Nearly Exact and Highly Efficient Elastic-Plastic Homogenization and/or Direct Numerical Simulation of Low-Mass Metallic Systems with Architected Cellular Microstructures

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Abstract

Additive manufacturing has enabled the fabrication of lightweight materials with intricate cellular architectures. These materials become interesting due to their properties which can be optimized upon the choice of the parent material and the topology of the architecture, making them appropriate for a wide range of applications including lightweight aerospace structures, energy absorption, thermal management, metamaterials, and bioscaffolds. In this paper we present, the simplest initial computational framework for the analysis, design, and topology optimization of low-mass metallic systems with architected cellular microstructures. A very efficient elastic-plastic homogenization of a repetitive Representative Volume Element (RVE) of the micro-lattice is proposed. Each member of the cellular microstructure undergoing large elastic-plastic deformations is modeled using only one nonlinear three-dimensional (3D) beam element with 6 degrees of freedom (DOF) at each of the 2 nodes of the beam. The nonlinear coupling of axial, torsional, and bidirectional-bending deformations is considered for each 3D spatial beam element. The plastic-hinge method, with arbitrary locations of the hinges along the beam, is utilized to study the effect of plasticity. We derive an explicit expression for the tangent stiffness matrix of each member of the cellular microstructure, using a mixed variational principle in the updated Lagrangian co-rotational reference frame. To solve the incremental tangent stiffness equations, a newly proposed Newton Homotopy method is employed. In contrast to the Newton’s method and the Newton-Raphson iteration method which require the inversion of the Jacobian matrix, our

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1 This paper is written in honor of Dr. Pedro Marçal, a true pioneer in Computational Mechanics, on the occasion of the celebration of his Lifetime Achievements at ICCES17, June 2017, in Madeira, Portugal

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Homotopy methods avoid inverting it. We have developed a code called CELLS/LIDS [CELLular Structures/Large Inelastic DeformationS] providing the capabilities to study the variation of the mechanical properties of the low-mass metallic cellular structures by changing their topology. Thus, due to the efficiency of this method we can employ it for topology optimization design, and for impact/energy absorption analyses.

**Keywords:** Architected Cellular Microstructures; Large Deformations; Plastic Hinge Approach; Nonlinear Coupling of Axial, Torsional, Bidirectional-Bending Deformations; Mixed Variational Principle; Homotopy Methods.

1. **Introduction**

A lot of natural structures such as hornbill bird beaks and bird wing bones are architected cellular materials to provide optimum strength and stiffness at low density. Humankind, over the past few years, has also fabricated cellular materials with more complex architectures in comparison with previously developed synthetic materials like open-cell metallic foams and honeycombs [1]. Properties of these cellular structures are determined based on their parent materials and the topology of the microarchitecture. Additive manufacturing technologies and progress in three-dimensional (3D) printing techniques enable the design of materials and structures with complex cellular microarchitectures, optimized for specific applications. In fact, one of the most interesting characteristics of cellular structures with pore network is that they can be designed with desirable properties, making them appropriate for lightweight structures, metamaterials, energy absorption, thermal management, and bioscaffolds [2]. For example, efforts are under way to fabricate bioscaffolds to repair and replace tissue, cartilage, and bone [3-6]. These architected materials should be fabricated in such a way that they can meet biocompatibility requirements in addition to the mechanical properties of the tissues at the site of implantation. Therefore, presentation of a highly efficient computational method to predict and optimize the mechanical properties of such structures is of interest. Herein, we present a nearly exact and highly efficient computational method to predict the elastic-plastic homogenized mechanical properties of low-mass metallic systems with architected cellular microstructures. The framework of the methods presented in this paper is also germane to the analysis under static as well as impact loads, design, and topology optimization of cellular solids.
The ultralow-density metallic cellular micro-lattices have been recently fabricated at HRL Laboratories [7, 8], suitable for thermal insulation, battery electrodes, catalyst supports, and acoustic, vibration, or shock energy damping [9-14]. They produced nickel cellular micro-lattices, consisting of hollow tubular members, by preparing a sacrificial polymeric template for electroless Ni deposition and, then, chemically etching the sacrificial template [7]. Using this process they fabricated novel nickel-based micro-lattice materials with structural hierarchy spanning three different length scales, nm, µm, and mm. They obtained a 93% Ni–7% P composition by weight for micro-lattices, using energy dispersive spectroscopic analysis. They employed quasi-static axial compression experiments to measure macroscopic mechanical properties such as Young’s moduli of nickel micro-lattices. The load, \( P \) was measured by SENSOTEC load cells, and the displacement, \( \delta \) was measured using an external LVDT for modulus extraction. Strain-stress curves were obtained based on engineering stress and strain defined, respectively, as \( \sigma = P/A_0 \) and \( \varepsilon = \delta/L_0 \). \( A_0 \) and \( L_0 \) are the initial cross-sectional area and length of the sample, respectively. Moreover, Salari-Sharif and Valdevit [15] extracted the Young’s modulus of a series of nickel ultralight micro-lattices by coupling experimental results obtained using Laser Doppler Vibrometry with finite element (ABAQUS) simulations. They [15] fabricated a sandwich configuration by attaching carbon/epoxy face sheets as the top and bottom layers of the ultralight nickel hollow micro-lattice thin film [7]. They [15] detected the resonant frequencies by scanning laser vibrometry and ABAQUS simulations and extracted the relation between Young’s modulus and the natural frequencies. Then, the effective Young’s moduli of samples were obtained in the direction normal to the face sheets [15]. It is worth noting that for finite element (FE) modeling, a Representative Volume Element (RVE) consisting of only four members of the cellular micro-lattice with at least ten thousands of 4-node shell FE\s was employed [15], resulting in at least 10,000 nodes and, thus, 60,000 degrees of freedom (DOF). We should emphasize that in our methodology each member can be modeled by a single spatial beam element. In other words, to perform a 4-member RVE analysis, we use only 4 spatial beam elements and 5 nodes, with a total of 30 DOF and, thus, at least 2000 times less DOF than in Ref. [15]. Since the cost of computation in a FE nonlinear analysis varies as the \( n/h \) power (\( n \) between 2 and 3) of the number of DOF, it is clear that we seek to present a far more efficient analysis procedure than any available commercial software. This provides the capability to simulate the cellular microstructure using repetitive
RVEs, consisting of an arbitrary number of members, enabling a very efficient homogenization and/or Direct Numerical Simulation (DNS) of a Cellular Macrostructure.

