = C''(q) + \lambda\sigma_\epsilon^2$$

A sufficient condition is that marginal costs increase at an increasing rate $C''(q) > 0$. The second-order condition can also be satisfied by a weaker condition:

$$-C''(q) < \lambda\sigma_\epsilon^2$$

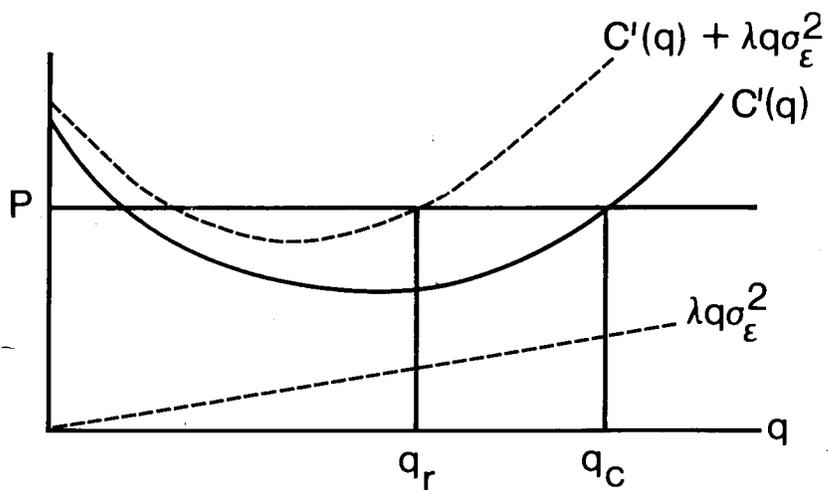


Figure 7.1 Marginal cost under certainty and risk.

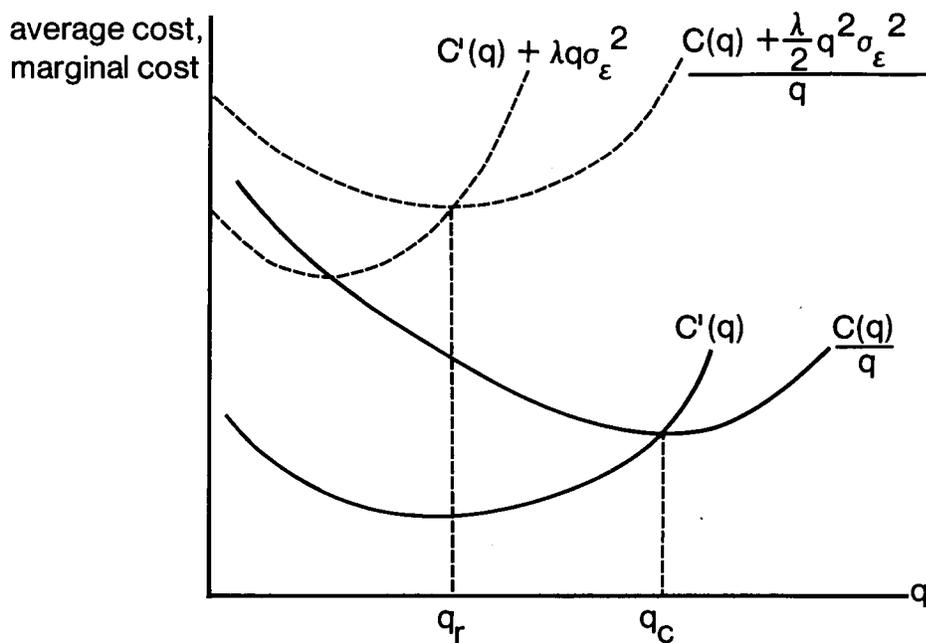


Figure 7.2 Average cost-minimizing output under certainty q_c compared to average cost-minimizing output under risk q_r .

A homogeneous cost function with costs constant over all output levels would satisfy this requirement. In such a case, the condition $C'''(q) = 0$ produces a unique optimal output under risk, although the output level under certainty is indeterminate.

These theoretical results concerning the reduction in optimal output under risk, relative to the certainty case, have important implications for empirical analyses. Studies on economies of scale, for example, may suggest that some types of firms are operating inefficiently because their size and levels of output appear to fall short of those which minimize average cost (Madden). However, as the above results show, including the costs of risk in the analysis will explain, at least in part, the discrepancy between optimal output under certainty and risk conditions. To illustrate this result graphically in Fig. 7.2, we represent average and marginal costs under certainty by solid lines, and average and marginal costs under risk by dotted lines. Clearly, the average cost minimizing output under risk q_r is less than the average cost minimizing output under certainty q_c .

THE EFFECT OF FIXED COSTS UNDER RISK

Recall that the certainty equivalent model of solutions from an EV set allows λ to remain fixed only if the EV set does not shift. Since λ equals the coefficient of absolute risk aversion $R(y)$, measured at the expected value of outcomes, any shift or rotation of the EV set requires a corresponding change in λ . DARA, CARA, and IARA attitudes imply that λ decreases, remains constant, or increases as the EV set shifts or rotates upward away from the risk axis toward higher-return solutions. The opposite effects occur for a shift or rotation in the EV set toward the risk axis.

An increase in fixed costs reduces the firm's expected profit at each level of output. This is equivalent to a downward parallel shift in the EV set (Fig. 7.3). The effect on the absolute risk aversion function is:

$$\frac{\partial \lambda}{\partial B} dB \gtrless 0 \quad \text{as} \quad R' [E(y)] \lesseqgtr 0 \quad (7.13)$$

This result is portrayed graphically in Fig. 7.3. The dotted EV frontier represents the EV frontier adjusted for an increase in fixed costs. The DARA decision maker responds to a downward parallel shift in the EV set by finding a new expected utility-maximizing choice that requires a higher rate of exchange between expected profits and variance. The movement from A_i , the initial equilibrium, to A_j , the new equilibrium, reflects the income response to higher fixed costs.

Whether or not decision makers respond as indicated in Fig. 7.3, it is plausible that changes in fixed costs should not be ignored. Increased fixed costs could lead some decision makers to a riskier location on the EV set in order to regain reduced profit margins. Increased fixed costs may result in more conservative output decisions to avoid the possibility of large operating losses. Or, fixed costs may have no income effect at all. The theory explicitly allows for any of these possibilities rather than ignoring the effects of fixed costs as it does under certainty.

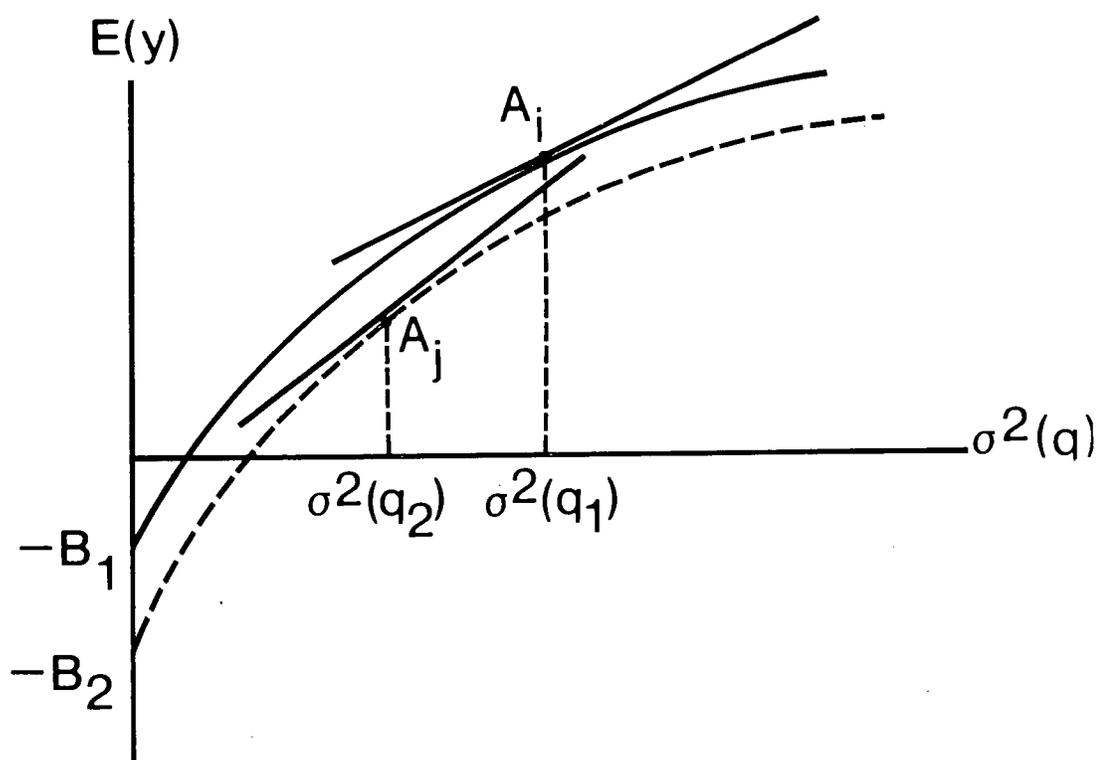


Figure 7.3 Income effect response to an increase in fixed costs.

The impacts of fixed costs are expressed mathematically by totally differentiating the first order conditions (7.12) with respect to q and B . The results are:

$$H dq + \frac{\partial \lambda}{\partial B} q \sigma_\epsilon^2 dB = 0 \quad (7.14)$$

Rearranging (7.14) results in the expression:

$$\frac{dq}{dB} = \frac{-(\partial \lambda / \partial B) q \sigma_\epsilon^2}{H} \quad (7.15)$$

The denominator on the right-hand side of (7.15) is positive, otherwise the second-order condition would fail. The sign of $\partial \lambda / \partial B \gtrless 0$ depends on $R'(y) \lesseqgtr 0$, so that for DARA decision makers $\partial \lambda / \partial B > 0$, implying that output will fall as fixed costs increase, $dq/dB < 0$. Similarly, if $R'(y) > 0$ (IARA decision makers), increased fixed costs will increase risky output, since $\partial \lambda / \partial B < 0$. Only for CARA decision makers [$R'(y) = 0$] will a change in fixed costs have no effect on output. Thus, except for the case of CARA decision makers, these results contradict those obtained for the certainty model.

EFFECTS OF TAXES

The effects of income taxes may also influence the firm's production organization under risk conditions. The expected after-tax profit y^* is:

$$E(y^*) = [pq - C(q) - B](1 - T) \quad (7.16)$$

and the variance is:

$$\sigma^2(y^*) = q^2(1 - T)^2\sigma_\epsilon^2 \quad (7.17)$$

The certainty equivalent model is:

$$y_{CE^*} = [pq - C(q) - B](1 - T) - \frac{\lambda}{2}q^2(1 - T)^2\sigma_\epsilon^2 \quad (7.18)$$

Maximizing the certainty equivalent income with respect to q yields:

$$p = C'(q) + \lambda q(1 - T)\sigma_\epsilon^2 \quad (7.19)$$

As (7.19) shows, proportional taxes affect the first-order condition under risk. The effect of the tax is to reduce the cost of risk. If λ is constant, output q is increased relative to the model results without taxes. But λ may not stay constant since introducing taxes will shift the EV set. Thus to analyze the effects of taxes on risky output q we must consider how taxes affect the distribution of profits.

Consider the impact of a proportional tax that has full offset provisions; that is, both positive and negative incomes are reduced by T percent. This has the effect of reducing negative expected income, reducing positive expected income, and reducing the variance of income.

