Ultralight cellular composite materials with architected geometrical structure

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A B S T R A C T

A novel computational approach is presented to predict the overall hyperelastic properties of ultralight cubic cellular lattices made of brittle carbon fiber-reinforced polymer composites, such as the ones recently fabricated at the MIT Media Lab-Center for Bits and Atoms. The repetitive unit cell (RUC) approach is employed to model the fabricated cellular micro-lattices. Each member of the cellular structure is modeled using only one finite beam element with 12 degrees of freedom, and the nonlinear coupling of axial, bidirectional-bending, and torsional deformations is studied for each 3D spatial beam element. Since the cellular composite material is fabricated via the assemblage of building blocks by mechanical interlocking connections, we utilize the standardized Ramberg-Osgood function for the moment-rotation relation at the ends of adjacent members to enable tuning the appropriate flexibility for connections between two extreme limits of pin-jointed or rigid-jointed connections. The mixed variational functional in the updated Lagrangian co-rotational reference frame is obtained to derive explicitly the stiffness matrix. Then, we use newly proposed homotopy methods to solve the algebraic equations.

1. Introduction

Composite materials, by offering weight-efficient structures for desirable strength and stiffness have been widely used for the engineered systems such as aerospace structures [1,2]. For example, Refs. [3,4] designed efficient structures for aircraft fuselage applications using high-performance composite components. On the other hand, three-dimensional (3D) cellular architected materials have offered specific mechanical properties (stiffness, strength, toughness and energy absorption), and the emergence of novel manufacturing techniques has also enabled their fabrication. Researchers have found a variety of engineering applications for these cellular structures with lattice truss topologies including lightweight, compact structural heat exchangers [5] and blast resistant sandwich panels [6,7]. Moreover, the actuation of members of such cellular materials enables them to deform arbitrarily, which makes them suitable for applications such as shape morphing, along with “smart-structural” concepts and active control [8,9]. Consequently, these observations have opened up research fields to build cellular materials using fiber-reinforced composites.

The application of fiber reinforced composites to build cellular materials with open-cell lattice topologies has introduced mechanical properties filling gaps between the existing materials and the unattainable materials in the low-density region of the Ashby’s chart [10–14]. Topologically structured cellular materials show mechanical properties which scale with the relative density of the open-cell lattices [7]. This scaling performance, which depends on the geometry of the nonstochastic lattice-based materials, follows a proportional law \( E \propto \rho \) for ideal stretch-dominated materials with \( (E \) being the initial elastic tangent modulus of the nonlinear bulk stress-strain curve of the lattice, and \( \rho \) being the open-cell lattice density) high coordination numbers [15–17], a quadratic law \( E \propto \rho^2 \) for bent-dominated cellular structures as well as carbon-based open-cell stochastic foams including carbon microtube aerographite and graphene cork [18–20], and a cubic scaling law \( E \propto \rho^3 \) for aerogels and aerogel composites [21–23]. Recently, Cheung and Gershenfeld [24] extended stretch-dominated carbon-fiber-reinforced cellular composites to the ultralight regime with a scaling relation \( E \propto \rho^{1.5} \).

Cheung and Gershenfeld [24,25] fabricated cross-shaped building blocks from carbon-fiber-reinforced composite materials and reversibly connected them, like chains, to form volume-filling cellular structures as shown in Fig. 1. Building blocks were constituted of four conjoined strut members to one locally central node. Strut members were integrating unidirectional carbon-fiber beams and looped carbon-fiber circular holes. Then, as displayed in Fig. 1(a) building blocks were assembled using shear clips through the four coincident connection holes such that cubic lattices of vertex-connected octahedrons, called cuboct,
were introduced as shown in Fig. 1(b). Cheung and Gershenfeld [24] examined lattice members with square cross section of width \( t \) and length \( l \) which is illustrated in Fig. 1(c). They tested only cellular lattices with slender strut members, the thickness-to-length ratio of which was smaller than 0.1 (\( \phi = t/l < 0.1 \)) [24]. The current study focuses on analyzing the overall hyperelastic mechanical properties of these recently fabricated cellular composite materials from a new and novel computational approach. Since building blocks tile space to generate a cellular structure, we consider connections to be flexible. To this end, nonlinear rotational springs are introduced at each of the member nodes.

We use the repetitive unit cell (RUC) approach to mimic the cuboct cellular structure. We seek to model each strut member of the cellular structure as a single nonlinear spatial beam finite element with 12 degrees of freedom (DOF). This will not only enable a very efficient homogenization of an RUC of the cellular material; but also the direct numerical simulation of a large macro cellular structure with an arbitrarily large number of members. By considering the frame-like behavior for the architected cellular composite material, the nonlinear coupling of axial, bidirectional-bending, and torsional deformations is studied for the large deformation analysis of the composite strut members in an updated Lagrangian co-rotational reference frame, for the first time in the literature. Hasanyan and Waas have recently employed higher order theories to derive continuum descriptions for truss and lattice type structures [26]. One of their focus was on the hexagonally packed circular celled honeycomb spanning three different length scales, the global length (macroscale), the intermediate length scale (mesoscale), and the microscale. In order to determine the correct closed form expressions of the homogenized constitutive relationship, they represented the cellular material by a representative volume element (RVE) in the mesoscale [26].

As mentioned earlier, Cheung and Gershenfeld [24] explained that their fabricated cellular composites with \( \phi < 0.1 \) are stretch-dominated lattice-based materials. In the current methodology, we assume frame-like behavior for the cellular structure, consider the fully nonlinear twisting-bending-stretching coupling for each strut member, and also study whether the response of cellular material is frame-like or truss-like for \( 0.01 < \phi < 0.2 \). The anisotropic property of each of the carbon-fiber-reinforced composite members is included by considering a transversely isotropic constitutive relation between the components of the second Piola-Kirchhoff stress tensor and the Green-Lagrang strain tensor in the co-rotational reference frame. Since the cellular composite material is fabricated by an assemblage of identical building blocks using shear clips, we account for the effects of flexible connections on the behavior of the material. What usually describes the behavior of the flexible connections is the moment-rotation relation at joints in such a way that the slope of this curve determines the instantaneous rotational rigidity of the connections. Therefore, an essential step in representing the flexible behavior of the connections is the consideration of an appropriate moment-rotation relation at the joints. Regarding the amount of forces in the connections, linear or nonlinear rotational springs can be incorporated. For example, when the magnitudes of moments are not small at the nodes, the structure may show different flexibilities with respect to the various loading conditions [27,28]. Many models consisting of bilinear or trilinear functions [29], polynomial [30], exponential [27], and Ramberg-Osgood functions [31] have been proposed to approximate the behavior of the nonlinear flexible connections. The standardized Ramberg-Osgood function [31] for the moment-rotation relation employed by Ang and Morris [32] not only expresses efficiently and accurately the nonlinear behavior of the connection based on only three parameters and guarantees a positive derivative, but also accounts for the additional moment at the connection due to the \( P-\Delta \) effect [32]. Using the Ramberg-Osgood function, Shi and Atluri [28] proposed a scheme to determine instantaneously the rigidity of the connections and showed the convenience and accuracy of the scheme for both static and dynamic analyses of space frames. In this study, we employ their proposed methodology in conjunction with an RUC approach to consider the effect of flexible connections on the overall mechanical properties of the cellular composite materials. It is noteworthy to mention that, in contrast to the previous work by Shi and Atluri [28] which used only the standard linear stiffness matrix, based on complementary energy, in the co-rotational frame for the tangent stiffness matrix, the current work derives explicitly the tangent stiffness matrix under the considerations of the full nonlinear couplings of axial, torsional, and bidirectional-bending deformations.

In the present study, we utilize a mixed variational principle [33], based on the assumptions of local trial functions for the generalized stress-resultants as well as deformations in each cellular member in the updated Lagrangian co-rotational reference-frame, to derive explicitly the tangent stiffness matrix of each member which undergoes nonlinear deformations, while also accounting for the nonlinear flexible connections. Softi et al. [34] have recently formulated the weak form of the equilibrium static equation and boundary conditions in the laminated composite beam and derived explicitly the tangent stiffness matrix under mechanical and hygrothermal loads. Cai et al. [35] also calculated the explicit form of the tangent stiffness matrix for space frames using a mixed variational principle [33]. In contrast to their study which considered only a very few macro members, the present analysis is able to mimic cellular composite materials with RUCs consisting of an arbitrarily large number of strut members. On the other hand, Cai et al. [35] employed Newton-type algorithm to solve the incremental tangent stiffness equations for only a few macro members. Newton-Raphson and

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**Fig. 1.** (a) Cross-shaped building blocks, (b) carbon fiber-reinforced polymer composite cuboct lattice recently fabricated at MIT Media Lab-Center for Bits and Atoms by Cheung and Gershenfeld [24], and (c) strut member geometry.
arc-length methods appear to fail in the current study due to the fact that buckling occurs simultaneously or sequentially in many hundreds of members of the lattice undergoing large strains. As an alternative to Newton-Raphson and arc-length methods, we use in the present paper the recently developed iterative-type Newton homotopy methods to solve the tangent-stiffness equations when buckling may occur in hundreds of members simultaneously or sequentially. Liu et al. [36–38] and Elghobary et al. [39] have recently developed homotopy methods, which transform the original nonlinear algebraic equations into an equivalent system of ODEs. Employing this method, the equilibrated state of the lattice structure is calculated by iteratively solving the algebraic equations without the inversion of the Jacobian (tangent stiffness) matrix. Thus, no arc-length methods are needed in the present study, at singular points. We have currently applied homotopy methods to analyze the large deformation elastic-plastic behavior of micro-architected cellular metallic materials [40,41].

In addition, an effort is devoted in the present paper, to study how the strain affects the tangent elastic modulus of the ultralight cellular composite. The strain energy density function and the tangent elastic moduli are calculated for cuboct cellular lattices with $0.01 < \phi < 0.2$ undergoing large strain levels of about 15%. We obtain that the tangent elastic modulus is a nonlinearly strain-dependent function and, thus, the strain energy cannot be represented by a simple quadratic function of the strain. Hence, we employ the Ogden’s hyperelasticity model [42,43] to fit the computed strain energies and to describe the mechanical behaviors of the architected material. Material parameters in such Ogden’s hyperelasticity models are calculated using the Levenberg-Marquardt nonlinear least squares optimization algorithm adapted by Twizell and Ogden [44].