In addition, Schaedler et al. [7] and Torrents et al. [8] showed experimentally that nickel-phosphorous cellular micro-lattices undergo large effective compressive strains through extensive rotations about remnant node ligaments. Unfortunately, there are no computational studies in the literature on the large-deformation elastic-plastic analysis of such metallic cellular structures, which is the major concern of the present study, although, there is a vast variety of studies on the large deformation analysis of space-frames [16-19], form the era of large space structures for use in the outer space. In the realm of the analysis of space-frames, numerous studies have been devoted to derive an explicit expression for the tangent stiffness matrix of each element, accounting for arbitrarily large rigid rotations, moderately large non-rigid point-wise rotations, and the stretching-bending coupling [20-24]. Some researchers employed displacement-based approaches using variants of a lagrangian for either geometrically or materially nonlinear analyses of frames [20-22]. Kondoh et al. [23] extended the displacement approach to evaluate explicitly the tangent stiffness matrix without employing either numerical or symbolic integration for a beam element undergoing large deformations. Later, Kondoh and Atluri [24] presented a formulation on the basis of assumed stress resultants and stress couples, satisfying the momentum balance conditions in the beam subjected to arbitrarily large deformations.

In order to study the elastic-plastic behavior of cellular members undergoing large deflections, we employ the mechanism of plastic hinge developed by Hodge [25], Ueda et al. [26], and Ueda and Yao [27]. In this mechanism, plastic hinge can be generated at any point along the member as well as its end nodes, everywhere the plasticity condition in terms of generalized stress resultants is satisfied. It is worthwhile to mention that contours of the Von Mises stress given in Ref. [15] for the 4-member RVE with PBCs show very high concentration of stress at the junction of four members. The stress contours were obtained based on linear elastic FE simulations [15]. Therefore, it clearly mandates an elastic-plastic analysis, which is undertaken in the present study. A complementary energy approach in conjunction with plastic-hinge method has been previously utilized to study elasto-plastic large deformations of space-framed structures [24, 28]. Shi and Atluri [28] derived the linearized tangent stiffness matrix of each finite element in the co-rotational reference frame in an explicit form and showed that this approach based on assumed stresses is
simpler in comparison with assumed-displacement type formulations. In contrast to Ref. [28] which presents the linearized tangent stiffness, the current work derives explicitly the tangent stiffness matrix under the nonlinear coupling of axial, torsional, and bidirectional-bending deformations.

One of the extensively employed approaches in the literature for the analysis of nonlinear problems with large deformations or rotations is based on variational principles. For instance, Cai et al. [29, 30] utilized the primal approach as well as mixed variational principle [31] in the updated Lagrangian co-rotational reference frame to obtain an explicit expression for the tangent stiffness matrix of the elastic beam elements. They [30] showed that the mixed variational principle in comparison to the primal approach, which requires $C^1$ continuous trial functions for displacements, needs simpler trial functions for the transverse bending moments and rotations. In fact, they [30] assumed linear trial functions within each element and obtained much simpler tangent stiffness matrices for each element than those previously presented in the literature [22, 23, 32]. While Ref. [30] considered only a few macro members, our analysis is applicable to metallic cellular micro-lattices with an extremely large number of repetitive RVEs. Since plasticity and buckling occur in many members of the micro-lattice, we found that the Newton-type algorithm that has been utilized in Ref. [30] fails. In the present study we discovered that only our Newton homotopy method provides convergent solutions in the presence of the plasticity and buckling in a large number of members of the micro-lattice.

To solve tangent stiffness equations, we use a Newton Homotopy method recently developed to solve a system of fully coupled nonlinear algebraic equations (NAEs) with as many unknowns as desired [33, 34]. By using these methods, displacements of the equilibrium state are iteratively solved without the inversion of the Jacobian (tangent stiffness) matrix. Newton Homotopy methods are advantageous, in particular, when the effect of plasticity is going to be studied. Since, it is well-known that the simple Newton’s method as well as the Newton-Raphson iteration method require the inversion of the Jacobian matrix, which fail to pass the limit load as the Jacobian matrix becomes singular, and require arc-length methodology which are commonly used in commercial off-the-shelf software such as ABAQUS. Furthermore, Homotopy methods are useful in the following cases, when the system of algebraic equations is very large in size, when the solution is
sensitive to the initial guess, and when the system of nonlinear algebraic equations is either over- or under-determined \[33, 34\].

The paper is organized as follows. The theoretical background including the nonlinear coupling of axial, torsional, and bidirectional-bending deformations for a typical cellular member under large deformation, mixed variational principle in the co-rotational updated Lagrangian reference frame, plastic hinge method, and equation-solving algorithm accompanying Newton Homotopy methods are summarized in Section 2. Section 3 is devoted to the validation of our methodology; a three-member rigid-knee frame, the Williams’ toggle problem, and a right-angle bent including the effect of plasticity are compared with the corresponding results given in the literature. Section 4 analyzes the mechanical behavior of two different cellular micro-lattices subjected to tensile, compressive, and shear loading. Throughout this section, it is shown that our calculated results (Young’s modulus and yield stress) under compressive loading are very comparable with those measured experimentally by Schaedler et al. \[7\] and Torrents et al. \[8\]. Moreover, the progressive development of plastic hinges in cellular micro-lattice as well as its deformed structure are presented. Finally, a summary and conclusion are given in Section 5. Appendices A, B, C, and D follow.

