An Elementarily Simple Galerkin Meshless Method: The Fragile Points Method (FPM) Using Point Stiffness Matrices, for 2D Elasticity Problems in Complex Domains

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SUMMARY

The Fragile Points Method (FPM) is a stable and elementarily simple, meshless Galerkin weak-form method, employing simple, local, polynomial, Point-based, discontinuous and identical trial and test functions. These functions to form the Galerkin weak form are derived from the Generalized Finite Difference method. Numerical Flux Corrections are introduced in the FPM to resolve the inconsistency caused by the discontinuous trial functions. Given the very simple polynomial characteristic of trial and test functions, all integrals in the Galerkin weak form can be calculated in the FPM without much effort. With the global matrix being sparse, symmetric and positive definitive, the FPM is suitable for large-scale simulations. Additionally, because of the inherent discontinuity of trial and test functions, we can easily cut off the interaction between Points and introduce cracks, rupture and fragmentation based on physical criteria. In this paper, we have studied the applications of the FPM to linear elastic mechanics and several numerical examples of 2D linear elasticity are computed. The results suggest that the FPM is accurate, robust, consistent and convergent. Volume locking does not occur in the FPM for nearly incompressible materials. Besides, a new, simple and efficient approach to tackle pre-existing cracks in the FPM is also illustrated in this paper and then applied to model-I crack problems. Physical formation of cracks and their propagation, rupture and fragmentation will be dealt with in our sequent papers.

KEY WORDS: Fragile Points Method; Numerical Flux Corrections; Large-scale simulations; Elastic mechanics; Model-I crack

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1. INTRODUCTION

Structural stress analysis is crucial and necessary in diverse engineering fields, such as aeronautics, astronautics, automobile engineering, etc. From the design, manufacture to maintenance of products, structural stress analysis plays a crucial role. Because of its significance, numerous researchers have been focusing on improving the accuracy and efficiency of this procedure for decades.

The Finite Element Method (FEM) is mature, reliable and widely used in structural stress analysis [1]. This method employs contiguous elements, and Element-based, local, polynomial, interelement-continuous trial and test functions. Because the trial and test functions are Element-based, the Galerkin weak form leads to Element Stiffness Matrices. Therefore, integrals in the Galerkin weak form underlying the FEM are easy to compute. The symmetry and sparsity of the global stiffness matrix make the FEM suitable and efficient in large-scale simulations. However, the accuracy of the FEM greatly depends on the quality of mesh. In order to obtain satisfactory solutions, much efforts are usually spent on meshing. Especially, even if simulations are initialized with a high-quality mesh structure, mesh distortion will occur in the case of large deformations and the precision of solutions decreases dramatically. In order to study the formation of cracks, rupture and fragmentation, methods such as remeshing, deleting elements, and Cohesive-Zone models are often used.

Meshless methods, which eliminate mesh structure partly or completely, have been invented and developed since the end of last century. Element Free Galerkin (EFG) [2] and Meshless Local Petrov-Galerkin (MLPG) [3] methods are two classical meshless weak-form methods based on the “Global Galerkin” and “Local Petrov-Galerkin” weak forms, respectively. While the EFG uses the same Node-based trial and test functions, the MLPG method uses different trial and test function spaces. The test function can be Dirac function or Heaviside function. These two meshless methods have mainly utilized Moving Least Squares (MLS) or Radial Basis Function (RBF) approximations to deduce Node-based trial functions. With MLS and RBF approximations, higher-order continuity is easy to achieve. Besides, since individual nodes have replaced mesh structure, EFG and MLPG can conveniently insert or remove additional nodes and bypass the influence of distortion even in large deformation and fracture cases (e.g. [4], [5]). However, on the other hand, the trial
functions obtained by the MLS or RBF method are rational functions and grossly complex. Therefore, the computation of integrals in the weak forms in either EFG or MLPG is extremely tedious, less accurate and even can influence the method’s stability. To reduce the computational cost and improve the accuracy of integration, some special, new types of numerical integration methods, for example, the series of nodal integration methods [6], are often adopted.

Smoothed Particle Hydrodynamics (SPH) method [7], as a kind of meshless particle method, is simpler in form and needs less computational cost. Nevertheless, the SPH approach is based on a strong form, instead of a weak form. Demonstrating this method’s stability is not an easy task. Besides, tensile instability will occur in the SPH, if we use Smoothed Kernel functions to calculate derivatives.

Peridynamics [8], as a relatively new method, is based on a different set of theories rather than the traditional continuum mechanics. Therefore, those classical constitutive models and engineering experience which scientists and engineers have developed for decades are difficult to be applied exactly and directly in Peridynamics, which is still far from being applicable to large-scale practical engineering situations.

From the above discussion, we can conclude that simple, local, polynomial, “Point-Based” shape functions are helpful in the calculation of integrals in the weak form. Besides, a weak-form method can have a better performance on stability. But with these requirements, it is difficult to keep the trial and test functions continuous over the entire domain. In our previous paper, we have developed the Fragile Points Method [9] for the first time. The FPM approach employs very simple, local, polynomial, Point-based and discontinuous shape functions constructed by the Generalized Finite Difference method. Since the FPM uses “Point-Based” trial and test functions in a Galerkin weak form, the method leads to “Point Stiffness Matrices” as opposed to the Element Stiffness Matrices in the FEM. Numerical Flux Corrections are introduced in FPM to solve the inconsistency caused by the method’s discontinuity of trial and test functions. Integrals in the Galerkin weak form can be computed easily with Gauss Integration method or just analytically. Like the FEM, since the FPM is based on a Galerkin weak form, a symmetric, sparse and positive definitive global matrix can be deduced in the FPM, which means the FPM can be easily used in large-scale simulations. More importantly, because of the discontinuity of functions, we can easily cut off the interaction between two Points and introduce cracks, rupture or fragmentation without much effort.
In this paper, we formulate and apply the FPM for solving linear elasticity problems on complex shaped domains. A new, simple and efficient approach to introduce cracks in the FPM is also illustrated in this paper. The procedure for constructing Point-based trial and test functions is introduced in Section 2. The Interior Penalty Numerical Fluxes and the numerical implementation of the FPM for elasticity are discussed in Section 3. In Section 4, several numerical examples for 2D linear elasticity problems are studied and specific steps to deal with cracks in the FPM are introduced. Last, a conclusion and some discussions for further studies on modeling the formation of cracks, rupture and fragmentation are given in Section 5.