The outline of the paper is as follows. First, a new computational approach is developed in Section 2 to predict the overall mechanical properties of cellular composite materials, such as the one recently fabricated at the MIT Media Lab-Center for Bits and Atoms [24]. The validity of the presented methodology is first verified in Section 3. To this end, two different problems are solved: (1) the effect of joint flexibility on the behavior of T-shaped frame and (2) the behavior of a Toggl with nonlinear rotational supports. Then, the calculated results are compared with the corresponding results given in the literature. Section 4 is devoted to solving the computational models of the cellular composite materials with $0.01 < \phi < 0.2$. The initial tangent elastic moduli of the bulk lattice, as calculated from the current approach are compared with the experimentally measured moduli by Cheung and Gershenfeld [24]. Furthermore, we model various cellular composites with different thickness-to-length ratios. Then, their calculated moduli are plotted on a map of initial tangent elastic moduli versus member aspect ratio in order to gauge the performance of these materials in terms of their geometry. In fact, the proposed methodology enables us to predict the relative initial tangent elastic modulus of the cellular composite material as a function of the member aspect ratio. Moreover, we study the variation of the strain energy density function and the tangent elastic moduli versus strain for cuboct cellular composites with $0.01 < \phi < 0.2$ and investigate if they obey hyperelastic constitutive models. The Ogden’s hyperelastic model is fitted to the calculated strain energy density functions for both pin-jointed and rigid-jointed lattices and, then, the corresponding material parameters are calculated using the Levenberg-Marquardt nonlinear least squares optimization algorithm. Finally, the paper is summarized in Section 5. Appendices A, B, C, and D follow.

2. The present computational approach

In this section, the fundamental concepts governing the current approach are explained. Modeling of each cellular composite member by using only a single nonlinear spatial 3D beam element with 12 DOF is discussed in Section 2.1. First, the member generalized strains and stresses are derived in the updated Lagrangian co-rotational reference by considering the nonlinear coupling of axial, bidirectional-bending, and torsional deformations. The member generalized stresses are calculated to account for the effect of the anisotropic properties of the cellular composite members. Next, the mixed variational functional corresponding to an RUC consisting of $N$ spatial beam finite elements is obtained on the basis of the nodal generalized displacement vectors and the member generalized stresses. Then, the trial functions considered for the generalized stress and displacement fields are given in Section 2.1. The tangent stiffness matrix is derived explicitly in Section 2.2 utilizing the mixed variational functional in the co-rotational updated Lagrangian reference frame, while also accounting for the nonlinear flexible connections. The algorithm of solution, which is the Newton homotopy method, is also expressed in Section 2.3.

2.1. Single 3D spatial beam element corresponding to each member of the cellular composite

We propose to model each strut member of the cellular structure by using only one single nonlinear spatial 3D beam element with 6 DOF at each of the 2 nodes of the beam. The thickness-to-length ratio ($\phi = t/l$) of strut members plays a role on the truss-like or frame-like behavior of the cellular materials. Thus, we consider the nonlinear coupling of axial, torsional, and bidirectional-bending deformations for each beam element and study the response of the cellular architected material for $0.01 < \phi < 0.2$. Cheung and Gershenfeld [24,25] examined samples with $\phi < 0.1$ and reported stretch-dominated response for these samples.

A typical 3D spatial beam element of the cellular lattice with length $l$ and arbitrary cross section is shown in Fig. 2. The positions of the member spanning between nodes 1 and 2 are defined in a fixed global reference with axes $\overline{x}$ and the orthonormal basis vectors $\overline{e}_l(i = 1,2,3)$. The initial undeformed state of the member is also described in a local coordinate system with axes $\overline{x}_i$ and base vectors $\overline{e}_l(i = 1,2,3)$ as shown in Fig. 2. It is assumed that the nodes 1 and 2 of the member undergo arbitrarily large displacements. The local coordinates of the deformed member (current configuration) are determined by $x_i$ according to the orthonormal basis vectors $\overline{e}_l(i = 1,2,3)$, as illustrated in Fig. 2. It is also assumed that deformations in the current coordinate system are moderate, and the local axial variation of $u_{0l}$, $\partial u_{0l}/\partial x_i$ is small as compared to the transverse rotations of $\partial u_{0l}/\partial x_l(i = 2,3)$. The components of $u_{0l}$, $i = 1,2,3$, are the local displacements at the centroid of the member cross section along $x_i$-axis, $i = 1,2,3$, respectively.

We consider that the member shown in Fig. 2 is under torsion $T$

![Fig. 2. A typical 3D spatial beam element of the cellular structure spanning between nodes 1 and 2, including the nomenclature of coordinates: global state, $\overline{e}_l$, undeformed local state, $\overline{e}_l$, and deformed local state, $\overline{e}_l(i = 1,2,3)$.](image)
around \( x_i \)-axis and the bending moments \( M_k \) and \( M_l \) around \( x_j \)- and \( x_m \)-axes, respectively, which result accordingly in the angle of twist \( \tilde{e}_2 \) and the bend angles \( \tilde{\beta}_{20} \) and \( \tilde{\beta}_{20} \). The warping displacement due to the torsion \( T \), \( \tilde{\beta}_{20}(x_i,x_l) \) is considered to be independent of the variable \( x_k \), and the axial and the transverse displacements at the centroid of the member cross section \( (x_k = x_l = 0) \) are represented by \( u_0(x_i) \) along \( \tilde{e}_1 \), \( u_2(x_i) \) along \( \tilde{e}_2 \), and \( u_3(x_i) \) along \( \tilde{e}_3 \) directions, respectively. Therefore, using the normality assumption of the Bernoulli-Euler beam theory the following 3D displacement field is considered for each spatial beam element in the current coordinate system, \( e \),

\[
\begin{align*}
u_1(x_1,x_2,x_3) &= u_1(x_1,x_2) + u_0(x_1) - x_0 \frac{\partial u_1(x_1)}{\partial x_0} - x_2 \frac{\partial u_2(x_1)}{\partial x_2} - x_3 \frac{\partial u_3(x_1)}{\partial x_3}, \\
u_2(x_1,x_2,x_3) &= u_0(x_1) - \tilde{e}_2 x_0, \\
u_3(x_1,x_2,x_3) &= u_3(x_1) + \tilde{e}_3 x_0.
\end{align*}
\]

(1)

The member generalized strain, \( E \) is determined based on the Green-Lagrange strain components in the updated Lagrangian co-rotational reference as follows

\[
E = \begin{bmatrix} \varepsilon_{11}^0 \varepsilon_{22}^0 \varepsilon_{33}^0 \varepsilon_{12}^0 \varepsilon_{13}^0 \varepsilon_{23}^0 \end{bmatrix} = DE.
\]

(2)

The details how the Green-Lagrange strain components are calculated from the displacement field given in Eq. (1) are presented in Appendix A. The member generalized stress, \( s \) is calculated with respect to the member generalized strains, \( E \) based on the increments of the second Piola-Kirchhoff stress tensor, \( S^i \) as follows

\[
s = \begin{bmatrix} N_{13} & M_{22} & M_{32} \\ M_{22} & M_{33} & T \\ T & 0 & 0 \end{bmatrix} = DE.
\]

(3)

The calculation of the components of the second Piola-Kirchhoff stress tensor accounting for the anisotropic properties of the constituent composite material is also included in Appendix A. \( C_{ij} \) \( i,j = 1,2,\ldots,6 \) are the elastic tensor components of the constituent material in Voigt notation, \( A \) is the area of the cross section, \( I_1 \) and \( I_2 \) \( (i,j = 2,3) \) are the first and second moments of inertia of the cross section, respectively, \( I_1 = \int x^2 dA \), \( I_2 = \int x^4 dA \), \( I_3 = \int x^6 dA \), \( I_3 = \int x^6 dA \), and \( I_4 = \int (c_{1} x^2 + c_{2} x^4) dA \).

The mixed variational functional for an RUC consisting of \( N \) strut members (as shown in Fig. 2) is obtained in the co-rotational updated Lagrangian reference-frame, based on the incremental components of the second Piola-Kirchhoff stress tensor and those of the displacement field, as well as the pre-existing Cauchy generalized resultants, as below

\[
x_{k}^e = \sum_{m=1}^{N} \int_{I_1} \left( 1 - \frac{1}{2} g^{2} D^{2} s \right) dI + \int_{I_1} N_{13} \frac{1}{2} (u_{20} + u_{30}) dI, \]

(4)

where \( s = [N_{13} M_{22} M_{32} T^{0}]^{T} \) is the initial member generalized Cauchy stresses in the co-rotational reference coordinates \( e_{s} \), \( s = s + S = [N_{13} M_{22} M_{32} T^{0}]^{T} \) is the total member generalized stress in the coordinates \( e_{s} \), \( \mathbf{Q} \) is the nodal external generalized force vector in the global reference frame \( e_{q} \), and \( q \) is the incremental nodal generalized displacement vector in the coordinates \( e_{q} \). Computational details of Eq. (4) are presented in Appendix B.