2. Theoretical Background

Throughout this section, concepts employed to derive nearly exact and highly efficient elastic-plastic homogenization of low-mass metallic systems with architected cellular microstructures are given. Considering the nonlinear coupling of axial, torsional, and bidirectional-bending deformations, strain-displacement and stress-strain relations in the updated Lagrangian co-rotational frame are described in Section 2.1. Section 2.2 is devoted to derive an explicit expression for the tangent stiffness matrix of each member of the cellular structure, accounting for large rigid rotations, moderate relative rotations, the bending-twisting-stretching coupling and elastic-plastic deformations. Solution algorithm is also given in Section 2.3.
2.1 The Nonlinear Coupling of Axial, Torsional, and Bidirectional-Bending Deformations for a Spatial Beam Element with a Tubular Cross-Section

A typical 3D member of a cellular structure is considered, spanning between nodes 1 and 2 as illustrated in Fig. 1. The element is initially straight with arbitrary cross section and is of the length $l$ before deformation. As seen from Fig. 1, three different coordinate systems are introduced: (1) the global coordinates (fixed global reference), $\mathbf{x}_i$ with the orthonormal basis vectors $\mathbf{e}_i$, (2) the local coordinates for the member in the undeformed state, $\bar{\mathbf{x}}_i$ with the orthonormal basis vectors $\bar{\mathbf{e}}_i$, and (3) the local coordinates for the member in the deformed state (current configuration), $\mathbf{x}_i$ with the orthonormal basis vectors of $\mathbf{e}_i$, ($i = 1, 2, 3$).

Local displacements at the centroidal axis of the deformed member along $\mathbf{e}_1$-directions are denoted as $u_{10}, (i = 1, 2, 3)$. Rotation about $x_1$-axis (angle of twist) is denoted by $\tilde{\theta}$, and those about $x_i$-axes, $i = 2, 3$, (bend angle) are denoted by $\theta_{10}, i = 2, 3$, respectively. It is assumed that nodes 1 and 2 of the member undergo arbitrarily large displacements, and rotations between the undeformed state of the member and its deformed state are arbitrarily finite. Moreover, it is supposed that local displacements in the current configuration ($\mathbf{x}_i$ coordinates system) are

![Fig. 1. Nomenclature for the reference frames corresponding to the global, undeformed, and deformed states.](image)

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moderate and the axial derivative of the axial deflection at the centroid, \( \frac{\partial u_{10}}{\partial x_1} \) is small in comparison with that of the transverse deflections at the centroid, \( \frac{\partial u_{i0}}{\partial x_1} \) \( (i = 2, 3) \).

We examine large deformations for a cylindrical member with an unsymmetrical cross section around \( x_2 \)- and \( x_3 \)-axes and constant cross section along \( x_1 \)-axis subjected to torsion around \( x_1 \)-axis, \( T \) and bending moments \( M_2 \) and \( M_3 \) around \( x_2 \)- and \( x_3 \)-axes, respectively. It is assumed that the warping displacement due to the torsion \( T \) is independent of \( x_1 \) variable, \( u_{1T}(x_2, x_3) \), the axial displacement at the centroid is \( u_{10}(x_1) \), and the transverse bending displacements at the origin \( (x_2 = x_3 = 0) \) are \( u_{20}(x_1) \) and \( u_{30}(x_1) \) along \( e_2 \)- and \( e_3 \)-directions, respectively. The reason for the consideration of the nonlinear axial, torsional, and bidirectional-bending coupling for each spatial beam element is the frame-like behavior of these cellular metallic micro-lattices. The scanning electron microscopy (SEM) images of micro-lattices given by Torrents et al. [8] notify the formation of partial fracture at nodes (for a micro-lattice with \( t = 500 \text{ nm} \)), localized buckling (for a micro-lattice with \( t = 1.3 \mu m \)), and plastic hinging at nodes (for a micro-lattice with \( t = 26 \mu m \)). Therefore, 3D displacement field for each spatial beam element in the current configuration is considered as follows using the normality assumption of the Bernoulli-Euler beam theory,

\[
\begin{align*}
  u_1(x_1, x_2, x_3) &= u_{1T}(x_2, x_3) + u_{10}(x_1) - x_2 \frac{\partial u_{20}(x_1)}{\partial x_1} - x_3 \frac{\partial u_{30}(x_1)}{\partial x_1}, \\
  u_2(x_1, x_2, x_3) &= u_{20}(x_1) - \hat{\theta} x_3, \\
  u_3(x_1, x_2, x_3) &= u_{30}(x_1) + \hat{\theta} x_2.
\end{align*}
\]

The Green-Lagrange strain components in the updated Lagrangian co-rotational frame \( e_i \), \( (i = 1, 2, 3) \) are

\[
\varepsilon_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \right),
\]

where the index notation \( \varepsilon_{ij} \) denotes \( \partial \varepsilon / \partial x_i \) and \( k \) is a dummy index. Replacement of Eqs. (1) into Eq. (2) results in the following strain components,

\[
\begin{align*}
  \varepsilon_{11} &= u_{1,1} + \frac{1}{2}(u_{1,1})^2 + \frac{1}{2}(u_{2,1})^2 + \frac{1}{2}(u_{3,1})^2 \approx u_{10,1} + \frac{1}{2}(u_{20,1})^2 + \frac{1}{2}(u_{30,1})^2 - x_2u_{20,11} - x_3u_{30,11}, \\
  \varepsilon_{22} &= u_{2,2} + \frac{1}{2}(u_{1,2})^2 + \frac{1}{2}(u_{2,2})^2 + \frac{1}{2}(u_{3,2})^2 = \frac{1}{2}(u_{1T,2} - u_{20,1})^2 + \frac{1}{2}\hat{\theta}^2 \approx 0,
\end{align*}
\]
\[ \varepsilon_{33} = u_{3,3} + \frac{1}{2}(u_{1,3})^2 + \frac{1}{2}(u_{2,3})^2 + \frac{1}{2}(u_{3,3})^2 \approx 0, \quad (3) \]

\[ \varepsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) + \frac{1}{2}u_{3,1} u_{3,2} \approx \frac{1}{2}(u_{1T,2} - \hat{\theta}_1 x_3), \]

\[ \varepsilon_{13} = \frac{1}{2}(u_{1,3} + u_{3,1}) + \frac{1}{2}u_{2,1} u_{2,3} \approx \frac{1}{2}(u_{1T,3} + \hat{\theta}_1 x_2), \]

\[ \varepsilon_{23} = \frac{1}{2}(u_{2,3} + u_{3,2}) + \frac{1}{2}u_{1,2} u_{1,3} \approx 0. \]