We demonstrate the impact of a full-offset proportional tax in Figs. 7.4 and 7.5. First consider a distribution $f(y)$ with an expected value of zero, as shown by the solid line in Fig. 7.4. The before-tax parameters of the distribution are $E(y) = 0$ and $\sigma^2 = q^2\sigma_\epsilon^2$. In a normal distribution, about two-thirds of the probability falls within 1 standard deviation $q\sigma_\epsilon$ of the mean.

The after-tax distribution $f^*(y^*)$ also has a zero expectation, since reductions of positive incomes are offset by reduced losses. The variance, however, is unambiguously reduced to:

$$\sigma^2(y^*) = (1 - T)^2q^2\sigma_\epsilon^2$$

Thus the before-tax standard deviation of q is reduced to the after-tax standard deviation of $(1 - T)q\sigma_\epsilon$, resulting in a more peaked distribution, as shown by the dashed line in Fig. 7.4.

Now consider the impact of full-offset proportional taxes when expected profits are positive. Again the variance of y is reduced, but so is the after-tax expected profit—from $E(y)$ to $E(y)(1 - T)$. Thus the tax shifts the distribution to the left and peaks it, as shown in Fig. 7.5. The effects of taxes on negative expected income are similar to those in the positive income case except that the

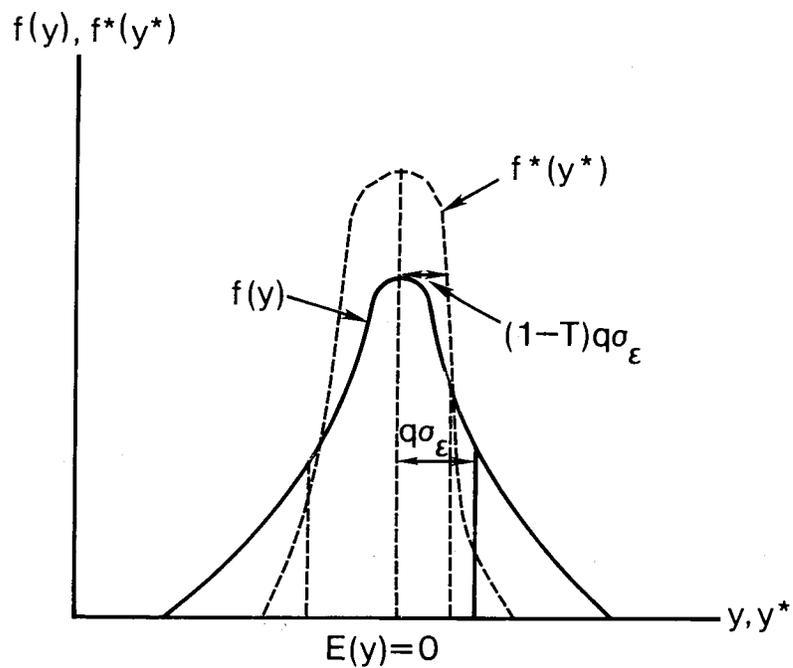


Figure 7.4 Probability density functions of before- and after-tax profits.

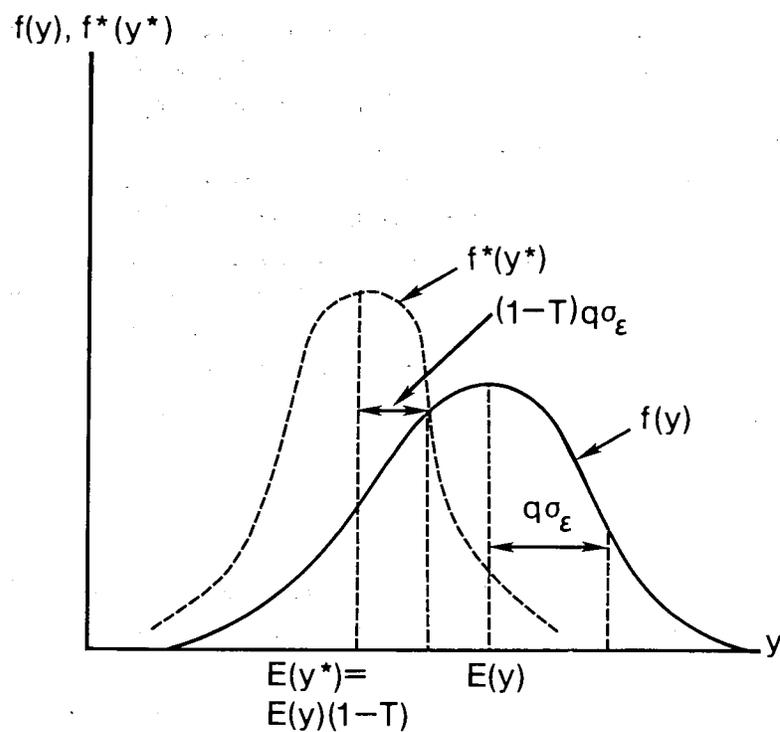


Figure 7.5 Probability density functions of before- and after-tax profits.

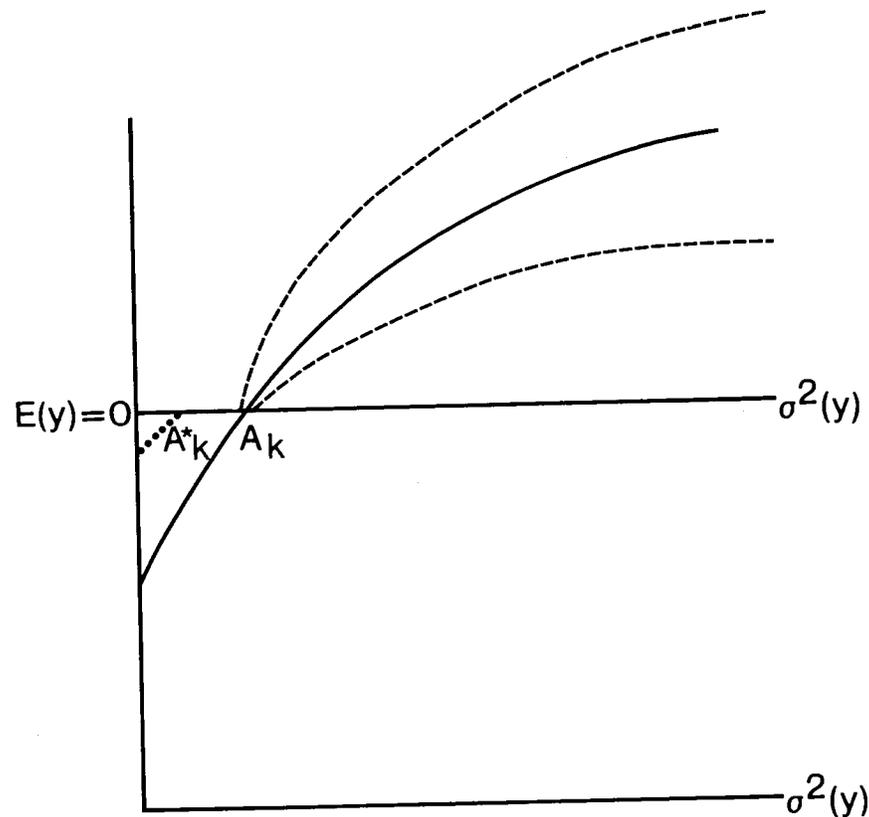


Figure 7.6 Before- and after-tax expected value-variance sets.

distribution shifts to the right and is peaked; a negative expectation is made less negative by the tax.

Next we will examine how the tax shifts the before-tax EV set. In Fig. 7.6, the solid curve represents the before-tax EV set, with the horizontal line representing zero expected profits. Choice A_k has zero before-tax profits. The introduction of a tax which reduces the variance without affecting expected profits implies that the before-tax choice is mapped on A_k^* , the after-tax EV choice. The before-tax distributions with negative expected profits shift the EV set upward, since they reduce variance and increase (reduce negative) expected values.

How distributions with positive expectations before taxes affect the EV efficient set is unclear. Taxes reduce both the variance and the expected value. Therefore the magnitude of these reductions determines whether the new portfolios lie below or above the original EV set. This ambiguity in the EV shift is reflected in Fig. 7.6 by the dotted interval around the original EV set in which the after-tax EV set may lie.

To find the effects of marginal increases in the proportional tax rate on the firm's risky output, the first-order conditions are totally differentiated with

respect to output q and the tax rate T . The total differential, including both the income and substitution effects, is:

$$-H^* dq + \left\{ \lambda q \sigma_\epsilon^2 - \frac{\partial \lambda}{\partial T} q (1 - T) \sigma_\epsilon^2 \right\} dT = 0 \quad (7.20)$$

$$\frac{dq}{dT} = - \frac{\left\{ \lambda q \sigma_\epsilon^2 - (\partial \lambda / \partial T) q (1 - T) \sigma_\epsilon^2 \right\}}{H^*} \quad (7.21)$$

where $H^* = [C''(q) + \lambda(1 - T)\sigma_\epsilon^2] > 0$ for a maximum.

We have already established that H^* is positive in order to meet the second-order condition. The sign of the numerator in (7.21), however, is ambiguous because the sign of $\partial \lambda / \partial T$ depends on the location of the initial equilibrium and the direction of the shift in the EV set. If the tax shifts the EV set upward and the decision maker is DARA, then $\partial \lambda / \partial T < 0$, but the derivative dq/dT is still ambiguous. This result also contradicts the certainty model result.

THE SUPPLY CURVE

The comparative statics for the supply curve are analyzed by differentiating the first-order condition (without taxes) with respect to q and p ; because of the EV shift we also differentiate λ . The result is:

$$\left(1 - \frac{\partial \lambda}{\partial p} q \sigma_\epsilon^2 \right) dp - H dq = 0 \quad (7.22)$$

where H is required to be positive to satisfy second-order conditions. Or, expressed as a differential:

$$\frac{dq}{dp} = \frac{1 - (\partial \lambda / \partial p) q \sigma_\epsilon^2}{H} > 0 \quad \text{for} \quad \frac{\partial \lambda}{\partial p} \leq 0 \quad (7.23)$$

We anticipated an unambiguous result, indicating that increases in the expected output price would increase the supply response. Instead, the slope of the supply curve depends on the derivative $\partial \lambda / \partial p$. If increasing p shifts the EV set upward, then the new EV equilibrium slope for the DARA decision maker does in fact decrease, and dq/dp is unambiguously positive. But dq/dp may be negative for the IARA decision maker. It is therefore reassuring that under the usual DARA or CARA assumptions the supply curve continues to slope upward.

Finally, we examine the effects of increased variance, for which no analogy exists in the certainty model. In certainty models, supply is characterized as a relationship between output and prices. Under uncertainty, however, the supply response must also be related to the probability density function of output prices—in particular to changes in variance.

To examine this situation, we differentiate totally the first-order condition with respect to q and σ_ϵ^2 . We also account for the effect on λ of an increase in σ_ϵ^2 which rotates the EV set downward toward the risk axis. The total derivative is:

$$H dq + \left(\lambda q + q \sigma_\epsilon^2 \frac{\partial \lambda}{\partial \sigma_\epsilon^2} \right) d\sigma_\epsilon^2 = 0 \quad (7.24)$$

and the differential is:

$$\frac{dq}{d\sigma_\epsilon^2} = \frac{-[\lambda q + q \sigma_\epsilon^2 (\partial \lambda / \partial \sigma_\epsilon^2)]}{H} \quad (7.25)$$

If $\partial \lambda / \partial \sigma_\epsilon \geq 0$, the sign of $dq/d\sigma_\epsilon^2$ is unambiguously negative since $H > 0$. Because the increase in σ_ϵ^2 rotates the EV set downward, $\partial \lambda / \partial \sigma_\epsilon^2 \geq 0$ corresponds to DARA and CARA, respectively, which are the more usual cases, and increasing the variance reduces output. In effect, for CARA and DARA decision makers, increasing σ_ϵ^2 increases the marginal cost of producing q and reduces output. Still, IARA results in ambiguity about the sign of $dq/d\sigma_\epsilon^2$.