From Eq. (4), it is found that only the squares of \( u_{20} \) and \( u_{30} \) appear in the mixed variational functional. Therefore, the trial functions for the displacement field of each cellular member are assumed such that \( u_{20} \) and \( u_{30} \) are linear in each member. Consequently, transverse rotations among the lattice member are related to the nodal ones via

\[
u_{0} = N_{0} a_{0} = \begin{bmatrix} 1 - \frac{x_{i}}{l} & 0 & \frac{x_{i}}{l} \\ 0 & 1 - \frac{x_{j}}{l} & 0 \\ \frac{x_{j}}{l} & 0 & 1 - \frac{x_{k}}{l} \end{bmatrix} \begin{bmatrix} d_{20} \\ d_{20} \\ d_{20} \end{bmatrix} = \begin{bmatrix} \beta_{20} \\ \beta_{20} \\ \beta_{20} \end{bmatrix},
\]

(5)

The nodal generalized displacement vector, \( a \) corresponding to each of the lattice members can be expressed in the updated Lagrangian co-rotational frame in terms of the displacement vectors at the member nodes, \( \nu(i = 1,2) \)

\[
\mathbf{a} = [a \ a^{T}],
\]

(6)

where,

\[
\nu = \begin{bmatrix} u_{10} & u_{20} & u_{30} & \beta_{20} & \beta_{20} \end{bmatrix}^{T}.
\]

(7)

The nodal generalized displacement vector of the member, \( a \) is related to the vector \( \mathbf{a} \) by

\[
\mathbf{a} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{T},
\]

(8)

The trial functions for the incremental second Piola-Kirchhoff generalized stress field of each cellular member are assumed such that the member generalized stress is determined as below

\[
s = \begin{bmatrix} N_{13} & M_{22} & M_{32} \\ M_{22} & M_{33} & T \\ T & 0 & 0 \end{bmatrix} = \begin{bmatrix} n_{m_{1}} & m_{1} & m_{2} & m_{3} \\ m_{1} & n_{m_{2}} & m_{3} & m_{4} \\ m_{2} & m_{3} & n_{m_{3}} & m_{4} \end{bmatrix} = P \beta.
\]

(9)

In the same way, the components of the initial member Cauchy generalized stress, \( s^{0} \) are determined

\[
s^{0} = P \beta^{0},
\]

(10)

in which,

\[
\beta^{0} = [n^{0} m_{1}^{0} m_{2}^{0} m_{3}^{0} m_{4}^{0}]^{T}.
\]

(11)

Subsequently, the incremental internal nodal force vector \( B \) for the strut member with nodes 1 and 2 at the ends (as depicted in Fig. 2) is presented by

\[
B = [N_{1} n_{m_{1}} m_{1} m_{2} N_{1} n_{m_{2}} m_{3} m_{4}]^{T},
\]

(12)

which relates to \( \beta \) via the following relation

\[
\mathbf{B} = \mathbf{Q} \beta.
\]

(13)

with

\[
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.
\]

(14)

The components of \( \nu_{m_{i}} \) correspond to the moments at the ith node, \( i = 1,2 \), along \( x_{j} \)-axis, \( j = 1,2,3 \), and those of \( N \) correspond to the axial force referring to the ith node.

2.2. Explicit derivation of tangent stiffness matrix accounting for nonlinear flexible connections

The use of the trial functions given in Section 2.1 into the Eq. (4), the mixed variational functional is rewritten in the co-rotational updated Lagrangian coordinates as follows.
\[ \mathbf{K} = G^2 \mathbf{H}^{-1} \mathbf{G} + \mathbf{K}_N, \]

where,

\[ \mathbf{H} = \int \left( \mathbf{P}^T \mathbf{C} \mathbf{P} \right) dl, \]
\[ G = \mathbf{N} \mathbf{\gamma}, \]
\[ \mathbf{K}_N = \mathbf{A}_{\text{int}} \int \mathbf{F} \mathbf{A}_{\text{int}} dl. \]

More details as how to calculate Eq. (15) and to derive the stiffness matrix, \( \mathbf{K} \), for the elastic large-deformation analysis of cellular composite with rigid connections is derived explicitly as

\[ \mathbf{K} = \mathbf{G}^2 \mathbf{H}^{-1} \mathbf{G} + \mathbf{K}_N. \]

The behavior of a nonlinear flexible connection can be modeled using a nonlinear rotational spring, the rotational rigidity of which corresponds to the slope of the moment-rotation curve. Therefore, a flexibly jointed lattice can be modeled as a collection of members connected with rotational springs at the ends of adjacent members. The moment-rotation relation based on the standardized Ramberg-Osgood function is described as

\[ \frac{\mathbf{S}}{\mathbf{\gamma}} = \mathbf{\mathcal{S}}_{\mathbf{M}} \left[ 1 + \left( \frac{\mathbf{S}}{\mathbf{\gamma}} \right)^{n} \right] \]

in which, \( \mathbf{\mathcal{S}}_{\mathbf{M}} \) is the rotational rigidity of the connection, \( \mathbf{S} \) is the relative rotation at the connection, and \( \mathbf{\gamma} \), \( \mathbf{\gamma}_0 \), \( \mathbf{\mathcal{S}}_{\mathbf{M}} \), and \( n \) are the three parameters describing the nonlinear behavior of the connection; \( n \) is a positive real number. Since the slope of the moment-rotation curve, \( \frac{d\mathbf{S}}{d\mathbf{\gamma}} \), corresponds to the instantaneous rotational rigidity of the connection denoted by \( S' \), it can be obtained by differentiating Eq. (23) with respect to the bending moment, \( M \),

\[ S' = \frac{d\mathbf{S}}{d\mathbf{\gamma}} = \frac{\mathbf{S}}{\mathbf{\gamma}} \left[ 1 + \left( \frac{\mathbf{S}}{\mathbf{\gamma}} \right)^n \right] \mathbf{M} > 0. \]

Consideration of the initial rigidity as \( S' \), the general form of Eq. (24) for loading cases (\( M \) \( d\mathbf{M} > 0 \)) becomes

\[ S' = \frac{S'}{1 + n \left( \frac{\mathbf{S}}{\mathbf{\gamma}} \right)^n}. \]

It is considered that the incremental nodal rotations, \( \Delta^\alpha \mathbf{\gamma}_0 \), at the ends of members in the co-rotational local coordinate are composed of \( \Delta\mathbf{\gamma}_0' \) and \( \Delta^\alpha \mathbf{\gamma} \), which are, respectively, due to the elastic behavior of the member and the contribution of the rotational spring. Here, \( \alpha = 1, 2 \) refers to the node, and \( i = 2, 3 \) refers to the direction along \( x_i \)-axis. The increments of the spring rotations are also expressed as

\[ \Delta^\alpha \mathbf{\gamma}_i = (-1)^i \Delta^\alpha \mathbf{M}_i \]

in which \( \mathbf{S} \) is the instantaneous rigidity of the spring defined by Eq. (29). Considering \( \Delta^\alpha \mathbf{M}_i \) as the variation of the incremental moment about \( x_i \)-axis at node \( \alpha \), \( \Delta^\alpha \mathbf{M}_i \), the effect of the rotational spring at the end of the member can be represented by its incremental energy as follows

\[ \int_0^1 (\Delta^\alpha \mathbf{M}_i \Delta^\alpha \mathbf{\gamma}_0) d\mathbf{x}_i = \int_0^1 (\Delta^\alpha \mathbf{M}_i \Delta^\alpha \mathbf{\gamma}_0) d\mathbf{x}_i + \Delta^\alpha \mathbf{M}_i \Delta^\alpha \mathbf{\gamma}_0, \]

\[ \int_0^1 (\Delta^\alpha \mathbf{M}_i \Delta^\alpha \mathbf{\gamma}_0) d\mathbf{x}_i = \int_0^1 (\Delta^\alpha \mathbf{M}_i \Delta^\alpha \mathbf{\gamma}_0) d\mathbf{x}_i + \Delta^\alpha \mathbf{M}_i \Delta^\alpha \mathbf{\gamma}_0. \]

Using the trial stress functions given in Eq. (9), the increments of the Cauchy stress resultants and stress couples in the co-rotational updated Lagrangian coordinate system become

\[ \Delta \mathbf{N}_{ij} = \Delta \mathbf{n}, \]
\[ \Delta \mathbf{M}_{ij} = \left( \begin{array}{c} 1 - \frac{x_i}{T} \\ \frac{1 - \frac{x_i}{T}}{T} \end{array} \right) \Delta \mathbf{m}_1 - \frac{x_i}{T} \Delta \mathbf{m}_2, \]
\[ \Delta \mathbf{T} = \Delta \mathbf{m}_3. \]

Substituting Eqs. (29–32) and (26) into the Eqs. (27) and (28) and considering the incremental forms for the energies due to the incremental uniaxial stretching, \( \Delta \mathbf{H} (\int_0^1 (\Delta \mathbf{M} \Delta \mathbf{\gamma}) d\mathbf{x}_i) \) and due to the incremental twisting angle, \( \Delta \mathbf{H} (\int_0^1 (\Delta \mathbf{M} \Delta \mathbf{\gamma}) d\mathbf{x}_i) \), we obtain the following relation

\[ \Delta \mathbf{H} = \mathbf{H}^{-1} \Delta \mathbf{a}. \]

for an arbitrary cross section, and

\[ \mathbf{H}^{-1} = \left[ \begin{array}{c} \mathbf{H}^{-1} \mathbf{a} \\ 0 \end{array} \right], \]

\[ \mathbf{H}^{-1} = \left[ \begin{array}{c} \mathbf{H}^{-1} \mathbf{a} \\ 0 \end{array} \right]. \]

for a symmetrical cross section. The details for the calculation of \( \mathbf{H}^{-1} \), \( \mathbf{H}^{-1} \), and \( \mathbf{H}^{-1} \), are provided in Appendix D. Consequently, to obtain the stiffness matrix accounting for the nonlinear flexible connections, \( \mathbf{H}^{-1} \) in Eq. (19) must be replaced by the matrix \( \mathbf{H}^{-1} \) given in Eq. (34) for arbitrary cross sections or Eq. (35) for symmetrical cross sections. Therefore, the explicit expression for the stiffness matrix, \( \mathbf{K} \), of the flexibly connected cellular composite undergoing large deformations is derived

\[ \mathbf{K} = \mathbf{G}^2 \mathbf{H}^{-1} \mathbf{G} + \mathbf{K}_N. \]

2.3. Newton homotopy method

After deriving the Jacobian matrix (stiffness matrix) the system of algebraic equations, \( F(X) = 0 \), should be solved, determining the vector \( X \in \mathbb{R}^n \) which is the displacement field of the equilibrated structure. Algorithms such as Newton-type iterative methods require to
invert the Jacobian matrix, which becomes problematic for the study of complex problems when the Jacobian is nearly singular or severely ill-conditioned. Researchers over the past three decades have enhanced the Newton-Raphson methods and introduced other methodologies such as arc-length which are commonly used in commercial off-the-shelf software such as ABAQUS. Liu et al. [36,37] have newly proposed homotopy methods to avoid the inversion of the Jacobian matrix, which are simpler to use when the Jacobian is nearly singular.