2. LOCAL, POLYNOMIAL, POINT-BASED, DISCONTINUOUS TRIAL AND TEST FUNCTIONS

For the linear elasticity, the governing equations are given in Eq. (2.1),

$$\begin{align*}
\varepsilon_{ij}(u) &= \frac{1}{2}(u_{i,j} + u_{j,i}) \\
\sigma_{ij}(u) + f_i &= 0 \\
\sigma_{ij}(u) &= D_{ijkl}\varepsilon_{kl}(u)
\end{align*}$$

in $\Omega$

where $\Omega$ is the problem domain; $\sigma_{ij}$, $\varepsilon_{ij}$ and $u_i$ stand for the stress tensor, strain tensor and displacement vector, respectively; $f_i$ is the body force and $D_{ijkl}$ connotes the fourth order linear elasticity tensor.

The corresponding boundary conditions are shown in Eq. (2.2), where $\Gamma_u$ and $\Gamma_t$ are displacement prescribed and traction prescribed boundaries, respectively; $\overline{u}_i$ and $\overline{t}_i$ denote the prescribed displacements and tractions on the corresponding boundary, respectively; $n_j$ stands for the unit vector outward to the external boundary $\partial\Omega$.

$$\begin{align*}
u_i &= \overline{u}_i \text{ on } \Gamma_u \\
\sigma_{ij}(u)n_j &= \overline{t}_i \text{ on } \Gamma_t
\end{align*}$$

Considering the problem domain $\Omega$, as shown in Figure 1(a), several Points are distributed randomly inside the domain or on its boundary. Utilizing these Points, the domain can be partitioned into conforming and nonoverlapping subdomains of arbitrary shape, with only one Point involved in each subdomain (shown in Figure 1(b)). Numerous methods can be used for this partition, and in this paper, the Voronoi
Diagram method is chosen. Thus, in the present Fragile Points Method, the construction of trial and test function is totally based on these randomly distributed Points. It should be noted that in contrast, in the FEM, the trial and test functions are Element-based and are continuous between Elements which are contiguous.
In each subdomain, we define the simple, local, polynomial, discontinuous displacement vector or trial function $\mathbf{u}^b$ with the values of $u_x$, $u_y$ (displacement in $x$ and $y$ directions, respectively) and their derivatives in $x$ and $y$ directions at the internal Point. For instance, the 2D linear displacement vector in the subdomain $E_0$ which contains the Point $P_0$ is given in Eq. (2.3),

$$
\mathbf{u}^b(x, y) = \begin{bmatrix} u_x^0 & u_y^0 \\
\frac{\partial u_x^0}{\partial x} + \frac{\partial u_x}{\partial x} (x-x_0) + \frac{\partial u_x}{\partial y} (y-y_0) & \frac{\partial u_y^0}{\partial x} + \frac{\partial u_y}{\partial x} (x-x_0) + \frac{\partial u_y}{\partial y} (y-y_0) \end{bmatrix} \quad (x, y) \in E_0
$$

where $(x_0, y_0)$ are the coordinates of the Point $P_0$; $[u_x^0 \quad u_y^0]^T$ is the value of $\mathbf{u}^b$ at $P_0$; Derivatives $\left[ \frac{\partial u_x}{\partial x} \quad \frac{\partial u_x}{\partial y} \quad \frac{\partial u_y}{\partial x} \quad \frac{\partial u_y}{\partial y} \right]^T \bigg|_{P_0}$ are the yet unknown coefficients.

It is a crucial step to determine the derivatives $\left[ \frac{\partial u_x}{\partial x} \quad \frac{\partial u_x}{\partial y} \quad \frac{\partial u_y}{\partial x} \quad \frac{\partial u_y}{\partial y} \right]^T \bigg|_{P_0}$. In our work, we have used the Generalized Finite Difference (GFD) method [10] to
calculate the derivatives at each Point.

The first step for the GFD method is to define the support of the Point $P_0$. Usually, we prefer to define the support by drawing a circle at $P_0$ and assume all the Points included in that circle have interaction with $P_0$ (shown in Figure 2(a)). Alternatively, we can replace the circular support with a square one or other shapes. In this paper, the support of $P_0$ is defined to contain all its nearest neighboring points in the Voronoi Diagram partition (shown in Figure 2(b)). These neighboring points are named as $P_1, P_2, \ldots, P_m$. 

\[\text{(a)}\]
Figure 2(a) (b). Two kinds of support of \( P_0 \)

After defining the support of \( P_0 \), we define a weighted discrete \( L^2 \) norm \( J \) in a matrix form,

\[
J = (Aa + u_0 - u_m)^T W (Aa + u_0 - u_m)
\]

(2.4)

where

\[
A = \begin{bmatrix}
x_1 - x_0 & y_1 - y_0 & 0 & 0 \\
0 & 0 & x_1 - x_0 & y_1 - y_0 \\
x_2 - x_0 & y_2 - y_0 & 0 & 0 \\
0 & 0 & x_2 - x_0 & y_2 - y_0 \\
\vdots & \vdots & \vdots & \vdots \\
x_m - x_0 & y_m - y_0 & 0 & 0 \\
0 & 0 & x_m - x_0 & y_m - y_0
\end{bmatrix}
\]

\[
a = \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y}
\end{bmatrix}^T |_{p_0}
\]

\[
u_0 = \begin{bmatrix}
u_x^0 & u_y^0 & u_x^0 & u_y^0 & \cdots & u_x^0 & u_y^0
\end{bmatrix}^T |_{j=1,2,m}
\]

\[
u_m = \begin{bmatrix}
u_x^1 & u_y^1 & u_x^2 & u_y^2 & \cdots & u_x^m & u_y^m
\end{bmatrix}^T
\]
\[
W = \begin{bmatrix}
w_x^i & 0 & 0 & \cdots & 0 \\
0 & w_y^i & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & w_m^x & 0 \\
0 & \cdots & 0 & w_m^y & 0 \\
\end{bmatrix}
\]

\((x_i, y_i)\) are the coordinates of \(P_i\); \([u_x^i \quad u_y^i]^T\) is the value of \(u^h\) at \(P_i\); \([w_x^i \quad w_y^i]^T\) is the value of the weight function at \(P_i\) \((i = 1, 2, \ldots, m)\). For convenience, weight functions are taken to be constants in this paper.