By defining the following parameters:

\[ \theta = \hat{\theta}_1, \]

\[ \mathcal{N}_{22} = -u_{20,11}, \]

\[ \mathcal{N}_{33} = -u_{30,11}, \]

\[ \varepsilon_{11}^0 = u_{10,1} + \frac{1}{2}(u_{20,1})^2 + \frac{1}{2}(u_{30,1})^2 = \varepsilon_{11}^0 L + \varepsilon_{11}^0 NL, \]

and employing them into the Eqs. (3), strain components can be rewritten as below:

\[ \varepsilon_{11} = \varepsilon_{11}^0 + x_2 \mathcal{N}_{22} + x_3 \mathcal{N}_{33}, \]

\[ \varepsilon_{12} = \frac{1}{2}(u_{1T,2} - \Theta x_3), \]

\[ \varepsilon_{13} = \frac{1}{2}(u_{1T,3} + \Theta x_2), \]

\[ \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{23} = 0, \quad (5) \]

and as follows in the matrix notation,

\[ \varepsilon = \varepsilon^L + \varepsilon^N, \quad (6) \]

where

\[ \varepsilon^L = \begin{bmatrix} \varepsilon_{11}^L \\ \varepsilon_{12}^L \\ \varepsilon_{13}^L \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}(u_{1T,2} - \Theta x_3) \\ \frac{1}{2}(u_{1T,3} + \Theta x_2) \end{bmatrix}, \quad (7) \]

\[ \varepsilon^N = \begin{bmatrix} \varepsilon_{11}^N \\ \varepsilon_{12}^N \\ \varepsilon_{13}^N \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(u_{20,1})^2 + \frac{1}{2}(u_{30,1})^2 \\ 0 \\ 0 \end{bmatrix}. \quad (8) \]

Similarly, the member generalized strains are determined in the matrix form as

\[ E = E^L + E^N = \begin{bmatrix} \varepsilon_{11}^0 \\ \mathcal{N}_{22} \\ \mathcal{N}_{33} \\ \Theta \end{bmatrix}, \quad (9) \]

where \( E^L = \begin{bmatrix} u_{10,1} & -u_{20,11} & -u_{30,11} & \hat{\theta}_1 \end{bmatrix}^T \) and \( E^N = \begin{bmatrix} \frac{1}{2}(u_{20,1})^2 + \frac{1}{2}(u_{30,1})^2 & 0 & 0 & 0 \end{bmatrix}^T \).
We consider for now that the member material is linear elastic, thus the total stress tensor (the second Piola-Kirchhoff stress tensor), $\mathbf{S}$ of which is calculated as

$$\mathbf{S} = \mathbf{S}^1 + \mathbf{\tau}^0,$$  \hspace{1cm} (10)

here, $\mathbf{\tau}^0$ is the pre-existing Cauchy stress tensor, and $\mathbf{S}^1$ is the incremental second Piola-Kirchhoff stress tensor in the updated Lagrangian co-rotational frame $\mathbf{e}_i$ given by

$$S_{11}^1 = E \varepsilon_{11},$$

$$S_{12}^1 = 2\mu \varepsilon_{12},$$

$$S_{13}^1 = 2\mu \varepsilon_{13},$$

$$S_{22}^1 = S_{33}^1 = S_{23}^1 \approx 0,$$ \hspace{1cm} (11)

in which $\mu$ is the shear modulus, $\mu = \frac{E}{2(1+\nu)}$, $E$ is the elastic modulus, and $\nu$ is the Poisson’s ratio.

Using equations (5) and (11), the generalized nodal forces for the member shown in Fig. 1 subjected to the twisting and bending moments are calculated as

$$N_{11} = \int_A S_{11}^1 dA = E (A \varepsilon_{11}^0 + I_2 \kappa_{22} + I_3 \kappa_{33}),$$

$$M_{22} = \int_A S_{11}^1 x_2 dA = E (I_2 \varepsilon_{11}^0 + I_2 \kappa_{22} + I_2 \kappa_{33}),$$

$$M_{33} = \int_A S_{11}^1 x_3 dA = E (I_3 \varepsilon_{11}^0 + I_3 \kappa_{22} + I_3 \kappa_{33}),$$

$$T = \int_A (S_{13}^1 x_2 - S_{12}^1 x_3) dA = \mu I_{rr} \Theta,$$ \hspace{1cm} (12)

where $A$ is the area of the cross section, $I_i$ and $I_{ij}$ ($i,j = 2,3$) are the first moment and the second moment of inertia of the cross section, respectively, $I_2 = \int_A x_2 dA$, $I_3 = \int_A x_3 dA$, $I_{22} = \int_A x_2^2 dA$, $I_{33} = \int_A x_3^2 dA$, $I_{23} = \int_A x_2 x_3 dA$, and $I_{rr}$ is the polar moment of inertia, $I_{rr} = \int_A (x_2^2 + x_3^2) dA$. Using the element generalized strains, $\mathbf{E}$, the element generalized stresses, $\mathbf{\sigma}$ are also determined in the matrix form as

$$\mathbf{\sigma} = \mathbf{D}\mathbf{E},$$ \hspace{1cm} (13)

in which,

$$\mathbf{\sigma} = \begin{bmatrix} N_{11} \\ M_{22} \\ M_{33} \\ T \end{bmatrix},$$ \hspace{1cm} (14)
\[ D = \begin{bmatrix} EA & EI_2 & EI_3 & 0 \\ EI_2 & EI_{22} & EI_{23} & 0 \\ EI_3 & EI_{23} & EI_{33} & 0 \\ 0 & 0 & 0 & \mu I_{rr} \end{bmatrix} \]  \quad (15)

2.2 Explicit Derivation of Tangent Stiffness Matrix Undergoing Large Elasto-Plastic Deformation

Throughout this section, mixed variational principle in the co-rotational updated Lagrangian reference frame and a plastic-hinge method are employed to obtain explicit expressions for the tangent stiffness matrix of each member shown in Fig. 1. Stiffness matrix is calculated for each member by accounting for large rigid rotations, moderate relative rotations, the nonlinear coupling of axial, torsional, and bidirectional-bending deformations, and the effect of plasticity. The functional of the mixed variational principle in the co-rotational updated Lagrangian reference frame and the trial functions for the stress and displacement fields within each element are given in Section 2.2.1. Plastic analysis using plastic hinge method is described in Section 2.2.2. The explicit expression of the stiffness matrix in the presence of plasticity for each cellular member is also presented in Section 2.2.3.