In the case of complex problems where buckling happens in a large number of strut members of cellular materials, Newton-type methods suffer from the accuracy of inverting the Jacobian matrix. In fact, these algorithms fail to converge by showing oscillatory non-convergent behavior for such complex problems. While homotopy methods, by introducing the best descent direction in searching the solution vector, provide convergent solutions with the required accuracy. On the other hand, in contrast to Newton-type algorithms which are sensitive to the initial guess, the scalar homotopy methods employed in the current study are insensitive to the guess of the initial solution vector. Moreover, their convergence speed has been validated in the literature [38,39].

Liu and Atluri [36] developed a fictitious time integration method in the form of a system of nonlinear first order ODEs as

\[ X(t) = -\frac{v}{q(t)} F(X), \]

in which \( v \) is a nonzero parameter and \( q(t) \) is a monotonically increasing function of \( t \). In their proposed approach, the term \( v/q(t) \) results in speeding up the convergence. Later, Liu et al. [37] proposed a scalar homotopy method transforming the original nonlinear algebraic equations into an equivalent system of ODEs. Considering a fictitious time function \( Q(t), t \in [0, \infty) \), they presented a generalized scalar Newton homotopy function [37],

\[ h(X, t) = \frac{1}{2} |F(X)|^2 - \frac{1}{2Q(t)} |F(X_0)|^2 = 0, \]

where \( t \) is the fictitious time and \( Q(t) \) is a monotonically increasing function of \( t \) which satisfies \( Q(0) = 1 \) and \( Q(\infty) \to \infty \). Multiplying both sides of Eq. (39) by \( Q(t) \), differentiating with respect to \( t \), and considering \( X = X(t) \) result in

\[ \frac{1}{2} \dot{Q}(t) |F(X)|^2 + Q(t) \dot{F}^T B F + \dot{X} = 0, \]

where \( B \) is the \( n \times n \) Jacobian matrix defined by \( B_{ij} = \partial F_i / \partial x_j \) \((i, j = 1, \cdots, n)\). To transform the original nonlinear algebraic equations to ODEs, \( X \) is considered as \( \dot{X} = s \dot{u} \) in which \( s \) is a scalar, and there are a variety of choices for the driving vector \( u \) such as \( X = -\frac{2B^T F}{\dot{Q}(t)} \). Finally, the general form equation for the scalar Newton homotopy methods is obtained

\[ \dot{X} = -\frac{\dot{Q}(t) |F(X)|^2}{2Q(t)} F^T B u. \]

Therefore, the original algebraic equation is solved by numerically integrating the equivalent dynamical ODEs or discretizing it using the forward Euler method and obtaining the general form of the iterative Newton homotopy methods as

\[ X(t + \Delta t) = X(t) - (1 - \gamma) \frac{F^T B u}{\|Bu\|^2} u, \]

where, \(-1 < \gamma < 1\).

3. Verification

To verify the efficiency and accuracy of the current methodology, the response of the T-shaped frame and the toggle problem with nonlinear flexible connections are calculated and compared with other solutions available in the literature.

3.1. Effect of joint flexibility on the behavior of T-shaped frame

In this section, a T-shaped frame with nonlinear flexible connections is studied. The frame is first subjected to the load sequence 1 as shown in Fig. 3(a) and permitted to relax. Then, the relaxed frame after the first sequence of loading is subjected to the second sequence of loading as illustrated in Fig. 3(b). At the load sequence 2, the amount of the point load \( P \) is changed, and its effect on the final deformation of the relaxed structure is studied. The geometry of the frame members and the site of the flexible connections are shown in Fig. 3. The Young’s modulus of frame members is considered to be \( E = 3 \times 10^7 (lb/in^2) \). The nonlinear behavior of the flexible connections is determined by the moment-rotation relation given in Fig. 4. Using the current methodology, the variation of the rotation-to-vertical displacement ratio, \( \theta/d \) at the point \( A \) corresponding to the point load \( P \) at the point \( A \) is calculated for both flexible and rigid connections and plotted in Fig. 5. It is obvious that further deflection happens for the frame with flexible connections as compared to the case with rigid connections. For the purpose of comparison, the corresponding results for both rigid and flexible connections given by Shi and Atluri [28] are also included in Fig. 5. As it is seen, there is a good agreement between the results calculated from the present computational approach and those presented in [28].

3.2. Behavior of a toggle with nonlinear rotational supports

In this section, the effect of the nonlinear connection flexibility on the behavior of a toggle, plotted in Fig. 6, is examined. To this end, connections at the supports of the toggle are considered to be flexible,
of the toggle is studied. The mechanical properties of the toggle members are considered to be $EI = 9.27 \times 10^6$ psi and $EA = 1.885 \times 10^6$ lb. The deflection of the vertex versus the point load is depicted in Fig. 7. For the sake of comparison, the behavior of the toggle with rigid supports is also included in Fig. 7. As it is seen, the load–deflection behavior of the toggle with flexible supports shows considerably more deflections in comparison with that of the toggle with rigid supports. In addition, in Fig. 7 both calculated results corresponding to the flexible and rigid connections are compared with the numerical solutions given by Chen and Lui [27] obtained using their developed computer program (PFAPC) [45]. It is observed that both sets of results compare favorably with the corresponding results presented in [27,45].

4. Cuboct carbon-fiber–reinforced cellular composite material

In this section, we present a computational approach to predict the bulk mechanical properties of the cuboct cellular lattice, consisting of individual members made of carbon fiber-reinforced polymer composite, recently fabricated at MIT Media Lab-Center for Bits and Atoms [24]. The fabrication process of the cuboct cellular composite has been explained in the Introduction. Since the 3D open-cell composite material is generated by reversible assemblage of building blocks [24], it is worthwhile to first consider that the lattice members are flexibly connected and then study how the thickness-to-length ratio of the strut members affects the flexibility of the lattice. Cheung and Gershenfeld [24] tested samples with the aspect ratio about 0.05 ($\phi = 1/l \sim 0.05$) and considered two different Young's moduli for the constituent composite material, $E_c = 37$ and 71 GPa. They defined the relative density ($\rho$) of 3D cellular lattice as the summation of the relative density of the strut members ($\rho_s / \rho_0 \propto C_{ts} (1/l)^{3}$) and that of the connections ($\rho_c / \rho_0 \propto C_{tc} (1/l)^{5}$). Where, $\rho$ is the mass of the lattice divided by the total bounding volume, and $\rho_c$ is the mass of the lattice divided by only the volume of the constituent solid material [24]. The connection and the strut members contribution constants ($C_{tc}$ and $C_{ts}$), which are determined based on the lattice geometry, were considered as $3/(2\sqrt{2})$ and $3\sqrt{2}$, respectively [24].

To study the experimental observations reported in Ref. [24], we model cuboct cellular composites with various thickness-to-length ratios, 0.01 $\leq \phi < 0.2$. We also examine three different $E_c$ ($E_c = E_c1, 1.5E_c$, and $2E_c$ where $E_c = 37$ GPa) for each value of $\phi$. Then, the flexibility of connections of the cellular composite simulated by a set of $\phi$ and $E_c$ is changed from rigid to pinned. Finally, we perform a complete nonlinear analysis up to a large overall strain level of about 15%, to calculate the tangent modulus of elasticity of the cuboct composites corresponding to each set of $\phi$, $E_c$, and joint rigidity. Furthermore, the initial tangent elastic modulus and the strain energy of the low-mass cellular composites are studied with respect to the variation of loading. Subsequently, the hyperelastic response of the cellular composite materials is investigated using the Ogden model, and the fitted material parameters are calculated.

4.1. Truss-like and frame-like behaviors of cuboct cellular composites: comparison with experimental results

Cheung and Gershenfeld [24] examined $4 \times 4 \times 4$ cuboct samples with the characteristic dimension of the repeating cell defined as pitch, $d = 5.1$ cm [24]. Since $d$ is proportional to the length of the lattice member, $l$, it is calculated to be $0.9$ cm ($d = 4\sqrt{2} l$ for a $4 \times 4 \times 4$ cuboct sample). Then, employing the amounts of $\rho$ and $\rho_c$ given in Ref. [25] into the relative density equation, $\rho / \rho_0 = 3 \phi^{\phi - 1} + 3 \sqrt{2} \phi^{\phi - 1}$, the amounts of $\phi = 1/l$ corresponding to the values of the open-cell lattice density, $\rho$, are calculated, resulting in the calculation of the width of the square cross section. We use the above-mentioned approach to calculate the length and the width of the lattice members for each value of the lattice density and, then, simulate $1 \times 1 \times 4$ cuboct samples with periodic
boundary conditions along x- and y-axes. As shown in Fig. 8, the RUC is composed of 21 nodes and 48 elements. To calculate the elastic modulus of the ultralight cellular composites, nodes on both top and bottom sides of RUC are loaded under compression.

As it is well-known, the relative initial tangent modulus of elasticity of cellular materials \( (E/E_c) \) depends on their relative density \( (\rho/\rho_c) \) \([7]\). For the samples modeled using the present approach, the relative density is proportional to the square of the member aspect ratio, \( \rho/\rho_c = 3\sqrt{2} \sqrt{\phi t} \). Therefore, we first seek to show that the relative elastic moduli of cubic cellular composites depend only on the thickness-to-length ratio of strut members. To this end, cellular composite structures with different length, \( l \) and width, \( w \) of strut members but the same member aspect ratio, \( \phi = t/l \) are modeled, and their overall initial tangent elastic modulus, \( E \) for different Young’s modulus of the constituent composite material, \( E_c \) are calculated and tabulated in Table 1.

As is found from Table 1, similar values of \( \phi \) lead to similar initial tangent elastic moduli, \( E/E_c \).