CONCLUDING COMMENTS

We have now analyzed a simple output model under conditions of risk. Contrasting the results with the certainty case is important. Under certainty, fixed costs and taxes do not affect output levels; under risk they can matter a lot or not at all. Under certainty, increasing marginal costs are required for profit maximization. This condition is sufficient, but not necessary, under risk. The supply curve slopes upward with price under certainty. It may do the same under risk, but under some conditions it can slope downward. Finally, changes in variance may increase, decrease, or leave constant the supply response.

The difference in results between the certainty and risk models is important in other ways as well. Under certainty, a single solution characterizes the response of all decision makers. With the conditions of certainty and more profits preferred to less, the choice with the highest profits is known and selected by all decision makers. Under risk, the unanimity of choice disappears and each decision maker can prefer a different choice because of his or her unique attitude toward risk. For analytic convenience, we categorize decision makers according to their response to increases in risk-free wealth—with DARA, CARA, and IARA decision makers requiring decreasing, constant, or increasing risk premiums, respectively, in response to an increase in risk-free wealth.

As evidenced by the solutions to the certainty-and-risk model described in this chapter, risk makes economic analysis more complicated. It also increases the information needed to solve the problem including attitudes toward risk held by the decision maker. But risk models also yield more believable results.

ENDNOTE

1. Hawawini later reformulated the Sandmo model within a mean-standard deviation framework.

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RISK-REDUCING INPUTS AS A RISK RESPONSE

Risk-reducing inputs such as irrigation systems, pesticides, subscriptions to market forecasting services, use of professional consultants, and buying new equipment represent another type of response to risk. Analytic models should consider the impacts of risk-reducing inputs on the expected utility of a decision maker and how the use of these inputs responds to changes in risk attitudes, expected returns, variances, and fixed costs. These impacts are addressed in this chapter, based on alternative modeling approaches.

THE JUST AND POPE ASSUMPTIONS

Just and Pope examined many important analytic issues involving risk-reducing inputs. They established seven conditions which an economic model should satisfy in order for input into the model to be risk-reducing. Letting q represent output and x_i be an input ($i = 1, \dots, n$), these conditions are:

1. $E(q) > 0$; the expected value of output is positive.
2. $\partial E(q)/\partial x_i > 0$ inputs provide positive contributions to the production process.
3. $\partial^2 E(q)/\partial x_i^2 < 0$; the marginal productivity of the inputs should diminish at some point.
4. $\partial E(q)/\partial \sigma^2 \sigma_\epsilon^2 = 0$; expected output can be held constant while reducing the variance of the random component.
5. $\partial \sigma^2(q)/\partial x_i \lesseqgtr 0$; the change in variance associated with a change in the risk-reducing input is not constant in sign.
6. $\partial \sigma^2(\partial q/\partial x_i)/\partial x_i \lesseqgtr 0$; the change in variance of a marginal product is not constant in sign.
7. $f(\theta x) = \theta f(x)$; constant stochastic returns to scale.

To evaluate the Just-Pope approach, we consider whether the standard production functions used in many types of risk analyses have the characteristics required above for risk-reducing inputs. Models containing empirical production functions usually have one of three standard forms.

$$\text{Model 1: } q = f(x)e^\epsilon; \epsilon \sim (0, \sigma_\epsilon^2)$$

$$\text{Model 2: } q = f(x)\epsilon; \epsilon \sim (1, \sigma_\epsilon^2)$$

$$\text{Model 3: } q = f(x) + \epsilon; \epsilon \sim (0, \sigma_\epsilon^2)$$

Consider how the distribution of q varies with x in each of the three models. In model 1, the exponentiated error term has a constant distribution. But since $f'(x) > 0$, the distribution is multiplied by an increasing factor $f(x)$ which spreads the distribution as x increases. Moreover, $e^\epsilon > 0$, and q is restricted in model 1 to positive values; Figure 8.1 illustrates how the distribution of q values changes with input values x_1, x_2 , and x_3 , where $x_1 < x_2 < x_3$.

Model 2 is similar to model 1 in that the distribution and variance of q increase with the input x . It differs from model 1 in that negative q values are possible when $\epsilon < 0$ and the expected value of q is $f(x_i)$. Figure 8.2 illustrates how the distribution of q values changes with input values x_1, x_2 , and x_3 , where $x_1 < x_2 < x_3$. Model 3 is the easiest to describe. Its variance is the same regardless of the choices of x . It is described graphically in Fig. 8.3.

We will consider each model using the criteria established by Just and Pope. Most often, model 1 assumes a normally distributed error term ϵ . Since $E(e^\epsilon)$ is the moment generating function $e^{t\epsilon}$ for $t = 1$, the expected value of q , $E(q)$, is:

$$\begin{aligned} E(q) &= f(x)E(e^\epsilon) \\ &= f(x)e^{\sigma_\epsilon^2/2} \end{aligned} \quad (8.1)$$

Assuming x has a positive marginal product (otherwise, none of it would be purchased), condition 1, $E(q) > 0$, is satisfied. Condition 2 is also met since:

$$\frac{\partial E(q)}{\partial x} = f'(x)e^{\sigma_\epsilon^2/2} > 0 \quad (8.2a)$$

and we may easily impose diminishing marginal productivity on f to meet condition 3. That is,

$$\frac{\partial^2 E(q)}{\partial x^2} = f''(x)e^{\sigma_\epsilon^2/2} < 0 \quad (8.2b)$$

To examine condition 4, we differentiate the expected value of q in Eq. 8.1 with respect to σ_ϵ^2 :

$$\frac{2E(q)}{2\sigma_\epsilon^2} = \frac{f(x)e^{\sigma_\epsilon^2/2}}{2} > 0 \quad (8.3)$$

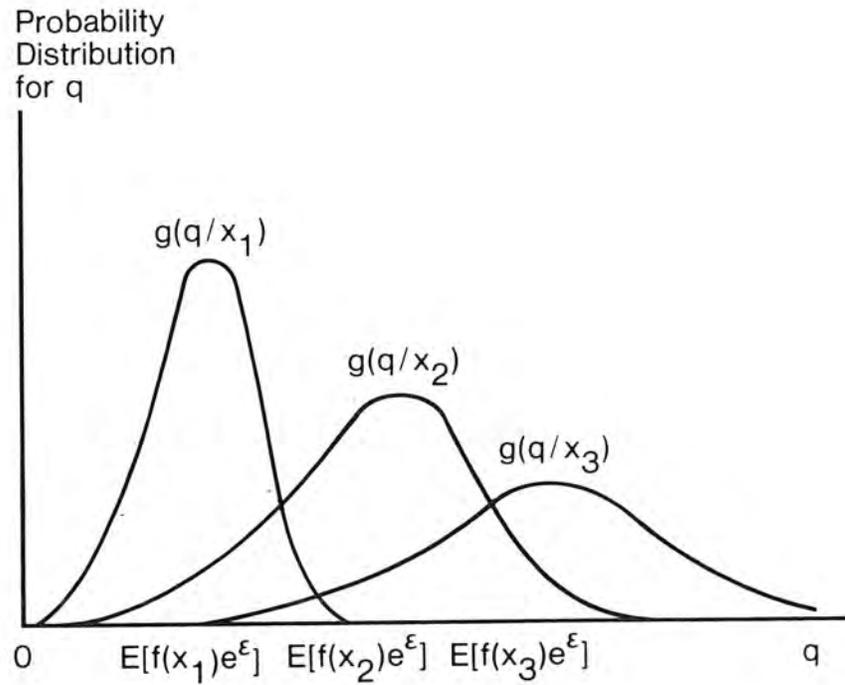


Figure 8.1 Probability distributions for alternative input values where $q = f(x_i)\epsilon$.

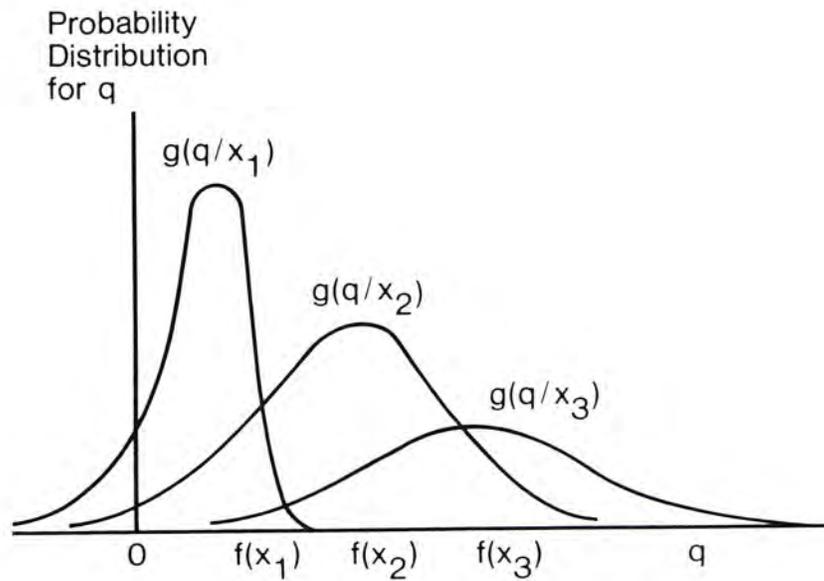


Figure 8.2 Probability distributions for alternative input values where $q = f(x_i)\epsilon$.

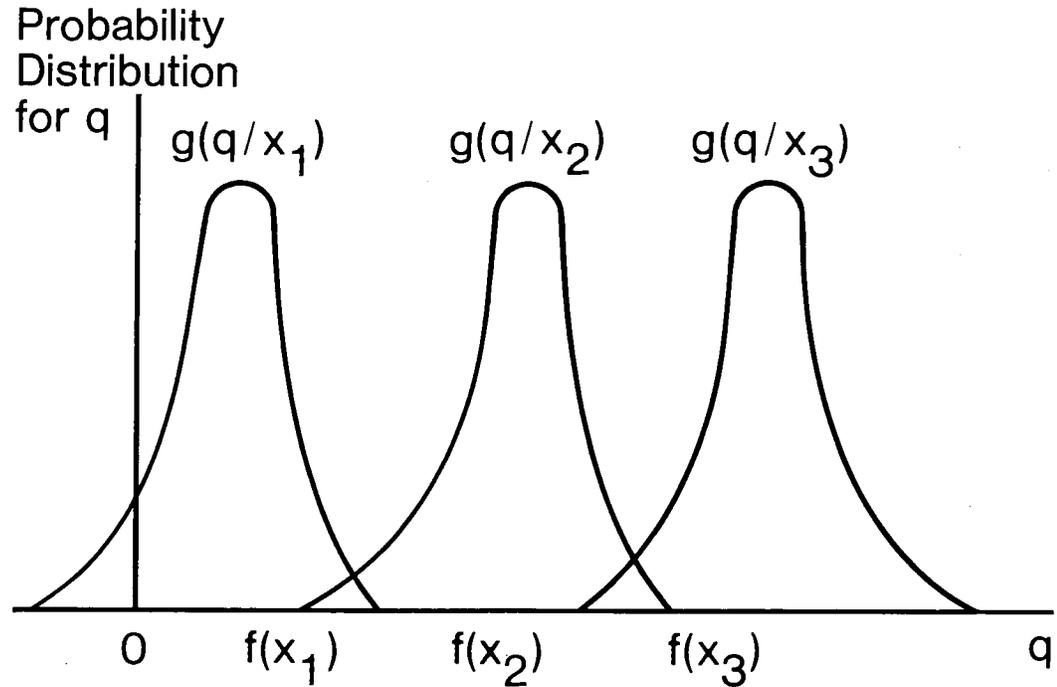


Figure 8.3 Probability distributions for alternative input values $x_1 < x_2 < x_3$ where $q = f(x) + \epsilon$.