By solving the stationarity of \(J\) in Eq. (2.4), we can derive the derivative vector \(a\) at \(P_0\).

\[
a = \left( A^T A \right)^{-1} A^T (u_m - u_0) \tag{2.5}
\]

Besides, \(u_m - u_0\) can be transformed into the following form,

\[
(u_m - u_0) = [I_1 \quad I_2] u_E \tag{2.6}
\]

where

\[
I_1 = \begin{bmatrix}
-1 & 0 \\
0 & -1 \\
-1 & 0 \\
\vdots & \vdots \\
0 & -1 \\
0 & -1 \\
\end{bmatrix}, \quad I_2 = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{bmatrix}_{2m \times 2m}
\]

Substituting Eq. (2.6) into Eq. (2.5), we obtain the relationship between \(a\) and \(u_E\).

\[
a = Cu_E \tag{2.7}
\]

where

\[
C = \left( A^T A \right)^{-1} A^T [I_1 \quad I_2]
\]

Finally, by substituting Eq. (2.7) into Eq. (2.3), the relationship between \(u^h\) and \(u_E\) is obtained in Eq. (2.8), where the matrix \(N\) is called the shape function of \(u^h\) in \(E_0\).

\[
u^h = Nu_E \tag{2.8}
\]
\[ N = \begin{bmatrix} x-x_0 & y-y_0 & 0 & 0 \\ 0 & 0 & x-x_0 & y-y_0 \end{bmatrix} C + I_3 \]
\[ = \begin{bmatrix} N_0 & 0 & N_1 & 0 & \cdots & N_m & 0 \\ 0 & N_0 & 0 & N_1 & \cdots & 0 & N_m \\ \end{bmatrix} n_{2(n+2)} \]
\[ I_3 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix} n_{2(n+2)} \]

where

According to the Eq. (2.1), the corresponding strain vector \( \varepsilon^h \) and stress vector \( \sigma^h \) in terms of \( u_E \) are given in Eq. (2.9),

\[ \begin{bmatrix} \varepsilon_x^h \\ \varepsilon_y^h \\ \gamma_{xy}^h \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} E = E \]

\[ \begin{bmatrix} \sigma_x^h \\ \sigma_y^h \\ \tau_{xy}^h \end{bmatrix} = DBu \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \end{bmatrix} \]

where \( D \) is the stress-strain matrix. In this paper we consider the material to be isotropic for simplicity.

\[ D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) / 2 \end{bmatrix} \]

\[ \bar{E} = \begin{cases} E \text{ (for plane stress)} \\ \frac{E}{1-\nu^2} \text{ (for plane strain)} \end{cases} \]

with

\[ \bar{\nu} = \begin{cases} \nu \text{ (for plane stress)} \\ \frac{\nu}{1-\nu} \text{ (for plane strain)} \end{cases} \]

Following the same procedure, we can derive \( u^h \) in each subdomain \( E_i \in \Omega \).

Eventually, the displacement vector \( u^h \) in the entire domain can be obtained. The corresponding test function \( v \) is prescribed to possess the same shape as the trial function in each subdomain, in the present FPM based on the Galerkin weak form.
Reviewing the process of constructing trial and test functions, we can see that no continuity requirement exists on the internal boundary between two neighboring subdomains. In other words, these two subdomains have their own function values on their common boundary. Therefore, only simple, local, polynomial, Point-based and piecewise-continuous trial and test functions are obtained in the problem domain.

To illustrate the discontinuity of trial functions and shape functions, a 2D example is shown here. We assume that 25 Points scattered irregularly in a 1×1 square. The graphical representation of all the shape functions about Point 13 (the subscripts in Eq. (2.8) equal 13) is given in Figure 3. The corresponding trial function of $u_*$ simulating the exponential function $e^{-10(x-0.5)^2-10(y-0.5)^2}$ is shown in Figure 4.

Figure 3. The shape functions about Point 13
Figure 4. The trial function simulating an exponential function

Unfortunately, because of this discontinuity of trial and test functions, if we directly use the trial and test functions in the traditional Galerkin weak form which is widely used in the FEM, EFG and other numerical methods, the solution will be inconsistent, inaccurate and cannot pass the patch tests [9]. In order to solve this inconsistency problem, Numerical Flux Corrections have been introduced to the FPM.

3. NUMERICAL FLUX CORRECTIONS AND NUMERICAL IMPLEMENTATION

3.1 Interior Penalty (IP) Numerical Flux Corrections for elasticity problems

Numerical Fluxes, frequently used in Discontinuous Galerkin Methods, are employed in the FPM to resolve the inconsistency caused by the discontinuity. A variety of Numerical Fluxes have been developed and studied by various researchers. These Numerical Fluxes have different effects on accuracy, stability and computational cost of a method. In our work, we prefer the Interior Penalty Numerical Fluxes which can lead to consistent and stable results and symmetric global stiffness matrices.

The governing equations of linear elasticity in 2D have been shown in Eq. (2.1). We multiply the second equation by the test function $v$ then integrate it on the subdomain $E$ by parts,

$$\int_E \sigma_{ij}(u)v_{ij} d\Omega - \int_{\partial E} \sigma_{ij}(u)n_j v_i d\Gamma = \int_E f_i v d\Omega$$  \hspace{1cm} (3.1)
where $\partial E$ is the boundary of the subdomain $E$, $\mathbf{n}$ is the unit vector outward to $\partial E$.