Table 1
Initial tangent elastic moduli for cellular composites with different member aspect ratios, computed using the current computational framework.

<table>
<thead>
<tr>
<th>( t ) (cm)</th>
<th>( l ) (cm)</th>
<th>( \phi = t/l )</th>
<th>( E/E_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1657</td>
<td>5.6870</td>
<td>2.9137 \times 10^{-2}</td>
<td>3.9905 \times 10^{-4}</td>
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<td>0.1485</td>
<td>5.1000</td>
<td>2.9118 \times 10^{-2}</td>
<td>3.9878 \times 10^{-4}</td>
</tr>
<tr>
<td>0.1657</td>
<td>3.6060</td>
<td>4.5951 \times 10^{-2}</td>
<td>9.9465 \times 10^{-4}</td>
</tr>
<tr>
<td>0.2409</td>
<td>5.1000</td>
<td>4.7225 \times 10^{-2}</td>
<td>9.9732 \times 10^{-4}</td>
</tr>
<tr>
<td>0.1657</td>
<td>4.0000</td>
<td>4.1425 \times 10^{-2}</td>
<td>8.0749 \times 10^{-4}</td>
</tr>
<tr>
<td>0.2113</td>
<td>5.1000</td>
<td>4.1425 \times 10^{-2}</td>
<td>8.0808 \times 10^{-4}</td>
</tr>
<tr>
<td>0.1657</td>
<td>5.5000</td>
<td>3.0127 \times 10^{-2}</td>
<td>4.2616 \times 10^{-4}</td>
</tr>
<tr>
<td>0.1536</td>
<td>5.1000</td>
<td>3.0127 \times 10^{-2}</td>
<td>4.2543 \times 10^{-4}</td>
</tr>
<tr>
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<td>6.0000</td>
<td>2.7617 \times 10^{-2}</td>
<td>3.5843 \times 10^{-4}</td>
</tr>
<tr>
<td>0.1408</td>
<td>5.1000</td>
<td>2.7617 \times 10^{-2}</td>
<td>3.5692 \times 10^{-4}</td>
</tr>
</tbody>
</table>

Next, the initial tangent elastic moduli of the ultralight cellular composite materials examined by Cheung and Gershenfeld \([24]\) with \( t/l \sim 0.05 \) and \( E_c = 37 \) GPa are calculated using the current computational methodology and compared with the corresponding experimental results in Table 2. As is seen from Table 2, there is a reasonable agreement between computational and experimental results. Moreover, it is found from Table 2 that there is no notable differences between our calculated initial tangent elastic moduli for the composite lattices with either pinned or fixed connections. This observation verifies the truss-like behavior of these lattice-based composites with \( \phi \sim 0.05 \), which is in accordance with the stretch-dominated behavior observed experimentally in Ref. \([24]\).

To investigate how the thickness-to-length ratio (\( \phi \)) of the cuboct lattice member affects the truss-like or frame-like behavior of the composite lattice, we model the cuboct cellular composites with \( \phi = 0.05, 0.08, 0.1, 0.15 \), and different Young’s moduli of the constituent composite material, \( E_c = 37, 55.5, 74 \) GPa for both rigid and pinned connections. To this end, \( 2 \times 2 \times 2 \) RUCs consisting of 36 nodes and 96 elements with periodic boundary conditions along \( x- \) and \( y- \) axes are considered. To calculate the overall initial tangent elastic modulus of the cellular composite material, the tensile loading is applied at the nodes on both top and bottom faces of the sample as shown in Fig. 9.

The nonlinear large deformation analysis of cellular composites with \( \phi = 0.05, 0.08, 0.1 \) and 0.15 results in the trend moduli of elasticity given in Figs. 10–13, respectively. For each case of \( \phi \), three different \( E_c \)
(\(E_1\), 1.5\(E_1\), and 2\(E_1\)) are examined, and the corresponding results are included in Figs. 10–13(a–c), respectively. Note that, for each set of \(\phi\) and \(E_c\), both pinned and fixed connections are studied. From the responses of cellular composite structures exhibited in Fig. 10(a–c) and 11(a–c), the lattices with \(\phi < 0.08\) can be introduced as truss-like structures due to the negligible differences between the behavior of materials with fixed and pinned connections. Although Fig. 12(a–c) and 13(a–c) corresponding to \(\phi > 0.08\) show considerable differences between the responses of material with fixed and pinned connections. Therefore, cuboct cellular composites with 0.08 < \(\phi\) < 0.2 can be

**Fig. 10.** Change of the tangent elastic modulus versus tensile strain for the cuboct cellular composite with \(\phi = 0.05\) and (a) \(E_c = E_1\), (b) \(E_c = 1.5E_1\), (c) \(E_c = 2E_1\). Both pinned and fixed connections are examined for each set of \(\phi\) and \(E_c\).

**Fig. 11.** Change of the tangent elastic modulus versus tensile strain for the cuboct cellular composite with \(\phi = 0.08\) and (a) \(E_c = E_1\), (b) \(E_c = 1.5E_1\), (c) \(E_c = 2E_1\). Both pinned and fixed connections are examined for each set of \(\phi\) and \(E_c\).
Fig. 12. Change of the tangent elastic modulus versus tensile strain for the cuboct cellular composite with $\phi = 0.1$ and (a) $E_s = E_c$, (b) $E_s = 1.5E_c$, (c) $E_s = 2E_c$. Both pinned and fixed connections are examined for each set of $\phi$ and $E_c$. 

Fig. 13. Change of the tangent elastic modulus versus tensile strain for the cuboct cellular composite with $\phi = 0.15$ and (a) $E_s = E_c$, (b) $E_s = 1.5E_c$, (c) $E_s = 2E_c$. Both pinned and fixed connections are examined for each set of $\phi$ and $E_c$. 

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considered as frame-like structures. The change in the values of the tangent elastic moduli by changing connections from fixed to pinned for the strain levels 1% and 15% are also included in Figs. 10–13(a–c). As is expected, consideration of the pinned connections leads to the smaller values of the overall elastic moduli for the cellular composite in comparison with the rigid connections.

From Figs. 10–13(a–c), it is seen that the tangent elastic modulus increases by rising strain until a critical strain level and, then, drops by going beyond the critical level. This critical strain level is calculated to be nearly 8 ~ 9.5%. This phenomenon has also been observed in single wall carbon nanotubes (SWNT) with the critical strain about 4 ~ 6% for the armchair tubes and 10 ~ 20% for the zigzag tubes [46]. For more illustration, the stress-strain curve which has been used to calculate the tangent elastic modulus given in Fig. 13(a) is presented in Fig. 14. The initial tangent, the maximum tangent and the tangent at εz = 14.1296% to the stress-strain graph are also included in Fig. 14. To show that the slope of the stress-strain curve changes, segments of the graph corresponding to the strain levels εz = 8.6887% and εz = 14.1296% are shifted to the strain level εz = 0.1482% and compared in the inset of Fig. 14.

The current methodology enables us to postulate that the relative initial tangent elastic modulus of the cuboct cellular composites with ϕ < 0.08 varies as a function of the aspect ratio of the strut member. We model various cuboct cellular lattices with ϕ = 0.01, 0.015, 0.02, 0.025, 0.03, 0.08 and with different Young’s moduli of the constituent composite material Ec = Ec, 1.5Ec, 2Ec. Using the present approach, the overall initial tangent elastic moduli, E of the cellular composite materials with respect to E and Ec are calculated and plotted in Fig. 15. If the overall initial tangent elastic moduli corresponding to Ec = Ec, 1.5Ec, and 2Ec are, respectively, called as Ec, Ec, and Ec, then the average of the relative initial elastic modulus for each value of ϕ is calculated as \( \bar{E} = \frac{E_{c} + Ec + 1.5Ec + 2Ec}{2} \). Next, using the method of least squares, we can explain that the average of the relative initial tangent elastic moduli of cuboct cellular composite materials with truss-like behavior, varies as a function of t/l, by the relation, 

\[ E/Ec = 0.2839\phi + 0.4445\phi^2 + 0.0018\phi^3 - 0.00002. \]

Utilizing this regressive function, the sum of squared errors (SSE) for Ec/Ec, Ec/(1.5Ec), and Ec/(2Ec) are calculated as SSE = 1.6622 × 10⁻⁸, 9.1883 × 10⁻⁹, and 2.2487 × 10⁻¹⁰, respectively. Since for the simulated samples \( \rho/\rho_c = 3\phi^2 \), we find that the relative initial tangent elastic moduli \( (E/Ec) \) of the cuboct composite lattices with \( \phi < 0.08 \) depend on the relative density \( (\rho/\rho_c) \) by an exponent of \( \frac{1}{2} \), such that \( E/Ec \propto (\rho/\rho_c)^{\frac{1}{2}} \), which is in accordance with the findings of Cheung and Gershenfeld [24] when they considered that the relative density of the cuboct cellular composite is only proportional to the square of the beam aspect ratio, namely \( (\rho/\rho_c) \propto \phi^2 \).

It is worthwhile to mention that for the cuboct cellular composites with \( 0.08 < \phi < 0.2 \), showing the frame-like behavior, the rigidity of the connections can change, depending on the geometry of the connection holes (such as circular, trapezoidal, or hexagonal holes). The current methodology is capable to study different types of connection flexibilities by tuning an appropriate moment-rotation relation to the nonlinear and/or linear rotational springs in such a way that the slope of the moment-rotation curve presents the flexibility of connections. Subsequently, the regression equation explaining the relative initial tangent elastic modulus of the cellular composite material versus the aspect ratio of members can be obtained using the present method for the interval from fixed to pinned connections.

### 4.2. A hyperelastic description of cellular composite material with architected structure

From Figs. 10–13(a–c) presented in Section 4.1, it is found that the cuboct cellular composite material behaves as a hyperelastic solid. This observation is consistent with the experimental observations reported by Cheung and Gershenfeld [24]. Since the tangent modulus of elasticity (Figs. 10–13(a–c)) depends nonlinearly on the strain, the strain energy will not be a perfect quadratic function in \( \varepsilon_z \). This fact clearly indicates that the ultralight cellular composite is a hyperelastic material and cannot be simply described by linear elasticity. Subsequently, we propose to employ Ogden’s hyperelasticity model to fit a strain energy function and to describe the mechanical response of the cellular composite material. Levenberg-Marquardt nonlinear least squares optimization algorithm [44] is used to calculate the material parameters in Ogden’s hyperelasticity model [42,43].