Since the form of model 1 precludes the possibility of $2E(q)/2\sigma_\epsilon^2 = 0$, it violates condition 4.

To examine condition 5, we first calculate the variance of output:

$$\begin{aligned}\sigma^2(q) &= E[f(x)e^\epsilon]^2 - [f(x)e^{\sigma_\epsilon^2/2}]^2 \\ &= [f(x)]^2\sigma^2(e^\epsilon)\end{aligned}\quad (8.4)$$

Next we differentiate the variance of q with respect to input x :

$$\frac{\partial \sigma^2(q)}{\partial x} = 2f(x)f'(x)\sigma^2(e^\epsilon) > 0 \quad (8.4)$$

Since $\partial \sigma^2(q)/\partial x$ can be signed only in one direction, condition 5 is also violated.

Model 1 also fails to satisfy condition 6. Letting the variance of marginal product under risk be:

$$\sigma^2[f'(x)] = [f'(x)]^2\sigma^2(e^\epsilon) \quad (8.6)$$

its derivative with respect to x is:

$$\frac{d\sigma^2[f'(x)]}{dx} = 2f'(x)f''(x)\sigma^2(e^\epsilon) < 0 \quad (8.7)$$

which is always negative since $f''(x) < 0$ by assumption. Thus the model $f(x)e^\epsilon$ is not suitable for evaluating risk-reducing inputs. Finally, condition 7 can be tested only after a specific functional form for $f(x)$ is chosen.

Model 2 also fails to meet Just and Pope's conditions. Output in model 2 is written as:

$$q = f(x)\epsilon \quad (8.8)$$

an expected output $E(q)$ is $f(x)$.

As in model 1, the first three conditions [$E(q) > 0$, $f'(x) > 0$, and $f''(x) < 0$] are satisfied by assumption. In contrast to model 1, model 2 satisfies condition 4. To examine 5, we again find the variance of output $\sigma^2(q)$:

$$\sigma^2(q) = f(x)^2\sigma_\epsilon^2 \quad (8.9)$$

Finding $\partial\sigma^2(q)/\partial x$, we test condition 5:

$$\frac{\partial\sigma^2(q)}{\partial x} = 2f(x)f'(x)\sigma_\epsilon^2 > 0 \quad (8.10)$$

Since $\partial\sigma^2(q)/\partial x$ can be signed only in one direction, condition 5 is also violated. The variance of the marginal product $\sigma^2[f'(x)]$ is equal to $[f'(x)]^2\sigma_\epsilon^2$. It is also unidirectional in sign as x increases:

$$\frac{d\sigma^2[f'(x)]}{dx} = 2f'(x)f''(x)\sigma_\epsilon^2 < 0$$

Thus condition 6 fails. Again, condition 7 can be tested only for a specific functional form of $f(x)$.

The reader can verify that model 3 also fails to meet several of Just and Pope's conditions. Thus none of the models typically used to examine production function relationships between outputs and inputs capture the essential features of risk-reducing inputs.

AN ALTERNATIVE MODEL

Failure of existing models to meet the marginal conditions for evaluating risk-reducing inputs led Just and Pope to formulate the following model:

$$q = f(x) + h(x)\epsilon \quad (8.11)$$

where it is assumed that $f'(x) > 0$, $f''(x) < 0$, $h'(x) \leq 0$, and ϵ is distributed $\epsilon \sim (0, \sigma_\epsilon^2)$. It can be shown that the model in (8.11) satisfies the seven conditions.¹

Checking condition 1, the expected value of q is:

$$E(q) = f(x) > 0 \quad (8.12)$$

as long as $f'(x) > 0$, a requirement that guarantees condition 2 as well. Condition 3 is also satisfied as long as $f''(x) < 0$. Conditions 4 and 5 require that we calculate the variance of q :

$$\sigma^2(q) = E[f(x) + h(x)\epsilon - f(x)]^2 = [h(x)]^2 E(\epsilon^2) = [h(x)]^2 \sigma_\epsilon^2 \quad (8.13)$$

Condition 5 is tested by differentiating $\sigma^2(q)$ with respect to x :

$$\frac{\partial \sigma^2(q)}{\partial x} = 2h(x)h'(x)\sigma^2 \quad (8.14)$$

Since $h'(x) \leq 0$, (8.14) satisfies condition 5.

To test condition 4, we begin by noting that:

$$\frac{\partial E(q)}{\partial x} = f'(x) \quad (8.15a)$$

Forming the ratio of the expressions in (8.15a) and (8.14) we can write:

$$\frac{\partial E(q)}{\partial \sigma^2(q)} = \frac{f'(x)}{2h(x)h'(x)\sigma^2} \quad (8.15b)$$

With an appropriate choice of $h'(x)$, Eq. (8.15b) can approach zero, hence satisfy condition 4.

Next we examine condition 6 by first finding the variance of the marginal product of x . The expected value of the marginal product is:

$$E[f'(x) + h'(x)\epsilon] = f'(x) \quad (8.16)$$

and the variance is:

$$E[h'(x)\epsilon]^2 = h'(x)^2\sigma_\epsilon^2 \quad (8.17)$$

Differentiating the variance of the marginal product with respect to x yields:

$$\frac{d\sigma^2 [f'(x) + h'(x)\epsilon]}{dx} = 2h'(x)h''(x)\sigma_\epsilon^2 \quad (8.18)$$

The sign of this expression depends on both $h'(x)$ and $h''(x)$, which have not been specified. Hence condition 6 can also be satisfied. Previous comments about condition 7 continue to hold.

Thus we find a model with a rather general form which satisfies assumptions made about the impact of risk-reducing inputs. We will apply this model in analyzing the effects of pesticides, an important risk-reducing input in many types of agricultural production.

THE PEST MANAGEMENT MODEL

The presence of pests in agricultural production poses an interesting problem for economic analysis. Pests are an uninvited input with a negative marginal product in the production model. Controlling the levels of pests reduces the likelihood of lost production. However, the pest population depends on stochastic factors such as heating degree days, humidity, pest populations in previous periods, and the host environment. Thus decision makers must make control decisions under risk.

The weather may eliminate the need for pest controls. On the other hand, waiting until control measures are clearly needed may result in extensive damage requiring difficult or costly control procedures. Determining the threshold level of a pest population which should trigger control procedures is an important problem for biological scientists and agricultural economists. We now consider a model for determining this threshold level.

We assume that the firm operates in a perfectly competitive market. Its input x is completely divisible and, nondurable and produces a completely divisible output q . The input-output relationship is expressed by the production function $q = f(x)$. Input purchases and output sales occur at certain prices p_x and p , respectively, accompanied by fixed cost B .

In this model the pest population is uncertain. It is described by the random variable $N + \epsilon$, where $\epsilon \sim (0, \sigma_\epsilon^2)$ and $-N \leq \epsilon$ for all possible values of ϵ . This stochastic population is estimated from samples taken within the infested area. The expected pest level and the variance of the pest level are expressed as $E(N + \epsilon) = N$ and $\sigma^2(N + \epsilon) = \sigma_\epsilon^2$, respectively.

The damage to the plant (or animal) without pest control is represented by a damage function $D(N + \epsilon)$. Since $N + \epsilon$ is a random variable, so is the outcome of the damage function.

The decision maker responds to the presence of the pest by applying a control measure. In our model, the control is a pesticide z which is purchased at price p_z . The effectiveness of the pesticide is described by a "kill" function $k(z)$, with diminishing marginal kill properties so that $k'(z) > 0$ and $k''(z) < 0$. Since the number of pests killed depends on $N + \epsilon$ at application, the pesticide effectiveness is expressed as the percentage of $N + \epsilon$ that is killed so that $0 \leq k(z) \leq 1$.

The expected postapplication damage of the pest is functionally related to the pest population still alive after the pesticide application; that population is $(N + \epsilon)[1 - k(z)]$. The damage function D then is:

$$D = D \{ (N + \epsilon) [1 - k(z)] \}$$

To simplify the analytic model, D and N are assumed to be linearly related by the function:

$$D = d(N + \epsilon)[1 - k(z)] \quad (8.19)$$

where d = some constant damage caused by each pest.

The profit function y is now formulated as:

$$y = p \{ f(x) - d(N + \epsilon) [1 - k(z)] \} - p_x x - p_z z - B \quad (8.20)$$

It is important to note that the description of output q is characterized by $f(x) - d(N + \epsilon) [1 - k(z)]$. This expression has the general form $f + h\epsilon$ suggested by Just and Pope, where

$$f = f(x) - dN [1 - k(z)] \quad \text{and} \quad h = -d [1 - k(z)] \epsilon$$

Thus this formulation is applicable to the analysis of a risk-reducing input.

Expected profit is:

$$E(y) = p \{ f(x) - dN [1 - k(z)] \} - p_x x - p_z z - B \quad (8.21)$$

and variance of profits, $\sigma^2(y)$, is:

$$\sigma^2(y) = \{ pd [1 - k(z)] \}^2 \sigma_\epsilon^2 \quad (8.22)$$

The certainty equivalent model is written as:

$$\begin{aligned} y_{CE} = & p \{ f(x) - dN [1 - k(z)] \} - p_x x - p_z z - B \\ & - \frac{\lambda}{2} \{ pd [1 - k(z)] \}^2 \sigma_\epsilon^2 \end{aligned} \quad (8.23)$$

The two control variables for this problem are z , the level of pesticides, and x , the inputs to the production function. The firm can produce a more certain output if it applies more pesticides. The cost effectiveness of the pesticide application, however, is uncertain because the pest population is a random variable. The decision maker might accept the risk that nature will reduce the pest population to a very low level. Or, a strongly risk-averse decision maker might pay a high price to eliminate the pests. In any case, the control decision should be independent of the production decision. The first-order conditions with respect to z and x confirm this hypothesis:

$$\frac{\partial y_{CE}}{\partial x} = p f'(x) - p_x = 0 \quad (8.24)$$

$$\frac{\partial y_{CE}}{\partial z} = pdN k'(z) - p_z + \lambda (pd)^2 [1 - k(z)] k'(z) \sigma_\epsilon^2 = 0 \quad (8.25)$$

The first-order condition with respect to x confirms that the output decision is independent of the pesticide control decision. It equates the value of marginal product to the marginal factor cost of x , the usual result of a certainty model. The first-order condition with respect to z suggests that the pesticide is applied until the reduction in profit variability due to uncertain pest damage weighted

by the risk aversion parameter λ plus the expected loss reduction just equals the cost of an additional unit of pesticide.

Thus the firm chooses its inputs based on the equality between its marginal factor costs and its marginal value products. But the risk-reducing input is chosen so that its marginal factor cost just equals the increase in the firm's certainty equivalent associated with a reduction in the variance and the level of damage caused by the pest.