For every subdomain $E_i \in \Omega$, Eq. (3.1) should be satisfied. Therefore, we sum Eq. (3.1) over all subdomains.

$$
\sum_{E \in \Omega} \int_{E} \sigma_{ij} (u) v_{ij} \, d\Omega - \sum_{E \in \Omega} \int_{E} \sigma_{ij} (u) n_j v_i \, d\Gamma = \sum_{E \in \Omega} \int_{E} f_{ij} v_i \, d\Omega
$$

(3.2)

Considering the symmetry of stress tensor ($\sigma_{ij} = \sigma_{ji}$), we can transform the first term of Eq. (3.2) into the following form,

$$
\sum_{E \in \Omega} \int_{E} \sigma_{ij} (u) v_{ij} \, d\Omega = \sum_{E \in \Omega} \int_{E} \sigma_{ij} (u) \epsilon_{ij} (v) \, d\Omega
$$

(3.3)

Let $\Gamma$ denote the set of all external and internal boundaries and $\Gamma_h = \Gamma - \Gamma_e - \Gamma_u$ stands for the set of all internal boundaries. For the convenience, we rewrite the second term in Eq. (3.2) with the jump operator $[]$ and average operator $\{\}$. 

$$
\sum_{E \in \Omega} \int_{E} \sigma_{ij} (u) n_j v_i \, d\Gamma = \sum_{e \in \Gamma_h} \int_{e} \left[ \sigma_{ij} (u) n_j^e \right] [v_i] \, d\Gamma + \sum_{e \in \Gamma_e} \int_{e} \left\{ \sigma_{ij} (u) n_j^e \right\} [v_i] \, d\Gamma
$$

(3.4)

When $e \in \Gamma_h$ (assuming $e$ is shared by subdomains $E_1$ and $E_2$), $n_j^e$ is a unit vector normal to $e$ and points from $E_1$ to $E_2$. The average $\{\}$ and jump $[]$ operator for any quantity $w$, at an internal boundary are defined as Eq. (3.5).

$$
[w] = w_{E_1}^{E_2} - w_{E_2}^{E_1} \quad \{w\} = \frac{1}{2} \left( w_{E_1}^{E_2} + w_{E_2}^{E_1} \right)
$$

(3.5)

When $e \in \partial \Omega$, $n_j^e$ is outward to $\partial \Omega$ and the average $\{\}$ and jump $[]$ operator are defined as below.

$$
[w] = w_{e}^{E_1} \quad \{w\} = w_{e}^{E_2}
$$

It should be noted that for two neighboring subdomains, no matter which one is chosen as $E_1$, the value of Eq (3.4) stays the same. If $\sigma_{ij} (u)$ is the exact solution in an intact contiguous domain, there should be no jump on internal boundaries, in other words, $\left[ \sigma_{ij} (u) n_j^e \right] = 0$ i.e., tractions are reciprocated at internal intact boundaries. Besides, with the traction boundary condition $\sigma_{ij} (u) n_j = \bar{t}_j$, Eq. (3.4) can be rewritten as below.

$$
\sum_{E \in \Omega} \int_{E} \sigma_{ij} (u) n_j v_i \, d\Gamma = \sum_{e \in \Gamma_h} \int_{e} \left[ \sigma_{ij} (u) n_j^e \right] [v_i] \, d\Gamma + \sum_{e \in \Gamma_e} \int_{e} \bar{t}_j v_i \, d\Gamma
$$

(3.6)
Eventually, we substitute Eq. (3.3), Eq. (3.6) into Eq. (3.2), and add two boundary integrals

\[ \sum_{e \in \Gamma_x} \int_{e} \left\{ \sigma_y(u) n^e_j \right\} [u_i] d \Gamma \]

and

\[ \sum_{e \in \Gamma_x} \frac{\eta}{h_e} \int_{e} [u_i] [v_j] d \Gamma \]

on both sides. Then we obtain the FPM with Interior Penalty Numerical Flux Corrections for elasticity problems,

\[
\begin{align*}
\sum_{e \in \Omega} \int_{e} \sigma_y(u) e_j (v) d \Omega - \sum_{e \in \Gamma_x} \int_{e} \left\{ \sigma_y(u) n^e_j \right\} [v_j] d \Gamma \\
- \sum_{e \in \Gamma_x} \int_{e} \left\{ \sigma_y(v) n^e_j \right\} [u_i] d \Gamma + \sum_{e \in \Gamma_x} \frac{\eta}{h_e} \int_{e} [u_i] [v_j] d \Gamma \\
= \sum_{e \in \Omega} \int_{e} f_e v_e d \Omega + \sum_{e \in \Gamma_x} \int_{e} \bar{u}_e v_e d \Gamma - \sum_{e \in \Gamma_x} \int_{e} \sigma_y(v) n^e_j \bar{u}_e d \Gamma + \sum_{e \in \Gamma_x} \frac{\eta}{h_e} \int_{e} \bar{u}_e v_e d \Gamma 
\end{align*}
\]

(3.7)

where \( h_e \) is an edge-dependent parameter and equal to the length of the boundary in this paper; \( \eta \) is a positive number independent of the edge size. It should be noted that with IP Numerical Flux Corrections, the method is only stable when the penalty parameter \( \eta \) is large enough [11]. A discussion about the effect of the penalty parameter is given in Section 4 and more information can be found in [12].

We can find that in Eq. (3.7), if the trial function and test function utilize the same shape functions, the FPM with IP Numerical Flux Corrections is a symmetric Galerkin weak-form approach. Moreover, displacement boundary conditions are imposed weakly in Eq. (3.7). If we impose \( u_i = \bar{u}_i \) strongly at the boundary points, Eq. (3.7) can be simplified as follows.

\[
\begin{align*}
\sum_{e \in \Omega} \int_{e} \sigma_y(u) e_j (v) d \Omega - \sum_{e \in \Gamma_x} \int_{e} \left\{ \sigma_y(u) n^e_j \right\} [v_j] d \Gamma \\
- \sum_{e \in \Gamma_x} \int_{e} \left\{ \sigma_y(v) n^e_j \right\} [u_i] d \Gamma + \sum_{e \in \Gamma_x} \frac{\eta}{h_e} \int_{e} [u_i] [v_j] d \Gamma \\
= \sum_{e \in \Omega} \int_{e} f_e v_e d \Omega + \sum_{e \in \Gamma_x} \int_{e} \bar{u}_e v_e d \Gamma 
\end{align*}
\]