2.2.1 Mixed Variational Principle in the Co-rotational Updated Lagrangian Reference Frame

Consideration of \( S_{ij}^1 \) and \( u_i \), respectively, as the components of the incremental second Piola-Kirchhoff stress tensor and the displacement field in the updated Lagrangian co-rotational frame, the functional of the mixed variational principle in the same reference frame with orthonormal basis vectors \( e_i \) is obtained as

\[ \mathcal{H}_R = \int_V \left\{ -B[S_{ij}^1] + \frac{1}{2} e^0_{ij} u_{k,i} u_{k,j} + \frac{1}{2} S_{ij} (u_{i,j} + u_{j,i}) - \rho b_i u_i \right\} dV - \int_{S_\sigma} \bar{T}_i u_idS, \]  \quad (16)

where, \( V \) is the volume in the current co-rotational reference state, \( S_\sigma \) is the part of the surface with the prescribed traction, \( \bar{T}_i = \bar{T}_{i0} + \bar{T}_{i1}^1 (i = 1, 2, 3) \) are the components of the boundary tractions, and \( b_i = b^0_i + b_i^1 \) (\( i = 1, 2, 3 \)) are the components of body forces per unit volume in the current...
configuration. The displacement boundary conditions prescribed at the surface \( S_u \) are also considered as \( \bar{u}_i \) (\( i = 1, 2, 3 \)), assumed to be satisfied a priori. Eq. (16) is a general variational principle governing stationary conditions, which with respect to variations \( \delta S^1_{ij} \) and \( \delta u_i \) results in the following incremental equations in the co-rotational updated Lagrangian reference frame,

\[
\frac{\partial B}{\partial S^1_{ij}} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right),
\]

\( \text{(17)} \)

\[
[S^1_{ij} + \tau^0_{ik} u_{k,j}]_{ij} + \rho b^1_i = -\tau^0_{ij,j} - \rho b^0_i,
\]

\( \text{(18)} \)

\[
n_j [S^1_{ij} + \tau^0_{ik} u_{k,j}] - \bar{T}^1_i = -n_j \tau^0_{ij} + \bar{T}^0_i \quad \text{on} \ S_\sigma,
\]

\( \text{(19)} \)

where \( n \) is the outward unit normal on the surface \( S_\sigma \). For a group of members, \( V_m \) (\( m = 1, 2, \ldots, N \)) with common surfaces \( \rho_m \), Eq. (16) can be written as

\[
\mathcal{H}_R = \sum_m \left\{ \int_{S_{\sigma,m}} -B[S^1_{ij}] + \frac{1}{2} \tau^0_{ik} u_{k,i} u_{j,k} + \frac{1}{2} S^1_{ij} (u_{i,j} + u_{j,i}) - \rho b_i u_i \right\} dV - \int_{S_{\sigma,m}} \bar{T}_i u_i dS \quad m = 1, 2, \ldots, N.
\]

\( \text{(20)} \)

If the trial function \( u_i \) and the test function \( \partial u_i \) for each member, \( V_m \) (\( m = 1, 2, \ldots, N \)) are chosen in such a way that the inter-element displacement continuity condition is satisfied at \( \rho_m \) a priori, then stationary conditions of \( \mathcal{H}_R \) for a group of finite elements lead to

\[
\frac{\partial B}{\partial S^1_{ij}} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \quad \text{in} \ V_m,
\]

\( \text{(21)} \)

\[
[S^1_{ij} + \tau^0_{ik} u_{k,j}]_{ij} + \rho b^1_i = -\tau^0_{ij,j} - \rho b^0_i \quad \text{in} \ V_m,
\]

\( \text{(22)} \)

\[
[n_i (S^1_{ij} + \tau^0_{ik} u_{k,j})]^+ + [n_i (S^1_{ij} + \tau^0_{ik} u_{k,j})]^- = -[n_i \tau^0_{ij}]^+ - [n_i \tau^0_{ij}]^- \quad \text{at} \ \rho_m,
\]

\( \text{(23)} \)

\[
n_j [S^1_{ij} + \tau^0_{ik} u_{k,j}] - \bar{T}^1_i = -n_j \tau^0_{ij} + \bar{T}^0_i \quad \text{on} \ S_{\sigma,m},
\]

\( \text{(24)} \)

here, + and − denote the outward and inward quantities at the interface, respectively. The continuity of the displacement at the common interface, \( \rho_m \) between elements, is determined by

\[
u_i^+ = u_i^- \quad \text{on} \ \rho_m.
\]

\( \text{(25)} \)

Applying Eqs. (5) and (13) into the Eq. (20) and integrating over the cross section area of each element, Eq. (20) is rewritten as

\[
\mathcal{H}_R = \sum_{m=1}^{N} \left\{ \int_{l} \left( -\frac{1}{2} \sigma^T D^{-1} \sigma \right) dl + \int_{l} N^0_{11} \frac{1}{2} (u_{20}^2 + u_{30}^2) dl + \int_{l} (\bar{N}_{11} \varepsilon_{11} + \bar{M}_{22} \kappa_{22} + \bar{M}_{33} \kappa_{33} + \bar{T} \Theta) dl - \bar{Q} q \right\},
\]