This model can be used to analyze several interesting questions. For example, how will pesticide applications change as risk aversion increases? We answer this question by totally differentiating the first-order condition (8.25) which produces the result:

$$\frac{dz}{d\lambda} = \frac{-(pd)[1-k(z)]k'(z)\sigma_\epsilon^2}{Nk''(z) + \lambda pd[1-k(z)]k''(z)\sigma_\epsilon^2 - \lambda pd[k'(z)]^2\sigma_\epsilon^2} > 0 \quad (8.26)$$

since $k''(z) < 0$.

The pest population where the decision maker is indifferent about applying the pesticide is called the *economic threshold* (Headley). To find the threshold population N^* we replace z in the first-order condition (8.25) by zero and solve for N^* .

$$N^* = \frac{p_z - \lambda(pd)^2[1-k(0)]k'(0)\sigma_\epsilon^2}{pdk'(0)} \quad (8.27)$$

The threshold population increases with the cost of pesticides and decreases with increases in output price p , damage function parameter d , variance of pest population σ_ϵ^2 , and risk aversion parameter λ .

$$\begin{aligned} \frac{dN^*}{dp_z} &> 0 \\ \frac{dN^*}{dp} &< 0 \\ \frac{dN^*}{dd} &< 0 \\ \frac{dN^*}{d\sigma_\epsilon^2} &< 0 \\ \frac{dN^*}{d\lambda} &< 0 \end{aligned}$$

The changes in threshold level N in response to an increase in fixed costs or input prices are less clear. Both affect the EV set in similar ways. Consider, for example, the increase in input price p_x . Because the output decision is independent of the pesticide decision, every EV efficient solution has the same level of input x . Thus increasing p_x shifts the EV set downward and reduces profits for every choice. The downward shift is parallel because the increased input price uniformly affects profits at all levels of output. Therefore the only impact on the threshold level N^* is an income effect:

$$\frac{dN^*}{dp_x} = \frac{-(\partial\lambda/\partial p_x)(pd)^2[1-k(0)]k'(0)\sigma_\epsilon^2}{pdk'(0)} \quad (8.28)$$

The sign of dN^*/dp_x depends on the numerator. First, recognize that increasing input costs reduce expected profits and shift the EV frontier downward. For DARA decision makers, a downward shift in the EV set makes the decision maker less willing to accept possible losses from pest damage. As a result, $\partial\lambda/\partial p_x > 0$, making $dN^*/dp_x < 0$. For IARA decision makers $dN^*/dp_x > 0$ and 0 for CARA decision makers.

To summarize, (8.26) suggests that the more risk-averse the individual decision maker (i.e., the larger λ is), the greater his or her application of the risk-reducing input z . It can also be shown, using Eq. (8.27), that the more risk-averse the decision maker (i.e., the larger λ is), the lower the economic threshold pest population N^* at which pesticide application is begun. Finally, the effect of an increase in the cost of the productive input x on the economic threshold depends on whether the decision maker is DARA, in which case N^* decreases, IARA, in which case N^* increases, or CARA, in which case N^* does not change.

ALTERNATIVE FORMULATION

The pest management model just described is patterned after a model developed by Feder. However, Feder's model has serious limitations for purposes other than analyzing risk-reducing inputs because of the limited number of risk sources it considers. In reality, the effectiveness of the pesticide, described by function $k(z)$, is probably stochastic, as is the output price p . The output price and the level of output may not be independent either. If pest control is effective and output increases, prices will likely decline. A more complete pest management model using simulation techniques has been developed by Cochran, et al. and by Talpaz and Borosh.

We can extend our model results by allowing the price of the output and the pest population to be stochastic. The earlier model described in Eq. (8.20) is modified by letting the output price be $p + v$, where $v \sim (0, \sigma_v^2)$ and $-p < v$. Then let the pest population be $N + \epsilon$, as before where $\epsilon \sim (0, \sigma_\epsilon^2)$ and $-N < \epsilon$. Finally, assume ϵ and v are independent. Now let the stochastic profit function be expressed as:

$$y = (p + v) \{f(x) - d(N + \epsilon) [1 - k(z)]\} - p_x x - p_z z - B \quad (8.29)$$

With two random variables, we can proceed as before using our EV model. However, with two random variables, consistency between EU and EV results is not guaranteed. On the other hand, results from the EV model are easily derived which is not the case with EU models.

In Chap. 10, we will analyze a two random variable model in our EV framework. Two random variables, however, quickly complicate our analysis, forcing us into numeric rather than analytic approaches. Furthermore, we could find the threshold level for N^* as before but the solution would require solving a quadratic formula with few deterministic results.

OTHER EXTENSIONS

The risk-reducing inputs model can also be expanded in the direction of the flexibility models to be considered in Chap. 9. Suppose, for instance, that the input z is chosen after ϵ has been revealed in the first pest management model discussed. How would the choice variables z and x be affected by this increased flexibility?

To answer this question, we express the ex post profit function y as:

$$y = p \{f(x) - d(N + \epsilon) [1 - k(z)]\} - p_x x - p_z z - B \quad (8.30)$$

The first- and second-order conditions for the choice of z , which optimizes y after ϵ is revealed, are:

$$pd(N + \epsilon)k'(z) - p_z = 0 \quad (8.31)$$

and

$$pd(N + \epsilon)k''(z) < 0 \quad \text{for } k''(z) < 0 \quad (8.32)$$

From (8.32) we solve for the optimal value of z :

$$z = z^*(p, p_z, d, N, \epsilon)$$

which is then substituted into the ex ante profit function given in (8.30). The equivalent model can now be written as:

$$y = p[f(x) - d(N + \epsilon) \{1 - k[z^*(p, p_z, d, N, \epsilon)]\}] - p_x x - p_z [z^*(p, p_z, d, N, \epsilon)] - B \quad (8.33)$$

Interestingly, the solution for x is not affected. It is still $pf'(x) = p_x$.

Moreover, since z is chosen ex post, its choice is affected by the realized value of the random variable ϵ , but not by risk or the distribution of ϵ . This result occurs because the ex post adjustment possibility in essence allows the firm to eliminate the unknown consequences of z on the pest population $N + \epsilon$.

The conclusion that both x and z are chosen without the distribution of ϵ influencing their choice is unique to the form of the model just examined, namely, that $f(x)$ and p are both certain. The way the model is described, the variance associated with ϵ is not influenced by the input value for x . Thus it should be clear that these results are sensitive to the model formulation, that general conclusions under risk are hard to obtain, and that whether errors enter the models in a multiplicative or additive manner is crucial to the outcome.

CONCLUDING COMMENTS

Risk-reducing input models are applicable to a wide range of practical problems. Irrigation, fertilization, insurance, labor utilization, and excess machinery investments all have risk-reducing capacities. What should be apparent is that, for inputs to have the risk-reducing impact suggested by Just and Pope, the models must have the recognizable form of (8.11a) or (8.11b). Understanding the risk-reducing characteristics helps explain this dimension of their use by various types of business firms. A certainty model lacks this explanatory power and fails to generate results consistent with the risk responses of many types of decision makers.

ENDNOTE

1. Clearly Just and Pope's condition could be generalized to more than one input. For example, if the two inputs were x and z , their condition is written as:

$$q = f(x, z) + h(x, z)\epsilon$$

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OPTIMAL INPUT USE UNDER RISK

In this chapter we discuss the impact of risk on the firm's demands for its factors of production. In particular we discuss how the firm's acquisition and use of inputs adjust to changes in risk. The firm's responses, of course, depend on the source of risk. Factor demands may be influenced by risk associated with input prices, input quality, output prices, production functions, input availability, and holding costs for inputs.

The most obvious response of factor demand to risk is to alter the relative amounts of inputs. Under certainty optimal input combinations are referred to as *least-cost combinations*. If risk alters these input combinations, then factors will no longer be in the least-cost combinations determined under certainty. Of interest here are comparisons of factor demands under risk to those under certainty conditions.

This chapter also examines the concept of flexibility and its influence on factor demands. *Flexibility* is defined as the ability to adjust to new information. It arises as a natural consequence of being able to make some input decisions after the random variable has been revealed. Sometimes flexibility is increased by holding inventories. It will be shown that flexibility almost always allows factors to adjust from their least-cost combinations determined under certainty.

In this chapter we will first review the theory of factor demand under certainty. Then we examine factor demands under the following conditions: (1) output price risk, (2) input price risk, (3) quality of input risk, and (4) production function risk. Then flexibility is introduced by allowing the firm to choose one of its inputs after the uncertainty is revealed. Finally, we show how the addition of flexibility results in a class of solutions falling between Sandmo's theory and the theory of the firm under certainty.

LEAST-COST INPUT COMBINATIONS UNDER CERTAINTY

Consider a production function which depends on two factors of production: L (for labor) and K (for capital). The usual assumption is that labor is available and can be used in perfectly divisible quantities. Many different assumptions have been employed regarding capital, including assuming it is an indestructible good or that it is perfectly divisible in acquisition and use. For now, assume that capital is fixed to the firm, so that its salvage value is less than its value to the firm. Thus the firm's only short-run decision for a given output q is to select the level of labor L . We write the profit function y as:

$$\max y = pf(L, K) - wL - rK \quad (9.1)$$

subject to $q = f(L, K)$, $f_L > 0$, $f_{LL} < 0$, $f_K > 0$, $f_{KK} < 0$, and $f_{KK}f_{LL} - f_{KL}f_{LK} > 0$, where w and r are the wage rate and rental price on capital, respectively, $f(L, K)$ is the production function for q with derivatives that satisfy second-order conditions for a maximum, and p is the output price. Since K is fixed to the firm, rK is also fixed. Thus the solution for L is:

$$pf_L = w \quad (9.2)$$

where the marginal value product equals the factor cost. Since $f_{LL} < 0$, second-order conditions are guaranteed, and from (9.2) we can find the short-run factor demand expression:

$$L = h(K, w, p) \quad (9.3)$$

Now assume that both K and L are divisible in acquisition and use. This implies that during the production process both inputs are converted to output q with no residual. Moreover, rK is now a variable cost. The profit expression is still (9.1), and the first-order condition for L is still (9.2). The first-order condition for K , however, is now found by differentiating (9.1). It equals:

$$pf_K = r \quad (9.4)$$

The ratio of the marginal value products for K and L define the tangency of the least-cost combination line for a given q equal to:

$$\frac{f_K}{f_L} = \frac{r}{w} \quad (9.5)$$

This solution is described graphically in Fig. 9.1. For a budget of B equal to:

$$B = wL + rK$$

the isocost line is:

$$L = \frac{B}{w} - \frac{r}{w}K$$

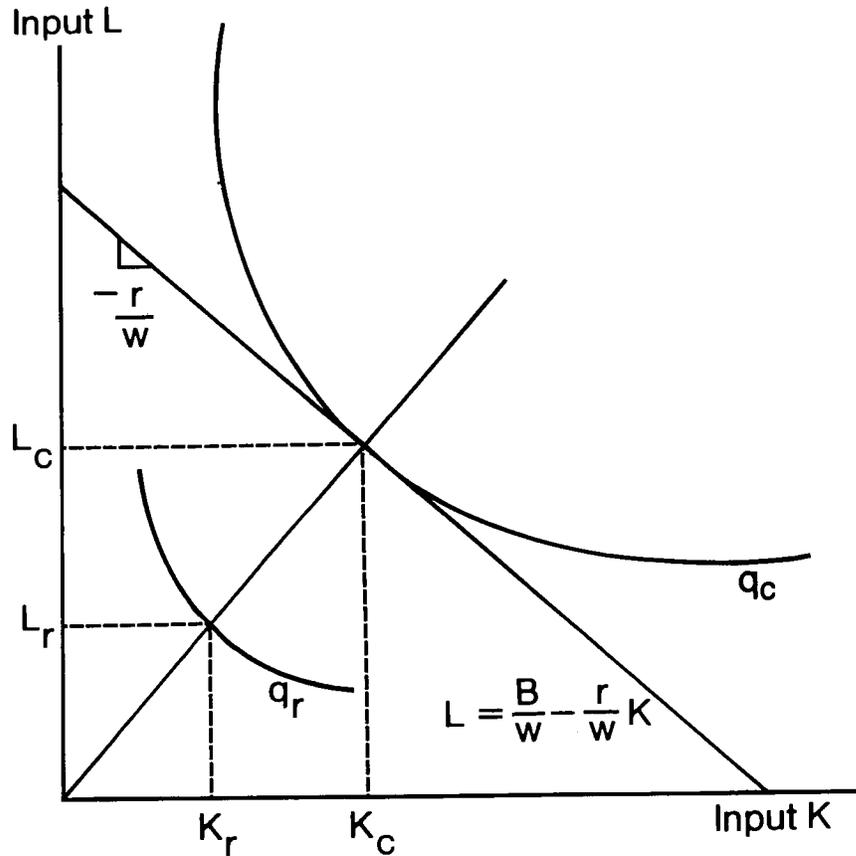


Figure 9.1 Isocost, isoquant, and least-cost solutions to a production problem with two variable inputs under certainty.