To study how the rigidity of the strut connections changes the strain energy density, \( U \) of the cellular composite material, the stress work densities corresponding to two different samples with \( \phi = 0.05 \) and 0.15 are represented by the area under the stress curve, \( \sigma (\varepsilon) \) in the \( \sigma - \varepsilon \)-diagram. For both samples, \( \phi = 0.05 \) (truss-like cellular structure) and \( \phi = 0.15 \) (frame-like cellular structure), both extreme cases consisting of pin-jointed members as well as rigid-jointed members are studied, and
two different Young’s moduli of the constituent composite material with $E_1 = E_c = 37 \text{ GPa}$ and $E_2 = 2E_c = 74 \text{ GPa}$ are examined. The calculated strain energy function of the sample with $\phi = 0.05$ for both pinned and fixed connections is given in Fig. 16(a) corresponding to $E_c = E_1$ and in Fig. 16(b) corresponding to $E_c = 2E_1$. The strain energy function of the sample with $\phi = 0.15$ is also exhibited in Fig. 17(a) and (b) corresponding to $E_c = E_1$ and $E_c = 2E_1$, respectively.

In the following, we fit Ogden’s hyperelasticity model to the calculated strain energy density functions and obtain the material parameters. The strain energy function of an isotropic elastic solid depends on the strain only through the principal stretches $\lambda_1$, $\lambda_2$ and $\lambda_3$ [42,43]. The Ogden unconstrained form of strain-energy potential based on the principal stretches is as [43]

$$U = \sum_{i=1}^{k} \frac{\mu_i}{\alpha_i} (\lambda_i^{\alpha_i} + \lambda_i^{-\alpha_i} - 2) + \sum_{j=1}^{m} \frac{1}{d_j} (J-1)^{d_j},$$

(42)

where, $\mu_i$, $\alpha_i$, and $d_j$ are the material parameters, and $J = \lambda_1\lambda_2\lambda_3$ is the Jacobian of deformation. Twizell and Ogden [44] adapted the Levenberg-Marquardt nonlinear least squares optimization algorithm to calculate the material constants in Ogden’s strain energy density function.

It is assumed that $K$ different values are available for the strain energy density function ($U_k$, $k = 1, \ldots, K$) at the $K$ different principal stretches ($\lambda_{k}$, $k = 1, \ldots, K$). The least squares criterion requires that

$$S = \sum_{k=1}^{K} \left( U_k - \sum_{i=1}^{M} \frac{\mu_i}{\alpha_i} (\lambda_{k1}^{\alpha_i} + \lambda_{k2}^{\alpha_i} + \lambda_{k3}^{-\alpha_i}) - \sum_{j=1}^{M} \frac{1}{d_j} (J_{k1}-1)^{d_j} \right)^2 = \sum_{k=1}^{K} E_k^2,$$

(43)

be minimized, in which, $U$ is obtained by taking $M$ terms, and $E_k$ is the error in $U_k$. Twizell and Ogden [44] implemented the Levenberg-Marquardt algorithm to minimize $S$ and obtain the optimal values of the material parameters $\mu_i$, $\alpha_i$, and $d_j$ ($i = 1, \ldots, M$). If the material parameters are introduced by a vector $x$

$$x = [\mu_1, \alpha_1, d_1, \ldots, \mu_M, \alpha_M, d_M]^T,$$

(44)

the Levenberg-Marquardt algorithm calculates the vector $x^{(r+1)}$ at $(r+1)th$ iteration from the vector $x^{(r)}$ at $(r)th$ iteration using the equation

$$x^{(r+1)} = x^{(r)} - \left[ \left( P^{(r)} \right)^T P^{(r)} + \gamma^{(r)} I \right]^{-1} P^{(r)} U^{(r)}, \quad r = 0, 1, 2, \ldots,$$

(45)

in which, $\gamma^{(r)}(r = 0, 1, 2, \ldots)$ is an arbitrary parameter, $E_k = E_1, E_2, \ldots, E_K$ is the errors vector, and $I$ is the identity matrix of order $3M$. The matrix $P$ is the first derivatives of order $K \times 3M$, the elements of which at the $i$th iteration are calculated as [44]

$$P_{ki}^{(r)} = \frac{\partial E_k}{\partial x_i} \bigg|_{x = x^{(r)}}, \quad (k = 1, \ldots, K; i = 1, \ldots, 3M; r = 0, 1, 2, \ldots).$$

(46)

We employ the nonlinear least-squares curve fitting of *lsqnonlin* in the Optimization Toolbox of MATLAB [47], choose the Levenberg-Marquardt method as the option for the iterative algorithm, and do not impose upper and lower bounds on the material parameters which should be identified. The set of optimal parameters for $M = 12$ is calculated for both truss-like ($\phi = 0.05$) and frame-like ($\phi = 0.15$) cellular composites and tabulated in Tables 3 and 4, respectively; the mean squared errors (MSE) are also included. For the frame-like case, the optimal material parameters corresponding to both pin-jointed and rigid-jointed connections are calculated and compared in Table 4(a) and (b). Moreover, the effect of the Young’s modulus of the constituent composite material, $E_c$ on the Ogden’s material properties is studied by considering two different $E_c = E_1$ and $E_c = 2E_1$. For further illustration, the Ogden curves are also compared with the present calculated results in Figs. 18 and 19. Fig. 18(a) and (b) correspond to $\phi = 0.05$ with $E_c = E_1$ and $E_c = 2E_1$, respectively. Fig. 19 presents the results corresponding to the sample with $\phi = 0.15$, pin-jointed connections, $E_c = 2E_1$, and the largest MSE ($7.1275 \times 10^{-3}$ for $M = 1$ and $6.2775 \times 10^{-3}$ for $M = 2$) in comparison with other samples studied through this section. As it is seen from Figs. 18 and 19, good
Table 3
Set of optimal material parameters in Ogden’s strain energy density function for a cuboct cellular composite with truss-like behavior.

<table>
<thead>
<tr>
<th>( \phi = 0.05 )</th>
<th>( E_0 = E_t )</th>
<th>( E_0 = 2E_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = 1</td>
<td>M = 2</td>
<td>M = 1</td>
</tr>
<tr>
<td>( \mu_1 )</td>
<td>2.2567 \times 10^3</td>
<td>1.8120 \times 10^3</td>
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<td>( \alpha_1 )</td>
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<td>( d_1 )</td>
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<td>8.6147 \times 10^{-5}</td>
</tr>
<tr>
<td>( \phi_1 = \phi_2 )</td>
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<td>1.9810 \times 10^3</td>
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<td>( \phi_3 = \phi_4 )</td>
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<tr>
<td>( \phi_5 = \phi_6 )</td>
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<tr>
<td>MSE</td>
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<td>2.9385 \times 10^{-5}</td>
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Table 4
Set of optimal material parameters in Ogden’s strain energy density function for a cuboct cellular composite with frame-like behavior: (a) \( E_t = E_0 \) and (b) \( E_t = 2E_0 \).

(a) \( \phi = 0.15 \)

<table>
<thead>
<tr>
<th></th>
<th>Pin-jointed</th>
<th>Rigid-jointed</th>
</tr>
</thead>
<tbody>
<tr>
<td>M = 1</td>
<td>M = 2</td>
<td>M = 1</td>
</tr>
<tr>
<td>( \mu_1 )</td>
<td>2.0512 \times 10^4</td>
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<td>( \alpha_1 )</td>
<td>4.1411</td>
<td>4.1390</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>6.7295 \times 10^{-5}</td>
<td>5.7737 \times 10^{-5}</td>
</tr>
<tr>
<td>( \phi_1 )</td>
<td>-</td>
<td>1.4709 \times 10^5</td>
</tr>
<tr>
<td>( \phi_2 )</td>
<td>-</td>
<td>6.1741 \times 10^2</td>
</tr>
<tr>
<td>( \phi_3 )</td>
<td>-</td>
<td>-9.9537 \times 10^{-4}</td>
</tr>
<tr>
<td>MSE</td>
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<td>2.1829 \times 10^{-3}</td>
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</table>

(b) \( \phi = 0.15 \)

<table>
<thead>
<tr>
<th></th>
<th>Pin-jointed</th>
<th>Rigid-jointed</th>
</tr>
</thead>
<tbody>
<tr>
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<td>M = 2</td>
<td>M = 1</td>
</tr>
<tr>
<td>( \mu_1 )</td>
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<td>4.0713 \times 10^3</td>
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<tr>
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<td>4.1413</td>
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<tr>
<td>( d_1 )</td>
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<td>2.8010 \times 10^{-3}</td>
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<tr>
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<tr>
<td>( \phi_2 )</td>
<td>-</td>
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<tr>
<td>( \phi_3 )</td>
<td>-</td>
<td>-1.6113 \times 10^{-4}</td>
</tr>
<tr>
<td>MSE</td>
<td>7.1275 \times 10^{-3}</td>
<td>6.2775 \times 10^{-3}</td>
</tr>
</tbody>
</table>

correspondence is obtained between Ogden models and the present results.