\( \text{(26)} \)

in which \( \sigma^0 = [N^0_{11}, \ M^0_{22}, \ M^0_{33}, \ T^0]^T \) is the initial member generalized stress in the co-rotational reference coordinates \( e_i \), \( \bar{\sigma} = \sigma + \sigma^0 = [\bar{N}_{11}, \ \bar{M}_{22}, \ \bar{M}_{33}, \ \bar{T}]^T \) is the total member
generalized stress in the coordinates \( e_i \), \( \vec{Q} \) is the nodal external generalized force vector in the global reference frame \( \vec{e}_i \), and \( q \) is the nodal generalized displacement vector in the coordinates \( \vec{e}_i \). Eq. (26) can be simplified by applying integration by parts to the third integral term on the right-hand side of the equation. More details as how to compute integration by parts are given in Appendix A. Stationary conditions for \( \mathcal{H}_R \) given in Eq. (26) result in

\[
D^{-1} \sigma = E,
\]

\[
\hat{N}_{11,1} = 0 \quad \text{in } V_m,
\]

\[
\hat{\tau}_{1,1} = 0 \quad \text{in } V_m,
\]

\[
\hat{M}_{22,11} + [N_{11}^0 u_{20,1}]_1 = 0 \quad \text{in } V_m,
\]

\[
\hat{M}_{33,11} + [N_{11}^0 u_{30,1}]_1 = 0 \quad \text{in } V_m,
\]

and the nodal equilibrium equations are obtained from the following relation,

\[
\sum_{m=1}^{N} \left\{ \hat{N}_{11} \delta u_{10}\big|_0^l + \hat{M}_{22} \delta u_{20}\big|_0^l - \hat{M}_{22} \delta u_{20}\big|_0^l + \hat{M}_{33} \delta u_{30}\big|_0^l - \hat{M}_{33} \delta u_{30}\big|_0^l + \hat{T} \delta \theta\big|_0^l + (N_{11}^0 u_{20,1}) \delta u_{20}\big|_0^l + (N_{11}^0 u_{30,1}) \delta u_{30}\big|_0^l - \vec{Q} \delta q \right\} = 0. \tag{28}
\]

Herein, the trial functions for the stress and displacement fields within each member, \( V_m (m = 1, 2, \cdots, N) \) are discussed. We assume that the components of the member generalized stress, \( \sigma \) obey the following relation

\[
\sigma = P \beta,
\]

where,

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 + \frac{x_1}{l} & -\frac{x_1}{l} & 0 & 0 & 0 \\
0 & 0 & 1 - \frac{x_1}{l} & \frac{x_1}{l} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
\beta = \begin{bmatrix}
n & 1 m_3 & 2 m_3 & 1 m_2 & 2 m_2 & m_1
\end{bmatrix}^T.
\]

Similarly, the components of the initial member generalized stress, \( \sigma^0 \) are determined

\[
\sigma^0 = P \beta^0,
\]

where,

\[
\beta^0 = \begin{bmatrix}
n^0 & 1 m_3^0 & 2 m_3^0 & 1 m_2^0 & 2 m_2^0 & m_1^0
\end{bmatrix}^T.
\]

Note that, \( ^i m_2 (\ ^i m_2^0) \) and \( ^i m_3 (\ ^i m_3^0) \) are, respectively, bending moments (initial ones) around the \( x_2 \)- and \( x_3 \)-axes at the \( i \)th node. \( n (n^0) \) and \( m_1 (m_1^0) \) are the (initial) axial force and the (initial)
twisting moment along the element, respectively. Therefore, the incremental internal nodal force vector $\mathbf{B}$ for the element shown in Fig. 1, with nodes 1 and 2 at the ends, can be expressed as

$$\mathbf{B} = \begin{bmatrix} N_1 & m_1 & m_2 & m_3 & N_2 & m_1 & m_2 & m_3 \end{bmatrix}^T,$$

which can be written as

$$\mathbf{B} = \mathbf{R} \beta,$$  \hspace{1cm} (34)

with

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (35)

From Eq. (26), it is seen that only the squares of $u_{20,1}$ and $u_{30,1}$ appear within each member. Therefore, we assume the trial functions for the displacement field in such a way that $u_{20,1}$ and $u_{30,1}$ become linear for each member. Moreover, we suppose that the bend angles around $x_2$- and $x_3$-axes along the member shown in Fig. 1 change with respect to the nodal rotations $\theta_{20}$ and $\theta_{30}$ ($i = 1, 2$) via the following relation

$$\mathbf{u}_\theta = \mathbf{N}_\theta \mathbf{a}_\theta = \begin{bmatrix} 1 - \frac{x_1}{l} & 0 & \frac{x_1}{l} & 0 \\ 0 & 1 - \frac{x_1}{l} & 0 & \frac{x_1}{l} \end{bmatrix} \begin{bmatrix} \theta_{20} \\ \theta_{30} \end{bmatrix}.$$  \hspace{1cm} (36)

Therefore, the nodal generalized displacement vector of the member can be expressed in the updated Lagrangian co-rotational frame $\mathbf{e}_i$ as

$$\mathbf{a} = \begin{bmatrix} \mathbf{1} \mathbf{a}^T \\ \mathbf{2} \mathbf{a}^T \end{bmatrix},$$  \hspace{1cm} (37)

where $\mathbf{1} \mathbf{a}$ ($i = 1, 2$) is the displacement vector of the $i$th node,

$$\mathbf{1} \mathbf{a} = \begin{bmatrix} u_{10} & u_{20} & u_{30} & \theta & \theta_{20} & \theta_{30} \end{bmatrix}^T.$$  \hspace{1cm} (38)

The nodal generalized displacement vector of the member, $\mathbf{a}$ is related to the vector $\mathbf{a}_\theta$ by

$$\mathbf{a}_\theta = \mathbf{T}_\theta \mathbf{a},$$  \hspace{1cm} (39)

in which,
\[
\mathbf{T}_\theta = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

(41)