(3.8)

For brevity, we can change Eq. (3.8) from a tensor form to a matrix form,

\[
\begin{align*}
\sum_{E \in \Omega} \int_{E} \mathbf{U}^T \mathbf{u} d \Omega - \sum_{e \in \Gamma_x} \int_{e} [ \mathbf{u} ] [ v ] d \Gamma \\
- \sum_{e \in \Gamma_x} \int_{e} [ \mathbf{u} ] [ v ] d \Gamma + \sum_{e \in \Gamma_x} \frac{\eta}{h_e} \int_{e} [ \mathbf{u} ] [ v ] d \Gamma \\
= \sum_{E \in \Omega} \int_{E} \mathbf{v}^T \mathbf{f} d \Omega + \sum_{e \in \Gamma_x} \int_{e} \mathbf{v}^T \bar{\mathbf{u}} d \Gamma 
\end{align*}
\]

(3.9)

where
Compared with the traditional Galerkin weak form [1], the Eq. (3.9) involves 3 extra boundary integrals on the left side, while the others stay identical. These additional boundary integrals are the contributions of the Interior Penalty Numerical Flux Corrections. To make the work clearer, if we define \( \mathbf{v} = \delta \mathbf{u} \), the integrals on a specific internal boundary (shown in Figure 5) are given in Eq. (3.10).

\[
\begin{align*}
\int_{\Gamma_e} \frac{\eta}{h_e} \mathbf{u} \mathbf{u}^T \mathbf{u} - (\mathbf{n}_e \mathbf{n})^T \{ \mathbf{u} \} - \{ \mathbf{u} \}^T \{ \mathbf{n}_e \} d\Gamma \\
= \delta \left( \int_{\Gamma_e} \frac{1}{2} \frac{\eta}{h_e} [\mathbf{a}]^T [\mathbf{u}] - \{ \mathbf{n}_e \}^T \{ \mathbf{u} \} d\Gamma \right) \\
= \delta \langle \text{Bonding Energy} \rangle
\end{align*}
\]  

Figure 5. The integrals on a specific internal boundary

In a physical view, \( \sum_{\Gamma_e} \int_{\Gamma_e} \{ \sigma_{y} (u) n_{y} \} [v] d\Gamma \) and \( \sum_{\Gamma_i} \int_{\Gamma_i} \{ \sigma_{y} (v) n_{y} \} [u] d\Gamma \) are the extra virtual work caused by the discontinuity of displacement and non-reciprocity of tractions on internal boundaries, and \( \sum_{\Gamma_e} \int_{\Gamma_e} [u] [v] d\Gamma \) play a role in restricting the jump of displacement as a kind of penalty functions. By postulating the physical quantities of energy required for the formation of cracks, rupture and fragmentation of unit length, criteria can be developed for rupture and fragmentation when these
physical quantities exceed these bonding energies, by breaking the bonds between neighboring Points. Alternatively, one could postulate criteria for rupture between two subdomains, based on the discontinuity of the Eshelby tractions [17] between the two segments. Hence the present method is named the Fragile Points Method (FPM).

3.2 Numerical implementation

This section will concentrate on the numerical implementation of the FPM. The FPM with IP Numerical Flux Corrections can be written finally in the following matrix form,

$$\mathbf{K} \mathbf{q} = \mathbf{Q}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{q}$ is the vector with nodal DOFs, $\mathbf{Q}$ is the load vector.

In Section 2, we have obtained the shape function $\mathbf{N}$ for $\mathbf{u}^h$ and $\mathbf{v}$, $\mathbf{B}$ for $\varepsilon$, $\mathbf{DB}$ for $\sigma$. By substituting them into the first term of Eq. (3.8), we derive the Point Stiffness Matrix $\mathbf{K}_E$, which is defined as the contribution of each Point to the global stiffness matrix.

$$\mathbf{K}_E = \int_E \mathbf{B}^T \mathbf{DB} d\Omega \quad \text{where } E \in \Omega \quad (3.9)$$

For the boundary integrals, the corresponding boundary stiffness matrix $\mathbf{K}_h$ is defined as below. The subscripts 1 and 2 denote which subdomain these shape functions belong to.

when $e \in \partial E_1 \cap \partial E_2$

$$\mathbf{K}_h = -\frac{1}{2} \int_e \mathbf{N}_e^T \mathbf{n} \mathbf{DB}_1 d\Gamma - \frac{1}{2} \int_e \mathbf{B}_1^T \mathbf{n} \mathbf{B}_1^T \mathbf{N}_e d\Gamma + \frac{\eta_e}{h_e} \int_e \mathbf{N}_e^T \mathbf{N}_e d\Gamma$$

$$- \frac{1}{2} \int_e \mathbf{N}_e^T \mathbf{n} \mathbf{DB}_2 d\Gamma + \frac{1}{2} \int_e \mathbf{B}_1^T \mathbf{n} \mathbf{B}_2^T \mathbf{N}_e d\Gamma - \frac{\eta_e}{h_e} \int_e \mathbf{N}_e^T \mathbf{N}_e d\Gamma \quad (3.10)$$

$$+ \frac{1}{2} \int_e \mathbf{N}_e^T \mathbf{n} \mathbf{DB}_2 d\Gamma - \frac{1}{2} \int_e \mathbf{B}_2^T \mathbf{n} \mathbf{B}_2^T \mathbf{N}_e d\Gamma - \frac{\eta_e}{h_e} \int_e \mathbf{N}_e^T \mathbf{N}_e d\Gamma$$

$$+ \frac{1}{2} \int_e \mathbf{N}_e^T \mathbf{n} \mathbf{DB}_2 d\Gamma + \frac{1}{2} \int_e \mathbf{B}_2^T \mathbf{n} \mathbf{B}_2^T \mathbf{N}_e d\Gamma + \frac{\eta_e}{h_e} \int_e \mathbf{N}_e^T \mathbf{N}_e d\Gamma$$

In the FPM, the global stiffness matrix $\mathbf{K}$ is obtained by assembling all the submatrices $\mathbf{K}_E$ and $\mathbf{K}_h$. This assembling process is the same as what we do in the FEM. Eventually, the FPM will lead to a sparse, symmetric and positive definitive global stiffness matrix.
When linear interpolations are employed for $\mathbf{u}^h$, the shape function $\mathbf{B}$ is constant and $\mathbf{N}$ is linear in each subdomain. Therefore, the integral for submatrix $\mathbf{K}_E$ can be calculated just multiplying the integrand by the area of the corresponding subdomain. For integrals on boundaries, numerical integration and direct analytic computation are both effective. For Eq. (3.10), 2 Points Gauss integration can lead to exact solutions. According to our results, reduced integration using only 1 Point can result in almost the same solutions as those obtained by 2 Points Gauss integration. In this paper, 1 Point Gauss integration method is used for boundary integrals.