Meanwhile, the isoquant for a given q_c is:

$$df_c = f_L dL + f_K dK = 0$$

or

$$\frac{dL}{dK} = -\frac{f_K}{f_L}$$

At the equilibrium tangency point, where the slope of the isoquant equals the slope of the isocost line, the equilibrium relationship is

$$\frac{f_K}{f_L} = \frac{r}{w}$$

which is the condition for (9.5). For the profit-maximizing output, found by solving (9.2) and (9.4) simultaneously, we associate the output q_c and input levels K_c and L_c . These will serve as reference values for later comparisons with factor demands and output levels derived under risk.

OUTPUT PRICE RISK AND FACTOR DEMANDS

Assume that, for the model with two variable inputs L and K described in Eq. (9.1), the output price is $p + \epsilon$ instead of p , where ϵ is a random variable distributed with mean zero and variance σ_ϵ^2 . Under this assumption, income y is written:

$$y = (p + \epsilon)f(L, K) - wL - rK \quad (9.6)$$

where the expected value of y , $E(y)$, and the variance of y , $\sigma^2(y)$, are written, respectively, as:

$$E(y) = pf(L, K) - wL - rK$$

and

$$\sigma^2(y) = f(L, K)^2 \sigma_\epsilon^2$$

This allows us to write the certainty equivalent model as:

$$\max_{K, L} y_{CE} = pf(L, K) - wL - rK - \frac{\lambda}{2} f(L, K)^2 \sigma_\epsilon^2 \quad (9.7)$$

The first-order conditions for L and K are written:

$$pf_L - w - \lambda f_L f \sigma_\epsilon^2 = 0 \quad (9.8a)$$

and

$$pf_K - r - \lambda f_K f \sigma_\epsilon^2 = 0 \quad (9.8b)$$

The relative input combinations are found from (9.8a) and (9.8b) by taking w and r to the respective right-hand sides and dividing the ratios. The result is:

$$\frac{w}{r} = \frac{f_L(p - \lambda q \sigma_\epsilon^2)}{f_K(p - \lambda q \sigma_\epsilon^2)} = \frac{f_L}{f_K} \quad (9.9)$$

which is the same condition as (9.5), the least-cost combination under certainty.

As Ishii and Sandmo (see Chap. 7) have shown, output responds to changes in risk, risk attitudes, and output price. Equation (9.9) indicates that the ratio of inputs remains the same despite the fact that the absolute level of inputs falls because output falls for risk averters when risk is introduced into the model. We conclude, therefore, that because risk changes output levels but leaves relative input ratios unaltered, production still occurs on the line of least-cost combinations. Similar conclusions were reached by Batra and Ullah and by Hartman (1975).¹

In addition to these results Batra and Ullah showed that with DARA and the derivative signs assumed in (9.1) an increase in σ_ϵ^2 reduces K and L . Hartman (1976) showed that only under risk aversion and with the signs already imposed on the derivative are both inputs less under risk than under certainty. The solution for factor levels and output under risk is less than the corresponding certainty result. This situation is portrayed in Fig. 9.1 as L_r , K_r , and q_r .

INPUT PRICE RISK

We next consider the case of input price risk. This condition may exist when not all inputs are purchased at the same time in the production process or when inputs are purchased in different types of markets. The agricultural production process is an example. The production of corn involves fertilizer, seed, and soil preparation in the spring when input prices are well-known. Costs to be incurred in the fall, including harvesting, drying, and transportation costs, are not, however, known with certainty. Another example is defense contracts established with contractors by the U.S. government using a "cost plus" pricing arrangement because the final production costs are subject to considerable risk.

To examine the effects of uncertain input prices, we define a production process, as before, with two inputs with stochastic prices and which are divisible in acquisition and use. Let the input price for labor L be $w + \epsilon$, where $\epsilon \sim (0, \sigma_\epsilon^2)$, and let the input price for capital K be $r + v$, where $v \sim (0, \sigma_v^2)$. The correlation between v and ϵ equals ρ . In this case, the output price p is certain. Finally, assume that the production function has the derivatives defined before to ensure that both inputs increase (decrease) with production.

The expected profit and variance of profit are expressed, respectively, as:

$$E(y) = pf(L, K) - wL - rK \quad (9.10)$$

$$\sigma^2(y) = L^2\sigma_\epsilon^2 + K^2\sigma_v^2 + 2LK\rho\sigma_v\sigma_\epsilon \quad (9.11)$$

The certainty equivalent model is:

$$\max_{K,L} y_{CE}(K, L) = E(y) - \frac{\lambda}{2}\sigma^2(y)$$

and the first-order conditions are:

$$\frac{dy_{CE}}{dL} = pf_L - w - \lambda(L\sigma_\epsilon^2 + K\rho\sigma_v\sigma_\epsilon) = 0 \quad (9.12a)$$

$$\frac{dy_{CE}}{dK} = pf_K - r - \lambda(K\sigma_v^2 + L\rho\sigma_v\sigma_\epsilon) = 0 \quad (9.12b)$$

Rearranging (9.12a) and (9.12b) yields the equivalent of (9.5) and (9.9):

$$\frac{w}{r} = \frac{pf_L - \lambda(L\sigma_\epsilon^2 + K\rho\sigma_v\sigma_\epsilon)}{pf_K - \lambda(K\sigma_v^2 + L\rho\sigma_v\sigma_\epsilon)} \quad (9.13)$$

In addition to input prices and marginal products, the ratio of factor combinations now depends on the output price, risk aversion, the risk variables σ_v , and σ_ϵ , and the correlation ρ associated with the two input prices.

While output price risk alters the level of factor demands, it does not change their relative demands because each is affected in the same way as long as the

production function is homothetic (the slope of the isoquants along a ray from the origin does not change).²

If a decision maker is not risk-neutral, factor price risk will change both the level and factor proportions relative to the line of least-cost combinations under certainty. This result stems from the fact that inputs incur differential price risks. Moreover, as the input price risk is reduced for one of the inputs, the relative factor demand moves in its favor. For example, if $\sigma_\epsilon^2 = 0$, then (9.13) becomes:

$$\frac{w}{r} = \frac{pf_{L^*}}{pf_{K^*} - \lambda K^* \sigma_v^2} \quad (9.14)$$

which can be achieved only if:

$$f_{L^*} < f_L \quad \text{or} \quad pf_{K^*} - \lambda K^* \sigma_v^2 > pf_K - \lambda(K\sigma_v^2 + L\rho\sigma_\epsilon\sigma_v)$$

This requires that L increase to L^* , or K decrease to K^* , or both. Thus eliminating the input price risk associated with labor L increases its demand relative to K .

QUALITY OF INPUT RISK

Suppose that now the firm faces a still different kind of risk, namely, that an input L can be purchased at a known price of w per unit but that the actual quality of L is stochastic, say $L + \epsilon$, where $\epsilon \sim (0, \sigma_\epsilon^2)$ and $\epsilon > -L$. Quality of input risk involves variability in services supplied by the input. For example, labor is hired to provide services, but health, incentives, previous training, and dependability leave uncertain the actual supply of available services.

Assuming the actual level of inputs supplied is $L + \epsilon$, let the production function for q with a fixed supply of capital be expressed as:

$$q = f(L + \epsilon|K)$$

subject to

$$f'_L > 0 \quad \text{and} \quad f''_L < 0$$

which ensures second-order conditions for certainty equivalent maximization.

The income function y , expected value of y , and variance of y now are written using the earlier notation as:

$$y = pf(L + \epsilon|K) - wL - rK$$

$$E(y) = pEf(L + \epsilon|K) - wL - rK$$

and

$$\sigma^2(y) = p^2 \sigma^2(f)$$

which allows us to form the certainty equivalent expression:

$$\max y_{CE} = pEf(L + \epsilon|K) - wL - rK - \frac{\lambda}{2}p^2\sigma^2(f) \quad (9.15)$$

Now consider the expressions for the variance and expected values of $f(L + \epsilon|K)$ if ϵ takes on the values ΔL and $-\Delta L$ with equal probability. The expected value of the production function is then:

$$E[f(L + \epsilon|K)] = \frac{1}{2}[f(L + \Delta L|K) + f(L - \Delta L|K)] < f(L|K) \quad (9.16)$$

The expected output is less than the output at the mean input level L for the same reason that $E[U(y)]$ is less than the utility of the mean income $U[E(y)]$. Both functions f and U are concave, so that marginal products to the left of L are greater than those to the right. With Jensen's inequality, we deduce that the expected value of a concave function is always less than the functional value of the mean level of input. Nevertheless, the expected value of the output depends critically on the value of L .

Now consider how increases in L affect the variance of output. Suppose, for example, that the firm selects input level $L^* > L$ while random factors determine that either $L^* - \Delta L$ or $L^* + \Delta L$ is actually available. At L^* , the range of outputs is:

$$\Delta f(L^*|K) = f(L^* + \Delta L|K) - f(L^* - \Delta L|K) \quad (9.17)$$

while at input level L the range of outputs is:

$$\Delta f(L|K) = f(L + \Delta L|K) - f(L - \Delta L|K) \quad (9.18)$$

But diminishing marginal productivity means that

$$\Delta f(L^*|K) < \Delta f(L|K)$$

as shown in Fig. 9.2. Because the variance in this case depends on the range determined by the end points, the variance of output associated with L^* , $\sigma^2[f(L^*)]$, is less than the variance associated with the input choice L , $\sigma^2[f(L)]$. Thus variance is reduced by increasing the input level. Moreover, this result is consistent with empirical observations. For example, farmers buying new equipment frequently keep their old machines to use when breakdowns occur. Extra parts are often held in inventory in case replacements are not available.

We complete the analysis of input service risk with the certainty equivalent model. To do so, let $Ef(L + \epsilon | K)$ be denoted $\bar{f}(L)$ and let the variance be $\sigma^2(L)$. Then assume that the input is purchased in a perfectly competitive market at constant price w .