Applying the trial functions of the stresses, Eq. (29) into the Eq. (26), the functional of the mixed variational principle in the co-rotational updated Lagrangian reference frame can be rewritten as

\[
\mathcal{H}_R = -\mathcal{H}_{R1} + \mathcal{H}_{R2} + \mathcal{H}_{R3} - \mathcal{H}_{R4},
\]

(42)

here,

\[
\mathcal{H}_{R1} = \sum_{m=1}^{N} \int_{l} \left( \frac{1}{2} \sigma^T D^{-1} \sigma \right) dl = \sum_{m=1}^{N} \int_{l} \left( \frac{1}{2} \mathbf{b}^T \mathbf{P}^T \mathbf{C} \mathbf{P} \mathbf{b} \right) dl,
\]

(43)

\[
\mathcal{H}_{R2} = \sum_{m=1}^{N} \left\{ \frac{2N}{l} u_{10} - \frac{2}{l} \left( 1 + m_3 \right) \left( \frac{2}{l} \right) u_{20} + \frac{2}{l} \theta_{30} - \right.
\]

\[
\left. \frac{1}{1} m_3 \frac{1}{l} \left( \frac{2}{l} u_{20} - \frac{1}{l} u_{30} \right) + \frac{2}{l} \theta_{20} - \frac{1}{l} \theta_{30} \right\} = \sum_{m=1}^{N} \left\{ \mathbf{B}^T \mathbf{F} \right\} = \sum_{m=1}^{N} \left\{ \mathbf{B}^T \mathbf{R}^T \mathbf{F} \right\},
\]

(44)

\[
\mathcal{H}_{R3} = \sum_{m=1}^{N} \int_{l} N_{11}^0 \left[ \frac{1}{2} (u_{20})^2 + \frac{1}{2} (u_{30})^2 \right] dl = \sum_{m=1}^{N} \int_{l} \sigma_1^0 \left[ \frac{1}{2} (\theta_{20})^2 + \frac{1}{2} (\theta_{30})^2 \right] dl =
\]

\[
\sum_{m=1}^{N} \int_{l} \frac{\sigma_1^0}{2} u_{10}^T u_{10} dl = \sum_{m=1}^{N} \int_{l} \frac{\sigma_1^0}{2} \mathbf{a}^T \mathbf{A}_{nn} \mathbf{a} dl,
\]

(45)

\[
\mathcal{H}_{R4} = \sum_{m=1}^{N} (\mathbf{a}^T \mathbf{F} - \mathbf{a}^T \mathbf{R}^T \mathbf{F}^0).
\]

(46)

where,

\[
\mathbf{C} = D^{-1}.
\]

(47)

\[
\mathbf{F} = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 1/l & 0 & -1 & 0 & 0 & 0 & -1/l & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & -1/l & 0 & 0 & 0 & -1 & 0 & 1/l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & -1/l & 0 & 0 & 0 & 0 & 0 & 1/l & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
0 & 1/l & 0 & 0 & 0 & 0 & 0 & -1/l & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

(48)

\[
\mathbf{A}_{nn} = \mathbf{T}_\theta^T \mathbf{N}_0^T \mathbf{N}_\theta \mathbf{T}_\theta.
\]

(49)

Invoking variational form for the functional of the mixed variational principle results in the following equation,

\[
\sum_{m=1}^{N} \delta \mathbf{b}^T \left( -\int_{l} (\mathbf{P}^T \mathbf{C} \mathbf{P} \mathbf{b}) dl + \mathbf{R}^T \mathbf{F} \right) + \sum_{m=1}^{N} \delta \mathbf{a}^T \left( \mathbf{F}^T \mathbf{R} \mathbf{b} + \sigma_1^0 \int_{l} \mathbf{A}_{nn} \mathbf{a} dl - \mathbf{F} + \mathbf{F}^T \mathbf{R} \mathbf{b}^0 \right) = 0,
\]

(50)
By letting \( H = \int_l P^T C P \, dl \), \( G = R^T \mathbf{Z} \), \( K_N = \sigma_0^2 \int_l A_{nn} \, dl \), \( F^0 = G^T \beta^0 \), Eq. (50) can be rewritten as
\[
\sum_{m=1}^N \delta \beta^T (-H \beta + G \alpha) + \sum_{m=1}^N \delta \alpha^T (G^T \beta + K_N \alpha - F + F^0) = 0.
\] (51)

### 2.2.2 Plasticity Effects in the Large Deformation Analysis of Members of a Cellular Microstructure

For an elastic-perfectly plastic material, the incremental work done on the material per unit volume is 
\[
dw = \sigma_{ij} \left( d \varepsilon_{ij}^p + d \varepsilon_{ij}^e \right)
\]
in which \( \varepsilon_{ij}^e \) and \( \varepsilon_{ij}^p \) are elastic and plastic components of strain, respectively, and \( \sigma_{ij} \) are the stress components. Using plastic hinge method, the plastic deformation is developed along the member wherever the plasticity condition is satisfied. Therefore, the total work expended in deforming the material of the body is
\[
W = \int_V \sigma_{ij} \left( d \varepsilon_{ij}^p + d \varepsilon_{ij}^e \right) \, dV = \int_V U(\varepsilon_{ij}^e) \, dV + \sum_i dW_i^p,
\] (52)
where, \( U(\varepsilon_{ij}^e) \) is the elastic strain energy density function, and \( dW_i^p \) is the increment of plastic work at the \( i \)th plastic hinge. When the theory of plastic potential is applied, the plasticity condition in terms of the stress components at the \( i \)th node is expressed as
\[
f_i(\sigma_x, \sigma_y, ..., \tau_{xy}, ..., \sigma_y) = 0,
\] (53)
the increment of plastic work at the \( i \)th node can be expressed as
\[
dW_i^p = d\mathbf{u}^p \mathbf{x}^T,
\] (54)
in which, \( d\mathbf{u}^p \), the increment of plastic nodal displacement at the \( i \)th node, is explained in term of the function \( f_i(x, \sigma_y) \),
\[
d\mathbf{u}^p = d\lambda_i \Phi_i,
\] (55)
\[
\Phi_i = \left[ \frac{\partial f_i(x, \sigma_y)}{\partial x} \right].
\] (56)
\( \mathbf{x} \) is the nodal force, and \( d\lambda_i \) is a positive scalar. Therefore, Eq. (52) can be rewritten as
\[
W = \int_V U(\varepsilon_{ij}^e) \, dV + \sum_i d\lambda_i \Phi_i^T \left| \int_p \right. \mathbf{x},
\] (57)