4. NUMERICAL EXAMPLES

In this section, a variety of problems in elastic mechanics are solved with the FPM. In order to estimate the errors of numerical results conveniently, we define two relative errors $r_u$ and $r_E$ with the displacement $L^2$ norm and the energy norm, respectively.

$$
\begin{align*}
    r_u &= \frac{\| \mathbf{u}^h - \mathbf{u}^{\text{exact}} \|_{L^2}}{\| \mathbf{u}^{\text{exact}} \|_{L^2}}, \\
    r_E &= \frac{\| \mathbf{E}^h - \mathbf{E}^{\text{exact}} \|_E}{\| \mathbf{E}^{\text{exact}} \|_E}
\end{align*}
$$

where

$$
\| \mathbf{u} \|_{L^2} = \left( \int_{\Omega} \mathbf{u}^T \mathbf{u} \, d\Omega \right)^{1/2}, \quad \| \mathbf{E} \|_E = \left( \frac{1}{2} \int_{\Omega} \mathbf{E}^T \mathbf{E} \, d\Omega \right)^{1/2}
$$

4.1 Patch test

In this part, we design the following patch test in a $1 \times 1$ domain (shown in Figure 6) to examine the consistency of the FPM. The patch test is considered in the plane stress case, with the exact displacement and stress prescribed as below

$$
\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} x + y \\ x + y \end{bmatrix},
$$

$$
\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E \\ \frac{E}{1-v} \\ \frac{E}{1-v} \\ \frac{E}{1+v} \end{bmatrix},
$$

$x, y \in (0,1)$
Displacement boundary conditions are imposed on four edges according to Eq. (4.2). Since the analytic solutions are linear for the displacement and constant for the stress, when linear interpolations are employed for displacement, the numerical solutions $\mathbf{u}^k$ and $\sigma^k$ should be equal to Eq. (4.2).

The distributions of Points in 3 different patterns are given in Figure 6. In these three cases, no matter how the Points are scattered uniformly or randomly, the relative errors $r_u$ and $r_E$ are both less than $5 \times 10^{-7}$ with the penalty parameter $\eta \geq 1000E$ to restrict the jumps on internal boundaries. And more accurate solutions can be obtained with larger penalty parameters. In fact, the distinctions between numerical solutions and analytic solutions are mainly round-off errors for the patch tests, which generate during the process of computation. Therefore, we can conclude that the present FPM is consistent and accurate enough to pass the patch tests.

(a) 9 regular points

(b) 9 irregular points
4.2 Cantilever beam

In this section, we employ the FPM to a cantilever beam problem with a parabolic-shear load on its end (shown in Figure 7). The corresponding analytic solutions of displacement and stress in plane stress case have been given in [13].

\[
\begin{align*}
  u_x &= -\frac{P}{6EI}\left(y - \frac{H}{2}\right)[3x(2L-x)+(2+v)y(y-H)] \\
  u_y &= \frac{P}{6EI}\left[x^2(3L-x) + 3v(L-x)\left(y - \frac{H}{2}\right)^2 + \frac{4+5v}{4}H^2x\right] \\
  \sigma_x &= -\frac{P}{I}(L-x)\left(y - \frac{H}{2}\right) \\
  \sigma_y &= 0 \\
  \tau_{xy} &= -\frac{Py}{2I}(y-H)
\end{align*}
\]

where

\[
I = \frac{H^3}{12}
\]
Figure 7. The cantilever beam with a parabolic-shear end load

Specifically, we prescribe that $P = 1$, $E = 1 \times 10^5$, $H = 1$ and $L = 8$. With $81 \times 11$ points distributed in the beam uniformly or randomly, the comparisons between numerical solutions $u_x$, $\sigma_x$ and analytical solutions along $x = L/2$ are given in Figure 8(a) and (b), respectively. The Poisson’s ratio $\nu$ is prescribed as 0.3 and the penalty parameter $\eta = E$.
Figure 8(a). Numerical solutions of $u_x$ along $x = L/2$

Figure 8(b). Numerical solutions of $\sigma_x$ along $x = L/2$

To study the performance of the FPM in convergence, the number of uniformly located Points is defined with 3 values: $41 \times 6$, $81 \times 11$ and $161 \times 21$. Besides, the Poisson’s ratio is defined as 0.3 or 0.4999 to test the locking property of the FPM for nearly incompressible materials.

The relations between $h$ (the distance of two neighboring Points in $x$-direction) and the relative errors $r_u, r_E$ are shown in Figure 9(a) and (b), respectively. Also, the corresponding convergence rate $R$ is given in Figure 9. Compared with the linear displacement-based FEM, whose convergence rates are 2 and 1 for displacement and strain energy, respectively [1], the FPM shows a better performance in the convergence of strain energy.