5. Summary

Lightweight cellular composite materials recently fabricated at MIT Media Lab-Center for Bits and Atoms have been introduced as an elastic solid with an extremely large measured modulus for an ultralight material [24]. One of the novel applications of these architected cellular composites is in adaptive structures, where certain members behave as actuators and control precisely the global shape of the structure. In this paper, we proposed a nearly exact and highly efficient methodology to study low-mass composite systems with architected cellular structures composed of anisotropic members with nonlinear flexible connections. The validity of the current computational approach is studied by solving different problems, T-shaped and toggle frames with nonlinear rotational connections/supports, and comparing with the available results in the literature. Moreover, \( 1 \times 1 \times 4 \) RUCs with periodic boundary conditions along x- and y-directions are modeled to mimic the 4 \( \times \) 4 \( \times \) 4 cuboct cellular composites fabricated by Cheung and Gershenfeld [24]. Using the present approach, the overall tangent elastic moduli of \( 1 \times 1 \times 4 \) RUCs are calculated and compared with the corresponding experimental results of 4 \( \times \) 4 \( \times \) 4 cuboct cellular composites. We observe a good correspondence between the calculated results and the experimental measurements. Further evidence for the validation of the present computational framework is provided by seeking the truss-like behavior of cuboct cellular composites, observed experimentally for the thickness-to-length ratio \( (\phi = t/l) \) of strut members less than 0.1. To this end, a series of 2 \( \times \) 2 \( \times \) 2 RUCs with \( \phi = 0.05, 0.08, 0.1, 0.15 \) and different Young’s moduli of the constituent composite material \( (E_t = 37, 55.5, 74 \text{ GPa}) \) are modeled, and the truss-like or frame-like response of the cellular composites is examined under tensile loading. Based on the results obtained for pin-jointed and rigid-jointed structures using the current methodology, we find that cellular composite lattices with \( \phi < 0.08 \) are truss-like cellular materials, while those with \( 0.08 < \phi < 0.2 \) show frame-like behavior. Taking the relative density of the cellular composite material to be proportional to the square of the member aspect ratio, we find that the relative overall initial tangential elastic moduli of the truss-like cellular composites are scaled with the relative density through an exponent of 3/2. This finding is consistent with the experimental observations reported in Ref. [24]. Furthermore, an effort is devoted to study the variation of the modulus of elasticity and the strain energy density function of the cuboct cellular composite as a function of the uniaxial tension. For this purpose, cellular composite materials constituted of strut members with \( \phi = 0.05 \) (truss-like lattice) and \( \phi = 0.15 \) (frame-like lattice) are studied. The obtained nonlinear strain-dependent behavior of the elastic modulus reveals that the strain energy density function cannot be described as a simple quadratic function of strain and, thus, the ultralight truss-like/frame-like cellular composite material obeys a hyperelastic constitutive model at moderately large strains. Then, we employed the Ogden’s hyperelasticity model to fit the calculated strain energy density function and evaluated the material parameters. Cheung and Gershenfeld [24] also reported a linear elastic response followed by a nonlinear superelastic response for their fabricated cellular composites. Therefore, the good correspondence between the results obtained using the current computational methodology and those of the experiment reveals that the presented method can capture the hyperelastic mechanical response of the cellular composite materials fabricated at MIT Media Lab-Center. Since we can easily change the topology of the cellular material by changing both coordinates and the connectivity of nodes, and also simply change the cross section of strut members as well as the constituent material, we have presented the simplest initial computational framework for the analysis, design, and topology optimization of cellular composite materials. It should be emphasized that the present method also provides the capability to tune the flexibility of connections depending on the quality of the assemblage of building blocks.

Acknowledgements

The authors gratefully acknowledge the support for the work provided by Texas Tech University.
Appendix A. Components of the Green-Lagrange strain and the second Piola-Kirchhoff stress tensor

Using the displacement field given in Eq. (1), the Green-Lagrange strain components in the updated Lagrangian co-rotational reference are calculated as

\[
\begin{align*}
\varepsilon_{u1} & = u_{1,1} + \frac{1}{2}(u_{2,1})^2 + \frac{1}{2}(u_{3,1})^2 \approx u_{10,1}, \\
\varepsilon_{u2} & = u_{2,2} + \frac{1}{2}(u_{3,2})^2 + \frac{1}{2}(u_{3,3})^2 = \frac{1}{2}(u_{1,2} - u_{3,1})^2 + \frac{1}{2}\varepsilon^2 \approx 0, \\
\varepsilon_{u3} & = u_{3,3} + \frac{1}{2}(u_{3,3})^2 + \frac{1}{2}(u_{3,3})^2 \approx 0, \\
\varepsilon_{u12} & = \frac{1}{2}(u_{2,1} + u_{2,1} + u_{3,2} - \varepsilon_{1,1}x_1), \\
\varepsilon_{u13} & = \frac{1}{2}(u_{1,3} + u_{1,3} + u_{3,3} + \varepsilon_{1,3}x_3), \\
\varepsilon_{u23} & = \frac{1}{2}(u_{2,3} + u_{2,3} + u_{3,2} + \varepsilon_{2,3}x_3),
\end{align*}
\]

where, the index notation \( \cdot \) denotes \( \partial / \partial x \).

The member generalized stress, \( s \) is calculated in such a way that it includes the anisotropic properties of the constituent composite material. Since the strut members are composed of unidirectional fiber composite beams, they are considered to be transversely isotropic material with the following elastic tensor

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\]

Therefore, the components of the second Piola-Kirchhoff stress tensor are calculated as...
in which \( C_{i0} = \frac{1}{2}(C_{11} - C_{12}) \). Substituting strain components from Eq. (A.1) into Eq. (A.3), the generalized nodal forces for each strut members are obtained

\[
\begin{align*}
N_{11} & = \int_S S_{11}^i dA = C_{11}(\Delta \varepsilon_{11}^i + I_2 N_{22}^1 + I_3 N_{33}^1), \\
M_{12} & = \int_S S_{12}^i x_0 dA = C_{12}(\Delta \varepsilon_{12}^i + I_2 N_{22}^1 + I_3 N_{33}^1), \\
M_{33} & = \int_S S_{33}^i x_0 dA = C_{11}(\Delta \varepsilon_{33}^i + I_2 N_{22}^1 + I_3 N_{33}^1), \\
T & = \int_S (S_{12}^i x_0 - S_{21}^i x_0) dA \approx \Theta,
\end{align*}
\]

(A.4)

Appendix B. The mixed variational functional for an RUC consisting of \( N \) strut members

The mixed variational functional in the co-rotational updated Lagrangian reference and for a group of members, \((m = 1,2,\ldots,N)\), is given by

\[
\mathcal{F}_m = \sum_m \left( \int_{V_m} \left\{ -B^i \frac{\partial \varepsilon_{ii}^m}{\partial \varepsilon_{ii}^m} - \int_{S_{nn}^m} \mathcal{T}_u dS \right\} dv - \int_{S_{nn}^m} \mathcal{T}_u dS \right) \quad m = 1,2,\ldots,N,
\]

(B.1)

where, \( S_{ii}^m \) is the incremental components of the second Piola-Kirchhoff stress tensor, \( u_i \) is the incremental components of the displacement field, \( V_m \) is the \( m \)th member volume in the current co-rotational reference, \( S_{nn}^m \) is the \( m \)th member surface where the traction is prescribed, and \( \mathcal{T}_u \) and \( b_i \) \((i = 1,2,3)\) are, respectively, the components of the boundary tractions and the body forces per unit volume in the current configuration. \( \tau_{ij}^m \) are the initial Cauchy stresses in the updated Lagrangian co-rotational frame, and \( \tau_{ij}^m + \sigma_{ij}^m \) are the total stresses. The prescribed displacement boundary conditions, \( \vec{n}_i \) \((i = 1,2,3)\) on the surface \( S_{nn}^m \), are assumed to be satisfied a priori. The conditions of stationarity of \( \mathcal{F}_m \) in Eq. (B.1) with respect to the variations \( \delta \varepsilon_{ii}^m \) and \( \delta u_i \) result in the following incremental equations in the co-rotational updated Lagrangian reference,

\[
\frac{\partial B}{\partial \varepsilon_{ii}^m} = \frac{1}{2}(u_{ii}^m + u_{ii}^m) \quad m = 1,2,\ldots,N,
\]

(B.2)

\[
\left[ \mathcal{S}_{ij}^m \right] + \tau_{ij}^m \frac{\partial u_{kk}^m}{\partial \varepsilon_{ii}^m} + \rho b_i = -\varepsilon_{ij}^m \quad m = 1,2,\ldots,N,
\]

(B.3)

\[
\left[ n_i (\mathcal{S}_{ij}^m + \tau_{ij}^m u_{kk}^m) \right]^+ + \left[ n_i (\mathcal{S}_{ij}^m + \tau_{ij}^m u_{kk}^m) \right]^+ = -[\mathcal{S}_{ij}^m]^+ - [\mathcal{S}_{ij}^m]^+ \quad m = 1,2,\ldots,N,
\]

(B.4)

\[
\left[ n_i \mathcal{S}_{ij}^m + \tau_{ij}^m u_{kk}^m - \mathcal{T} = -n_i \tau_{ij}^m \quad m = 1,2,\ldots,N
\]

(B.5)

Eq. (B.3) leads to the equilibrium correction of iterations, and Eq. (B.4) is the condition of the traction reciprocity at the inter-element boundary. Here, \( n \) is the outward unit normal on the surface \( S_{nn}^m, \rho_m \) is the common interface between adjacent members, and + and − denote the outward and inward quantities at the interface \( \rho_m \), respectively. The continuity of the displacement at \( \rho_m \) is also determined by \( u_{ii}^m = u_{ii}^m \). Substitution of the components of the second Piola-Kirchhoff stress tensor from Eq. (A.3) into the Eq. (B.1) and integration over the cross section area of each member result in Eq. (4). This equation (Eq. (4)) can be simplified by applying the following integration by parts to its third term on the right-hand side

\[
\begin{align*}
\int \vec{N}_{11} \varepsilon_{ii}^m dl & = \int \vec{N}_{11} u_{ii}^m dl = -\int \vec{N}_{11} u_{ii}^m dl + \vec{N}_{11} u_{ii}^m dl, \\
\int \vec{M}_{12} N_{22}^1 dl & = -\int \vec{M}_{12} u_{22}^m dl = -\int \vec{M}_{13} u_{22}^m dl + \vec{M}_{13} u_{22}^m dl, \\
\int \vec{M}_{33} N_{33}^1 dl & = -\int \vec{M}_{33} u_{33}^m dl = -\int \vec{M}_{33} u_{33}^m dl + \vec{M}_{33} u_{33}^m dl, \\
\int \vec{T} \Theta dl & = \int \vec{T} \Theta dl = -\int \vec{T} \Theta dl + \vec{T} \Theta dl,
\end{align*}
\]