Expected profit for the firm facing input service risk is:

$$E(y) = p\bar{f}(L) - wL \quad (9.19)$$

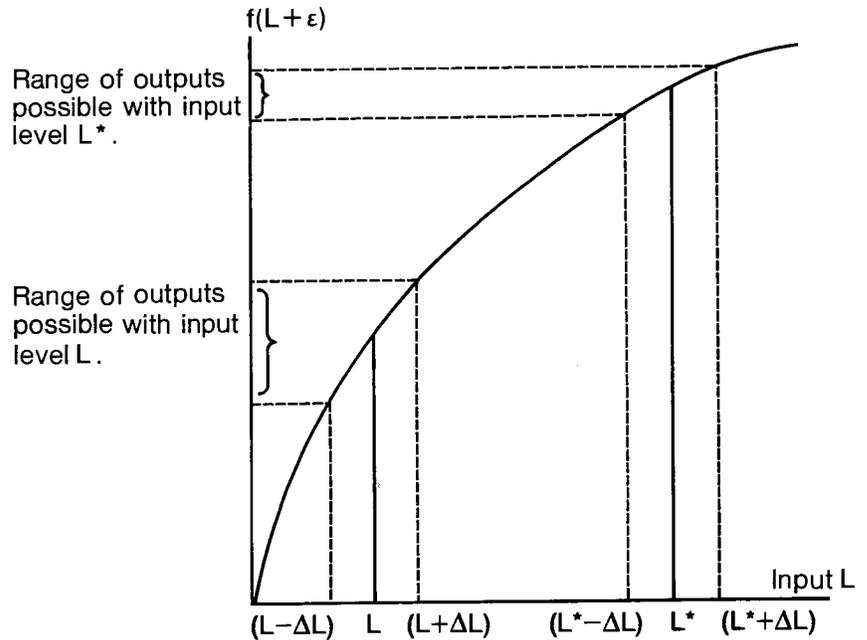


Figure 9.2 Comparison of the range of outputs at two levels of input $L^* > L$ with quality of input risk ϵ equal to $-L$ and L .

The first-order condition requires:

$$p\bar{f}_L(L_c) = w \tag{9.20}$$

where L_c is the value of L satisfying (9.20) which is also the certainty solution.

Now consider the solution for the risk-averse certainty equivalent maximizer whose objective function is:

$$\max y_{CE} = p\bar{f}(L_r) - wL_r - \frac{\lambda}{2}p^2\sigma^2[f(L_r)] \tag{9.21}$$

where L_r = optimal solution for L under risk. The first-order condition is:

$$p\bar{f}_L(L_r) - w - \frac{\lambda}{2}p^2 \frac{\partial \sigma^2[f(L_r)]}{\partial L} = 0 \tag{9.22}$$

But the earlier discussion deduced that $\partial \sigma^2[f(L_r)]/\partial L < 0$. Thus risk is decreasing with L , so that:

$$p\bar{f}_L(L_r) < p\bar{f}_L(L_c) \tag{9.23}$$

which, given a production function with diminishing marginal productivity, is true only when $L_r > L_c$. In general then, risk associated with the quality of inputs induces a decision maker to hold larger levels of the input than would

be held if the quality of the input were known with certainty or if the decision maker were less risk-averse.

Overstocking as a response to input service risk depends critically on the additivity of input service risk. Suppose, for example, that the actual input available to the firm is $L\epsilon$ rather than $L + \epsilon$, where the expected value of ϵ is 1 and the variance is σ_ϵ^2 . At higher input levels, the variance will then be $L^2\sigma_\epsilon^2$ rather than the constant σ_ϵ^2 for the additive case. In this case overstocking is not necessarily a proper response to quality of input risk. The case of multiplicative input quality risk is reviewed by Batra and Ullah who obtain results opposite those above, but for a firm facing multiplicative risks.

PRODUCTION FUNCTION RISK

Finally, consider the case where the quality of the input is certain but the input-output process is uncertain. Moreover, let output q be a function of two inputs K and L . Then:

$$q = f(L, K) + \epsilon \quad (9.24)$$

where $\epsilon \sim (0, \sigma_\epsilon^2)$ and where the derivatives of f have the signs described in (9.1).

Finally, assume as before that q is sold at a certain price p and that L and K are purchased at certain input prices w and r , respectively. Profit y is written as:

$$y = p[f(L, K) + \epsilon] - wL - rK \quad (9.25)$$

where $E(y) = pf(L, K) - wL - rK$ and $\sigma^2(y) = p^2\sigma_\epsilon^2$. Then the certainty equivalent model is:

$$\max y_{CE}(L, K) = pf(L, K) - wL - rK - \frac{\lambda}{2}p^2\sigma_\epsilon^2 \quad (9.26)$$

The derivatives of f ensure that certainty equivalent-maximizing values of L and K can be found. In this unusual model, risk is independent of the choices of L and K , so that the relative factor demands are least-cost combinations and the factor levels are those of the certainty model which satisfy:

$$pf_L - w = 0, \quad (9.27a)$$

and

$$pf_K - r = 0 \quad (9.27b)$$

If, however, the production function risk is multiplicative with

$$q = f(L, K)\epsilon$$

where $\epsilon \sim (1, \sigma_\epsilon^2)$, then the certainty equivalent model will be:

$$\max y_{CE}(L, K) = pf(L, K) - wL - rK - \frac{\lambda}{2}p^2f(L, K)^2\sigma_\epsilon^2 \quad (9.28)$$

Now the first-order conditions are written as:

$$pf_L - w - \lambda p^2 f(L, K) f_L \sigma_\epsilon^2 = 0, \quad (9.29a)$$

and

$$pf_K - r - \lambda p^2 f(L, K) f_K \sigma_\epsilon^2 = 0 \quad (9.29b)$$

Forming the ratio r/w results in the expression

$$\begin{aligned} \frac{r}{w} &= \frac{f_L [p - \lambda p^2 f(L, K) \sigma_\epsilon^2]}{f_K [p - \lambda p^2 f(L, K) \sigma_\epsilon^2]} \\ &= \frac{f_L}{f_K} \end{aligned} \quad (9.30)$$

which is the least-cost combination deduced under conditions of certainty in Eq. (9.5). Risk, however, alters the factor levels because, contrary to the additive risk case, risk is not independent of the factor choices. If the production function is homothetic, so that the input combination does not change with the level of production, the relative factor demands will be constant.

FLEXIBILITY AND INPUT SELECTIONS

Optimal input use is influenced not only by risk but also by flexibility. In everyday use "flexibility" has come to mean the characteristics associated with the initial choice which measures the ability to respond or conform to new or changing conditions (*Webster's New Collegiate Dictionary*, 1977).

Information and flexibility are critically linked. Information determines the importance of flexibility to the firm. If no new information is available, flexibility is not important. That is, the characteristics of the initial choice which allow it to adjust to new information are not useful. On the other hand, without the ability to adjust, or if the cost of adjustment is exorbitant, new information is of little use to the firm.

In the rest of this chapter we review efforts to incorporate risk and flexibility into the selection of divisible inputs and then construct a certainty equivalent model to illustrate the important relationship between a class of flexibility models and Sandmo's model.

In similar articles by Holthausen and by Hartman (1976), flexibility is introduced into a two-input production model by choosing one input, call it capital K , in the presence of risk and then later choosing the second input, call it labor L , after the risk is eliminated. Thus the second input is chosen under certainty conditions.

In Hartman's paper, the firm chooses the amount of capital it will employ when it is uncertain what the output price will be. Before the firm decides on how much labor to use, it receives information which reveals the output price. Let the firm's ex post income function y equal

$$y = (p + \epsilon) f(L, K) - wL - rK \quad (9.31)$$

where ϵ is a known output of the distribution of ϵ 's. Then the ex post solution of labor for a given capital level K^* is:

$$(p + \epsilon)f_L(L, K^*) = w \quad (9.32)$$

from which the factor demand for labor is derived as:

$$L = L(K^*, p, w, \epsilon) \quad (9.33)$$

This in turn leads to the ex ante profit function:

$$y = (p + \epsilon)f[L(K^*, p, w, \epsilon), K^*] - wL(K^*, p, w, \epsilon) - rK^* \quad (9.34)$$

Hartman then examines the expected utility-maximizing choice of K^* for the cases of risk aversion and risk neutrality. He finds that a risk-neutral firm's response to a Rothschild-Stiglitz mean-preserving spread of the distribution of ϵ 's is to increase or decrease the optimal capital input depending on whether $\partial(f_{LK}/f_{LL})/\partial L$ is less than or greater than zero. Thus the effect of an increase in risk on the demand for labor is ambiguous.

Holthausen and Perrakis each perform a similar analysis for a monopolistic firm. Holthausen's first model chooses the output ex post but the two inputs ex ante. This model implies that labor is in fact chosen ex post to adjust output. Holthausen's price-setting model is more plausible. He shows that, when capital use increases with output, a risk-averse firm uses a smaller amount than the expected cost-minimizing level for the level of output produced. Perrakis extends Holthausen's results by considering a firm with $n + m$ factors of production of which n are determined ex post and m determined ex ante.

Smith (1970) introduces flexibility in a different way. He assumes a competitive firm using two factors of production, capital and labor. Capital, he assumes, is chosen ex ante in the face of risk. Labor and the rate of capital utilization, on the other hand, are chosen ex post under conditions of certainty. The problem is complicated by quadratic depreciation functions of capital stock which depend on the rate of utilization. For higher utilization rates, the capital stock depreciates at a faster rate. In a similar paper, Smith (1969) examines similar consequences for a price-setting monopolist. Because of our focus on the competitive firm, we do not pursue the results of Smith's earlier paper.

Turnovsky introduces still a different flexibility model. He assumes that total costs are a function of output planned ex ante q , and ex post output $q + z$, where $-q < -z$. The amount $q + z$ is the amount the firm decides to produce after prices become known. Turnovsky penalizes the firm for adjusting output from its planned level y , and this penalty depends on the size of z .

FLEXIBILITY AND INVENTORIES

We now introduce a competitive firm model which captures and illustrates some essential features of the flexibility models described above.³ The model will also show the difference between ex ante models which contain no ex post controls (or flexibility) and ex post (instability) models which contain complete flexibility.

To begin, we assume that the competitive firm faces ex ante price risk but, depending on its flexibility, can make factor adjustments ex post when the actual output price is revealed. The level of flexibility, and thus the cost of making ex post adjustments, depend on its ex ante choices. The firm chooses its level of flexibility and places limits on its ability to choose output levels ex post once the output price is known with certainty.

The basic conclusion we draw from our model is that flexibility results in an output level somewhere between that of the ex ante model, in which all input choices are made under risk, and the price instability model, in which all choices are made under conditions of certainty. If adjustment costs are high (and flexibility low), output levels tend toward those of the ex ante firm. If adjustment costs are low (and flexibility high), the solution tends toward that of the firm whose decisions are made ex post.

First consider the competitive firm facing price risk whose output is chosen ex ante. Let output q depend on the cost function $C(q)$, where $C'(q) > 0$ and $C''(q) > 0$. Moreover, let the output price be $p + \epsilon$, where $\epsilon \sim (0, \sigma_\epsilon^2)$ and fixed costs equal B . This is the Sandmo model in which the certainty equivalent expression is:

$$\max y_{CE} = pq - C(q) - B - \frac{\lambda}{2} q^2 \sigma_\epsilon^2 \quad (9.35)$$

To simplify matters we assume that $C(q) = bq^2$, where $b > 0$ and is constant. With this assumption, (9.35) is solved by substituting for $C(q)$ and finding the first-order conditions. The first-order condition is:

$$p - 2bq - \lambda q \sigma_\epsilon^2 = 0 \quad (9.36)$$

Therefore:

$$q = \frac{p}{2b + \lambda \sigma_\epsilon^2} \quad (9.37)$$

Second-order conditions are guaranteed by the quadratic cost function and the quadratic nature of risk costs.