For the traditional FEM, when materials are nearly incompressible, the volume locking phenomenon occurs which provides a smaller displacement solution. However, from Figure 9, it is obvious that the FPM performs well when $\nu = 0.4999$, which means the FPM is a locking-free method for nearly incompressible materials.
In Eq. (3.8), the penalty parameter $\eta$ has a significant effect on the accuracy and stability of the FPM. To study its influence, Figure 10 is given to show the relations between relative errors and $\eta$. The penalty parameter is defined in the range of $\left(10^{-3} \times E, 10^3 \times E\right)$ with $161 \times 21$ Points distributed regularly in the beam.
A larger penalty coefficient can result in smaller jumps of displacements on internal boundaries, but also increase the condition number of the global matrix, thus affecting the precision of solutions [12]. From Figure 10, we can find that with the penalty parameter changing from a small value to a large one, the relative errors stay steady at first then increase gradually. Based on the fact that the penalty parameter needs to be large enough to maintain the stability of the method [11], it is better to define the parameter’s value within the range of $10^{-1} \times E$ to $10^{2} \times E$.

4.3 Ring with radial tension

In this part, a ring with radial tension is chosen as the model problem (shown in Figure 11(a)). The ring is defined as $\{(x, y) | a^2 \leq x^2 + y^2 \leq b^2\}$ and subjected to the uniform radial tension $p$. Since the ring is symmetric in geometry, we only simulate the upper right quarter of it (shown in Figure 11(b)). Symmetry boundary conditions are imposed on the left and bottom edges, which means $u_x = 0$, $t_y = 0$ for the left edge and $u_y = 0$, $t_x = 0$ for the bottom edge.
Specifically, we prescribe that $a = 1$, $b = 2$ and $p = 1$ for the ring. The exact solutions for stress and displacement are given in Eqs. (4.4) and (4.5) respectively, where $(r, \theta)$...
are the polar coordinates and $\theta$ is measured from the positive $x$-axis anticlockwise. The problem is solved in plane stress case with $E = 1 \times 10^5$ and $\nu = 0.3$. The penalty parameter $\eta = E$.

\[
\sigma_r = \frac{b^2}{b^2 - a^2} \left( 1 - \frac{a^2}{r^2} \right) p \\
\sigma_\theta = \frac{b^2}{b^2 - a^2} \left( 1 + \frac{a^2}{r^2} \right) p \\
\tau_{r\theta} = 0 \\
u_r = \frac{1}{E} \left( \frac{(1 - \nu)b^2}{b^2 - a^2} p + \frac{(1 + \nu)a^2b^2}{b^2 - a^2} \frac{1}{r} \right) \\
u_\theta = 0
\] (4.4)

To study the convergence of the FPM for displacement and strain energy in this problem, Points are distributed regularly in the domain with its number varying from $15 \times 11$, $15 \times 16$ to $15 \times 21$ (shown in Figure 12(a)). The relation between $h$, defined as the longest distance of two adjoining Points, and relative errors are given in Figure 13. where $R$ stands for the convergence rate.
Additionally, the numerical solutions of $u_r$ and $\sigma_r$ along $\theta = 0$ with $15 \times 21$ Points distributed regularly and randomly are shown in Figure 14(a) and (b) respectively. The irregular distribution of Points is illustrated in Figure 12(b). It can be seen that the FPM has approximated the displacement and stress satisfactorily in both the situations.
4.4 Infinite plate with a circular hole

In this part, we employ the FPM to simulate an infinite plate with a circular hole. As shown in Figure 15(a), the circular hole (radius equals \(a\)) locates at the plate’s center and the uniform tension \(p\) is imposed in the \(x\) direction at infinity. The exact solutions for stress and displacement are given in Eqs. (4.6) and (4.7) respectively.

\[
\sigma_r = p \left[ 1 - \frac{a^2}{r^2} \left( \frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3a^4}{2r^4} \cos 4\theta \right]
\]

\[
\tau_{xy} = p \left[ -\frac{a^2}{r^2} \left( \frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3a^4}{2r^4} \sin 4\theta \right]
\]
Based on the symmetry of the problem, we simplify the model in a local range: 
\( 0 \leq x, y \leq 2 \), as shown in Figure 15(b). Symmetry boundary conditions that \( u_x = 0 \), \( t_y = 0 \) at \( x = 0 \) and \( u_y = 0 \), \( t_x = 0 \) at \( y = 0 \), are imposed. Displacement boundary conditions are imposed on the upper side \((y = 2)\) and right side \((x = 2)\) according to Eq. (4.7). Free boundary conditions are imposed at \( r = 1 \).
Figure 15 (a). An infinite plate with a circular hole under remote tension

(b). The simplified model on fourfold symmetry

The problem is solved in plane stress case with $E = 1$ and $\nu$ prescribed as 0.3. The penalty coefficient $\eta$ is equal to $E$. 805 Points are distributed randomly in the domain (shown in Figure 16).

Figure 16. The Random distribution of Points

The numerical solution of $u_x$ and the corresponding error compared with the exact solution are given in Figure 17(a) and (b), respectively. Relative errors are $r_u = 3.83 \times 10^{-4}$ and $r_E = 1.84 \times 10^{-2}$.
Figure 17(a). The numerical solution of $u_x$

(b). The error of $u_x$

Additionally, the numerical solutions of $u_y$ and $\sigma_x$ along $x = 0$, as compared with the exact solutions are present in Figure 18(a) and (b), respectively.
Figure 18(a). The numerical solution of $u_y$ along $x = 0$

(b). The numerical solution of $\sigma_x$ along $x = 0$

It can be seen that the FPM can produce desirable solutions to simulate this stress concentration phenomenon.

4.5 Infinite plate with an elliptic hole

An infinite plate with an elliptic hole at its center and subjected to the uniform tension in $y$ direction remotely is studied in this section (shown in Figure 19(a)). The tension $p$ is prescribed as 1 in this paper. A $3 \times 3$ local part of the upper right quadrant of this plate is simulated, as shown in Figure 19(b), with the major axis $a = 2$ and the
minor axis $b = 1$. Exact solutions of this problem have been given in [14]. Because of
the symmetry of the model, boundary conditions that $u_x = 0$, $t_y = 0$ and $u_y = 0$, $t_x = 0$ are prescribed on the left and bottom edges, respectively. Free boundary
condition is imposed on the elliptic edge. For the upper and right edges, displacement
boundary conditions are imposed according to the exact solutions.
Figure 19(a). An infinite plate with an elliptic hole under remote tension
d(b). The simplified model on fourfold symmetry

The irregular distribution of 825 Points is illustrated in Figure 20. This problem is solved in plane stress case with $E = 1$ and $v = 0.3$. The penalty coefficient $\eta$ is equal to $E$. Under these conditions, we obtain that the relative errors derived by the FPM, $r_u = 1.74 \times 10^{-3}$ and $r_E = 3.69 \times 10^{-2}$.
The numerical solutions of \( \mu_x \) and \( \sigma_y \) along \( y = 0 \) as compared with the exact solutions are present in Figure 21(a) and (b), respectively.