(B.6) (B.7) (B.8) (B.9)

Appendix C. Derivation of the stiffness matrix, \( K \)

\[
\mathcal{K} = -\mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 - \mathcal{K}_4.
\]

(C.1)

\[
\mathcal{K}_1 = \sum_{m=1}^{N} \int \frac{1}{2} \varepsilon_{ii}^m D_{ii}^m \varepsilon_{ii}^m \quad \mathcal{K}_2 = \sum_{m=1}^{N} \left\{ \frac{1}{2} \varepsilon_{ii}^m \right\} \varepsilon_{ii}^m \varepsilon_{ii}^m \mathcal{K}_3 = \sum_{m=1}^{N} \left\{ \frac{1}{2} \varepsilon_{ii}^m \right\} \varepsilon_{ii}^m \mathcal{K}_4 = \sum_{m=1}^{N} \left\{ \frac{1}{2} \varepsilon_{ii}^m \right\} \varepsilon_{ii}^m.
\]

(C.2) (C.3) (C.4)
\[ \mathcal{U}_r = \sum_{m=1}^{N} \left( a^T F - a^T T^r \mathbf{B}^r \right). \]  

(C.5)

Invoking \( \delta \mathcal{U}_r = 0 \), results in the following equation

\[ \delta \mathcal{U}_r = \sum_{m=1}^{N} \delta \mathbf{F}^T (-H^r \mathbf{F} + \mathbf{G}^r \mathbf{B} + K^r \mathbf{a} - \mathbf{F}^r) = 0, \]  

(C.6)

where,

\[ F^r = G^r \mathbf{B}^r. \]  

(C.7)

In Eq. (C.6), \( \delta \mathbf{F}^T \) is independent and arbitrary for each member, \( m = 1, 2, \ldots, N \), leading to

\[ \mathbf{B} = H^r \mathbf{G}^r \mathbf{B}^r. \]  

(C.8)

By equating to zero the second summation of Eq. (C.6) and substituting \( \mathbf{B} \) from Eq. (C.8), the stiffness matrix, \( \mathbf{K} \), for the elastic large-deformation analysis of cellular composite with rigid connections is derived explicitly.

Appendix D. Derivation of the stiffness matrix, \( \mathbf{K} \) accounting for the nonlinear flexible connections

Substituting Eqs. (29–32) and (26) into the Eqs. (27) and (28) and using the increment of the strain energy density function, \( \Delta \mathcal{U} \)

\[ \Delta \mathcal{U} = \frac{1}{2} \Delta \mathbf{m}^T \Delta \mathbf{D} \Delta \mathbf{m} = \frac{1}{2} \left[ \begin{array}{c} \Delta \mathbf{m}^T \\
\end{array} \right] \left[ \begin{array}{c} \Delta \mathbf{D} \\
\end{array} \right] \left[ \begin{array}{c} \Delta \mathbf{m} \\
\end{array} \right], \]  

lead to

\[ \Delta \mathbf{D} \mathbf{m}_j \Delta \mathbf{G}^T_j = \Delta \mathbf{D} \mathbf{m}_j \Delta \mathbf{G}^r_j = \delta \Delta \mathbf{m}_j \frac{\Delta \mathbf{m}_j}{\mathbf{S}_j} + \delta \Delta \mathbf{m}_j \frac{\Delta \mathbf{m}_j}{\mathbf{S}_j} = \delta \Delta \mathbf{m}_j \frac{\Delta \mathbf{m}_j}{\mathbf{S}_j} + \delta \Delta \mathbf{m}_j \frac{\Delta \mathbf{m}_j}{\mathbf{S}_j} \]  

(C.1)

\[ \Delta \mathbf{D} \mathbf{m}_j \Delta \mathbf{G}^T_j = \Delta \mathbf{D} \mathbf{m}_j \Delta \mathbf{G}^T_j = \delta \Delta \mathbf{m}_j \frac{\Delta \mathbf{m}_j}{\mathbf{S}_j} + \delta \Delta \mathbf{m}_j \frac{\Delta \mathbf{m}_j}{\mathbf{S}_j}. \]  

(C.2)

which can be rewritten in the matrix form as below

\[ \begin{bmatrix} \Delta \mathbf{G}^{(2)}_20 \\ - \Delta \mathbf{G}^{(2)}_20 \end{bmatrix} = \frac{l}{6C_1(1_3^T \mathbf{I}^2 - 2_3^T \mathbf{I}_3 + l_2^T \mathbf{I} + \mathbf{A}_e^T - \mathbf{A}_l^T \mathbf{I}_3)}, \]  

where

\[ C_1 = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right) \]  

\[ C_2 = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right) \]  

\[ C_3 = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right). \]  

(C.3)
Using Eqs. (D.4), (D.5) and (D.8), it is obtained results, respectively, in the following relations to the incremental uniaxial stretching, \( \Delta (\Delta \Delta) \),

\[
\begin{bmatrix}
\Delta n \\
\Delta m_1 \\
\Delta m_2 \\
\Delta m_3
\end{bmatrix} = \frac{l}{6C_{11}((I_{I})^2 - 2l_{I}l_{II} + l_{I}^2) + A_{II}^{2} - Al_{II}l_{I}III)} \left[
\begin{array}{c}
6(I_{I}^2 - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I})
\end{array}
\right]
\]

\[= \begin{bmatrix}
\Delta n \\
\Delta m_1 \\
\Delta m_2 \\
\Delta m_3
\end{bmatrix},
\]

where \( a_i = \frac{6C_{11}(I_{I}I - 2l_{I}l_{II} + l_{I}^2) + A_{II}^{2} - Al_{II}l_{I}III)}{5l} \) \( i = 1 \), \( 2 \), \( 3 \). Considering the incremental forms for the energies due to the incremental uniaxial stretching, \( \Delta h \) and the incremental twisting angle, \( \Delta \theta \)

\[
\int_0^l \Delta h \Delta \delta N dx = \int_0^l (\Delta M_1 \Delta \theta) dx,
\]

results, respectively, in the following relations

\[
\Delta H = \frac{l}{2C_{11}((I_{I})^2 - 2l_{I}l_{II} + l_{I}^2) + A_{II}^{2} - Al_{II}l_{I}III)} \left[
\begin{array}{c}
6(I_{I}^2 - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I})
\end{array}
\right]
\]

\[= \begin{bmatrix}
\Delta n \\
\Delta m_1 \\
\Delta m_2 \\
\Delta m_3
\end{bmatrix},
\]

\[
\Delta^2 M_1 - \Delta^2 M_3 = \frac{T}{l} (\Delta^2 \theta - \Delta^2 \delta).
\]

Using Eqs. (D.4), (D.5) and (D.8), it is obtained

\[
\begin{bmatrix}
\Delta H \\
\Delta \delta \theta_{20} \\
\Delta \delta \theta_{30} \\
\Delta \delta \theta_{20}
\end{bmatrix} = \begin{bmatrix}
6C_{11}(I_{I}I^2 - 2l_{I}l_{II} + l_{I}^2) + A_{II}^{2} - Al_{II}l_{I}III) \\
\left[
\begin{array}{c}
6(I_{I}^2 - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I})
\end{array}
\right]
\end{bmatrix}
\]

\[= \begin{bmatrix}
\Delta n \\
\Delta m_1 \\
\Delta m_2 \\
\Delta m_3
\end{bmatrix},
\]

\[
\Delta^2 M_1 - \Delta^2 M_3 = \frac{T}{l} (\Delta^2 \theta - \Delta^2 \delta).
\]

For a symmetrical cross section, Eqs. (D.4), (D.5), and (D.8) are, respectively, reduced to

\[
\begin{bmatrix}
\Delta m_1 \\
\Delta m_2 \\
\Delta m_3
\end{bmatrix} = \begin{bmatrix}
6C_{11}(I_{I}I^2) \\
\left[
\begin{array}{c}
6(I_{I}^2 - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I})
\end{array}
\right]
\end{bmatrix}
\]

\[= \begin{bmatrix}
\Delta n \\
\Delta m_1 \\
\Delta m_2 \\
\Delta m_3
\end{bmatrix} = \begin{bmatrix}
\Delta \delta \theta_{20} \\
\Delta \delta \theta_{30} \\
\Delta \delta \theta_{20}
\end{bmatrix}.
\]

\[
\begin{bmatrix}
\Delta m_1 \\
\Delta m_2 \\
\Delta m_3
\end{bmatrix} = \begin{bmatrix}
6C_{11}(I_{I}I^2) \\
\left[
\begin{array}{c}
6(I_{I}^2 - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I}) \\
3(I_{I}I_{I} - I_{II}I_{I})
\end{array}
\right]
\end{bmatrix}
\]

\[= \begin{bmatrix}
\Delta n \\
\Delta m_1 \\
\Delta m_2 \\
\Delta m_3
\end{bmatrix} = \begin{bmatrix}
\Delta \delta \theta_{20} \\
\Delta \delta \theta_{30} \\
\Delta \delta \theta_{20}
\end{bmatrix}.
\]
\[
\Delta \mathbf{m}_1 = \frac{6C_1}{l} \mathbf{I}_{33} \begin{pmatrix} 1 & (2 + a_j)(2 + b_j) - 1 & -1 & 2 + a_j \end{pmatrix} \begin{pmatrix} -\Delta \mathbf{v}_{23} & \Delta \mathbf{v}_{33} \end{pmatrix} = \left( \mathbf{H}^{-1} \right)_{ij} \begin{pmatrix} -\Delta \mathbf{v}_{23} & \Delta \mathbf{v}_{33} \end{pmatrix}
\]
(D.12)

\[
\Delta N = \frac{C_{11} A}{l} \Delta \mathbf{H},
\]
(D.13)

where \( a_j = \frac{6C_1}{l} \mathbf{I}_{33} \) and \( b_j = \frac{6C_1}{l} \mathbf{I}_{33} \) (\( i = 2,3 \)).

References