In contrast to the results of the ex ante model, suppose q is chosen in the model above after output price $p + \epsilon$ is revealed. Then income y is written:

$$y = (p + \epsilon)q - bq^2 - B \quad (9.38)$$

Maximizing (9.38) with respect to q results in the solution for q equal to:

$$q = \frac{p + \epsilon}{2b} \quad (9.39)$$

so that on average q equals:

$$E(q) = \frac{p}{2b} > \frac{p}{2b + \lambda\sigma_\epsilon^2} \quad (9.40)$$

That $E(q)$ is larger than the q chosen in the ex ante model is not a general result but depends on the nature of the cost function $C(q)$, in this case assumed to be a quadratic.⁴

Now we introduce flexibility into the model. Instead of requiring the firm to choose q ex ante in the face of risk, let it choose an inventory of q equal to I . Then, after the output price is known, the firm can sell q equal to I , hold the unused portion of I as inventory ($q < I$), or produce more ($q > I$). However, we assume that for $q < I$ or for $q > I$, the firm pays an inventory or adjustment cost which is a quadratic function of the difference $I - q$. Thus we capture the essence of Turnovsky's adjustment cost, which allows us to examine the effects of increased or reduced flexibility (as a function of adjustment costs) yet continue to preserve the ex ante or ex post decision-making process described by Hartman and by Holthausen.

To find the optimal inventory level I , like Hartman we first find the firm's ex post supply of q . Let the ex post profit function y be:

$$y = (p + \epsilon)q - bq^2 - B - c(q - I)^2 \quad (9.41)$$

where $c(I - q)^2 =$ quadratic adjustment cost, where $c > 0$ and is constant.

The first-order condition for q given I in (9.41) is:

$$p + \epsilon - 2(b + c)q + 2cI = 0 \quad (9.42)$$

from which the demand for q is found:

$$q = \frac{p + \epsilon + 2cI}{2(b + c)} = q(I) \quad (9.43)$$

Next we substitute the right-hand side of (9.43) for q in (9.41) to obtain the ex ante income as a function of I . This is expressed as:

$$\begin{aligned} y &= (p + \epsilon)q(I) - b[q(I)]^2 - c[q(I) - I]^2 - B \\ &= \alpha_0(I) + \alpha_1(I)\epsilon + \alpha_2\epsilon^2 \end{aligned} \quad (9.44)$$

where

$$\alpha_0(I) = [(p + 2cI)^2/4(b + c)] - B - cI^2$$

$$\alpha_1(I) = (p + 2cI)/2(b + c)$$

$$\alpha_2 = \frac{1}{4(b + c)}$$

To find the optimal I , we find the certainty equivalent expression for (9.44). The expected value and variance of y , recalling that $E(\epsilon) = 0$, can be written:

$$E(y) = \alpha_0(I) + \alpha_2\sigma_\epsilon^2 \quad (9.45)$$

and

$$\sigma^2(y) = [\alpha_1(I)]^2 \sigma_\epsilon^2 + \alpha_2^2 \sigma^2(\epsilon^2) + 2\alpha_1(I)\alpha_2\mu_3 \quad (9.46)$$

where $\sigma^2(\epsilon^2) = E(\epsilon^4) - (\sigma_\epsilon^2)^2$ and $\mu_3 = E(\epsilon^3)$.

The certainty equivalent model, which now depends only on I , is:

$$\begin{aligned} \max y_{CE(I)} = & \alpha_0(I) + \alpha_2\sigma_\epsilon^2 - \frac{\lambda}{2} \left\{ [\alpha_1(I)]^2 \sigma_\epsilon^2 \right. \\ & \left. + \alpha_2^2 \sigma^2(\epsilon^2) + 2\alpha_1(I)\alpha_2\mu_3 \right\} \end{aligned} \quad (9.47)$$

The first-order condition for (9.47) is:

$$\frac{dy_{CE(I)}}{dI} = \frac{\partial \alpha_0(I)}{\partial I} - \lambda [\alpha_1(I)\sigma_\epsilon^2 + \alpha_2\mu_3] \frac{\partial \alpha_1(I)}{\partial I} = 0 \quad (9.48)$$

Since

$$\frac{\partial \alpha_0(I)}{\partial I} = \frac{c(p - 2bI)}{b + c}$$

and

$$\frac{\partial \alpha_1(I)}{\partial I} = \frac{c}{b + c}$$

we can substitute into (9.48) and obtain the first-order condition in terms of the original parameters used in (9.41). The first-order condition is:

$$(p - 2bI) - \lambda \left[\frac{(p + 2cI)\sigma_\epsilon^2}{2(b + c)} + \frac{\mu_3}{4(b + c)} \right] = 0 \quad (9.49)$$

The second-order conditions are satisfied, which guarantees that the solution to (9.49) is a maximum. The explicit solution for I is found from (9.49):

$$I = \frac{p(b + c) - (\lambda/4)(\mu_3 + 2p\sigma_\epsilon^2)}{2b(b + c) + \lambda c\sigma_\epsilon^2} \quad (9.50)$$

Notice that, when the output price variance σ_ϵ^2 increases, the optimal inventory level decreases. This response is due to the increase in risk costs. In Chap. 7, we showed that ex ante output decreases with increases in risk costs. Corresponding to the decrease in the optimal q is a decrease in the optimal inventory needed to supply q . Increasing the skewness has an effect similar to that of increasing σ_ϵ^2 ; it increases risk costs, and decreases optimal q and I . On

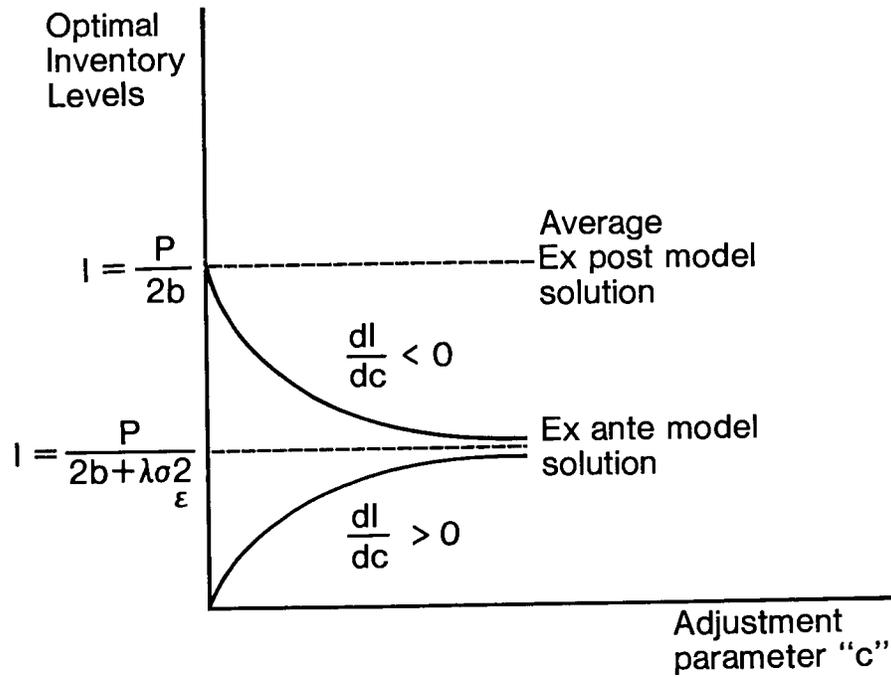


Figure 9.3 Optimal inventory levels as ex post adjustment costs increase.

the other hand, increasing p increases the optimal inventory level required by the firm:

$$\frac{dI}{d\sigma_\epsilon^2} < 0 \quad \text{and} \quad \frac{dI}{d\mu_3} < 0 \quad \frac{I}{dp} > 0 \quad \text{for} \quad 2(b+c) > \lambda\sigma_\epsilon^2$$

But for increases in c , the results are not entirely clear. For instance, the derivative of I with respect to c equals:

$$\frac{dI}{dc} = \frac{(\lambda b \mu_3 / 2) + (\lambda^2 / 4) [\mu_3 \sigma_\epsilon^2 + 2p(\sigma_\epsilon^2)^2]}{[2b(b+c) + \lambda c \sigma_\epsilon^2]^2} \begin{matrix} \leq 0 \\ \geq 0 \end{matrix} \quad (9.51)$$

When the distribution is symmetric (and $\mu_3 = 0$), then dI/dc is positive. When the distribution of ϵ 's is negatively skewed, then dI/dc can be negative.

An important question, however, is: What is the limit of I as c increases? The answer is: If c becomes very large, then ex post adjustments become more costly and less likely to deviate from the ex ante solution given in (9.37). This is the result of our model as c approaches infinity:

$$\lim_{c \rightarrow \infty} I = \frac{p}{2b + \lambda\sigma_\epsilon^2}$$

Thus I approaches the ex ante solution to the Sandmo model as c becomes large and adjustments from I costly. However, this limit may be approached either from below or from above, since the derivative in (9.51) does not change sign with increases in c .

On the other hand, when c becomes small or approaches zero, risk costs are eliminated ($\lambda = 0$) and the solution for I (on average) is:

$$\lim_{c \rightarrow 0} I = \frac{pb - (\lambda/4)(\mu_3 + 2p\sigma_c^2)}{2b^2} = \frac{p}{2b}$$

which is the average solution to the average output of the firm whose decisions are made ex post in the face of certainty [see Eq.(9.40)]. These results are described graphically in Fig. 9.3.

CONCLUDING COMMENTS

Input and output adjustments as responses to risk are closely linked. For normal inputs, those whose use decreases (increases) as risk increases (decreases), risk affects their levels in the same way that risk affects output decisions. That is, higher risk reduces the levels of input use and output. But even for normal inputs, some sources of risk may alter relative factor demands from their least-cost combinations derived under certainty.

If the risk affects two inputs in the same manner, the factor combinations relative to expected prices are the same as those of least-cost combinations derived under certainty. On the other hand, quality of input risk or input price risk which discriminates between inputs may significantly alter relative factor demands from their least-cost combinations.

One response to risk when the firm has flexible choices for output (and input) levels and when it expects new information is to hold inventories. As the cost of holding inventories or adjusting output from its ex ante planned level increases, flexibility is reduced, and in the limit the firm's solution is that of Sandmo's competitive firm model without flexibility. On the other hand, with perfect flexibility, the solution is the same as the price instability solution discussed by Shalit et al. In between complete flexibility and no flexibility, the inventory level may be greater than or less than the ex ante level of planned output.

The inventory level is ambiguous relative to the ex ante output solution. First, it may be ambiguous even if adjustment costs are symmetric, because diminishing marginal utility suggests that shortages reduce expected utility more than surpluses. On the other hand, costs of production increase at an increasing rate, so that overproduction is less likely than output reductions. Finally, skewness may dominate both these results. Once again we find that unambiguous results are harder to find in the world of risk.

ENDNOTES

1. Batra and Ullah concluded (p. 547) that "the risk-averse firm utilizes smaller quantities of inputs . . . than a firm operating under certainty." This statement, as

Hartman (1975) showed, claims too much. Only if the inputs are normal (Hartman, 1976), so that both increase (decrease) along the least-cost combination line as output is increased (decreased), does Batra and Ullah's claim hold. This condition is guaranteed in our example by the restriction placed on the derivative associated with $q(L, K)$, namely, that $q_L, q_K, q_{KL} > 0$ and $q_{LL}, q_{KK} < 0$.

2. Pope and Kramer also point out that risk-reducing inputs may not result in the cost-minimizing solution.
3. This section is considerably more difficult than earlier sections and can be skipped without loss of continuity.
4. See Shalit, et al. for a more detailed discussion of this point.

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