Figure 21(a). The numerical solution of \( \mu_x \) along \( y = 0 \)

Figure 21(b). The numerical solution of \( \sigma_y \) at \( y = 0 \)

It can be seen that the stress concentration phenomenon is simulated satisfactorily in Figure 21 (b). The stress concentration factor at \( (x, y) = (2, 0) \) is almost equal to the
4.6 Infinite plate with a pre-existing crack

In this section, we apply the FPM to solve linear elastic fracture mechanics problems with a pre-existing crack. Specifically, model-I crack problems are considered. The analytical displacement and stress fields near the crack tip for the model-I crack problems are given in Eq. (4.8) and Eq. (4.9), respectively [15], where $(r, \theta)$ are polar coordinates measured from the crack tip and $K_I$ is the model-I stress intensity factor.

\[
\begin{align*}
\begin{bmatrix} u_x \\ u_y \end{bmatrix} &= K_I \frac{r}{2\mu \sqrt{2\pi}} \begin{bmatrix} \cos \frac{\theta}{2} \left[ \kappa - 1 + 2 \sin^2 \frac{\theta}{2} \right] \\ \sin \frac{\theta}{2} \left[ \kappa + 1 - 2 \cos^2 \frac{\theta}{2} \right] \end{bmatrix} \\
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \begin{bmatrix} 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{bmatrix}
\end{align*}
\]

(4.8)

where \( \kappa = \frac{3 - \nu}{1 + \nu} \quad \mu = \frac{E}{2(1 + \nu)} \)

(4.9)

As shown in Figure 22, a single edge-cracked square plate with width $= b$ and crack length $= a$ is studied. Displacement boundary conditions are imposed on its four sides according to Eq. (4.8) with $K_I$ prescribed as 1. This problem is analyzed in plane stress case and we prescribe that $b = 2a = 10$, $E = 1$, $\nu = 0.3$. The penalty coefficient $\eta$ is set to be equal to $E$. 
Figure 22. An edge-cracked square plate

40×40 Points are scattered regularly in the domain (shown in Figure 23). To simulate a pre-existing crack in the FPM, we define that all the boundaries which coincide with the crack are free boundaries. Besides, for those Points located on one side of the crack, we prescribe that they will not be included in the supports of Points located on the other side of the crack, in other words, they will not interact with each other when computing shape functions. For example, in Figure 24, we assume a crack exists at boundaries $\Gamma_{26}$, $\Gamma_{15}$ and $\Gamma_{47}$. Therefore, they are defined as free boundaries. The support of Point 1 only contains Points 2,3,4 without Points 5 and the support of Point 5 does not include Point 1.
The numerical solutions of $u_y$ and $\sigma_y$ along $x = 5 + 0.5h$ ($h$ is the distance between 2 adjoining Points) as compared with exact solutions are demonstrated in Figure 25(a) and (b), respectively. It can be seen that the numerical solutions obtained by the FPM can achieve excellent accuracy.
Figure 25(a). The numerical solution of $u_y$ along $x = 20.5h$ 

(b). The numerical solution of $\sigma_y$ along $x = 20.5h$

Besides, we further evaluate the mode-I stress intensity factor $K_I$ through calculating the $J$-integral. The contour for the $J$-integral is shown in Figure 26 while $w = 2.5h$ and $l = 4.5h$. The solution of $K_I$ through the $J$-integral is shown in Table I, which is very accurate as compared with the exact solution.

Table I. Stress intensity factor for the pre-existing edge crack problem

<table>
<thead>
<tr>
<th>$K_I$ (J-integral)</th>
<th>$K_{exact}$</th>
<th>Error</th>
</tr>
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5. CONCLUSIONS

In this paper, we have employed the FPM to solve linear elasticity problems. Simple, Local, polynomial, discontinuous and Point-based trial and test functions are constructed in the FPM and Numerical Flux Corrections are introduced to surmount the inconsistency caused by the discontinuity. Eventually, in the FPM, based on the Galerkin method, a sparse, symmetric and positive definitive global stiffness matrix can be derived by assembling all the local Point Stiffness Matrices. Since trial and test functions are polynomial, all integrals in the Galerkin weak form can be easily and exactly computed in the FPM. From the examples provided in Section 4, we demonstrate the convergence, robustness, consistency and high accuracy of the FPM in elasticity problems. Volume locking does not occur in the FPM for nearly incompressible materials and stress concentration phenomena are also simulated exactly.

Besides all these advantages we mentioned, the FPM also possesses the ability to model either pre-existing cracks or physically forming cracks, rupture and fragmentation exactly and easily. In the FEM, when we introduce a crack between two adjoining elements, we have to divide them into unconnected elements. Adding new points is usually inevitable for this remeshing work [16]. Since the number of DOFs changes and the connectivity of elements is modified in each calculation step, the dimensions of the global stiffness matrix and the load vector varies constantly. We can see that simulating cracks which either form physically or propagate is such a complex task for the FEM. However, due to the discontinuity of the FPM, it will cost
less effort to model a crack between two adjoining subdomains. If a crack emerges at an internal boundary or rupture forms physically, we only need to redefine it as a free boundary and change relevant shape functions. According to Eq. (3.8), we only need to delete three boundary integrals about it on the left side while the right side stays the same. Just a slight modification of the global stiffness matrix can complete this operation without adding new Points.

From the above analysis, we can conclude that the FPM is an accurate, robust and consistent meshless Galerkin weak-form method with less computational costs. It also possesses great potential to solve extreme problems of rupture, and fragmentation. The simulation of crack propagation and the phenomena of rupture and fragmentation by the FPM will be discussed in our forthcoming paper.

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