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\( \hat{x}_l = l_p \) is the location of the plastic hinge. A variational form for the plastic work can be written as below:

\[
\delta \left\{ \sum_{m=1}^{N} \left( \sum_{i} \lambda_i \phi_i^T \right) \left( P \beta^0 + P \beta \right) \right\} = \sum_{m=1}^{N} \delta \left( \sum_{i} \lambda_i \phi_i^T \right) \left( P \beta^0 + P \beta \right) + \left( \sum_{i} \lambda_i \phi_i^T \right) P \delta \beta = \sum_{m=1}^{N} \delta \lambda_i \phi_i^T \left( P \beta^0 + P \beta \right) + \delta \beta^T P^T \left( \sum_{i} \lambda_i \phi_i^T \right)^T.
\]

\[ (58) \]

### 2.2.3 Explicit Derivation of Tangent Stiffness Accompanying Plasticity Effects

Using the functional of the mixed variational principle given in Section 2.2.1, Eqs. (42-46), Eq. (57) is expressed as

\[
W = \sum_{m=1}^{N} \left\{ - \int_{l} \left( \frac{1}{2} \beta^T P^T C \beta \right) dl + \left( \beta^T H T \mathfrak{X} a \right) + \int_{l} \frac{1}{2} \alpha^T T^T N^T T \alpha dl - (\alpha^T F - \alpha^T \mathfrak{X} \beta^0 ) \left( \mathfrak{G}^T \beta + K_N a - F + F^0 \right) \right\}.
\]

Then, invoking \( \delta W = 0 \) and using Eqs. (51) and (58), Eq. (51) can be modified to include the effect of plasticity by introducing new determined matrices \( \hat{\beta}, \hat{H}, \) and \( \hat{G} \) as follows,

\[
\sum_{m=1}^{N} \delta \hat{\beta}^T \left( - \hat{H} \hat{\beta} + \hat{G} a \right) + \sum_{m=1}^{N} \delta a^T \left( \hat{G}^T \hat{\beta} + K_N a - F + F^0 \right) = 0,
\]

in which,

\[
\hat{\beta}^T = [\beta^T \ d\lambda],
\]

\[
\hat{H} = \begin{bmatrix} H & A_{12} \\ A_{12}^T & 0 \end{bmatrix},
\]

\[
\hat{G}^T = [G^T \ 0],
\]

where,

\[
A_{12}^T = \begin{bmatrix} \frac{\partial f}{\partial N} & \frac{\partial f}{\partial M_3} (-1 + \frac{l_p}{l}) & \frac{\partial f}{\partial M_3} (\frac{l_p}{l}) & \frac{\partial f}{\partial M_2} (1 - \frac{l_p}{l}) & \frac{\partial f}{\partial M_2} \frac{l_p}{l} & \frac{\partial f}{\partial M_1} \frac{l_p}{l} \end{bmatrix}\]

(64)

Since \( \delta \hat{\beta}^T \) in Eq. (60) are independent and arbitrary in each element,

\[
\hat{H} \hat{\beta} = \hat{G} a,
\]

\[
\hat{\beta} = \hat{H}^{-1} \hat{G} a,
\]

By letting \( \sum_{m=1}^{N} \delta a^T \left( \hat{G}^T \hat{\beta} + K_N a - F + F^0 \right) = 0 \) and substituting \( \hat{\beta} \) from Eq. (66), we obtain...
Therefore, the stiffness matrix, \( K \), in the presence of plasticity is derived explicitly as
\[
K = \tilde{G}^T \tilde{H}^{-1} \tilde{G} + K_N = G^T H^{-1} G + K_N - G^T H^{-1} A_{12} C^T G = K - G^T H^{-1} A_{12} C^T G = K - K_P, \tag{68}
\]
where,
\[
K = G^T H^{-1} G + K_N = K_L + K_N, \tag{69}
\]
\[
K_P = G^T H^{-1} A_{12} C^T G, \tag{70}
\]
and,
\[
C^T = (A_{12}^T H^{-1} A_{12})^{-1} A_{12}^T H^{-1}. \tag{71}
\]
Since we study the nonlinear coupling of axial, torsional, and bidirectional-bending deformations for each element, plasticity condition is introduced by \( f_i(N, \bar{M}_1, \bar{M}_2, \bar{M}_3) = 0 \) at the location of the \( i \)th plastic hinge, then
\[
\phi_i = \begin{bmatrix} \frac{\partial f_i}{\partial N} & \frac{\partial f_i}{\partial M_1} & \frac{\partial f_i}{\partial M_2} & \frac{\partial f_i}{\partial M_3} \end{bmatrix}^T, \tag{72}
\]
and
\[
\sum d\lambda_i \phi_i^T |_{lp} = \begin{bmatrix} \sum d\lambda_i \frac{\partial f_i}{\partial N} |_{lp} & \sum d\lambda_i \frac{\partial f_i}{\partial M_1} |_{lp} & \sum d\lambda_i \frac{\partial f_i}{\partial M_2} |_{lp} & \sum d\lambda_i \frac{\partial f_i}{\partial M_3} |_{lp} \end{bmatrix} = [H_p \ \theta_{p1}^* \ \theta_{p2}^* \ \theta_{p3}^*], \tag{73}
\]
in which, \( H_p \) is the plastic elongation and \( \theta_{pi}^*, i = 1, 2, 3 \) are the plastic rotations at the location of plastic hinges. Components of the element tangent stiffness matrices, \( K_N, K_L, \) and \( K_P \) are presented in Appendix B. Moreover, transformation matrices relating coordinate systems corresponding to the deformed and undeformed states to the global coordinates system (Fig. 1) are also given in Appendix C.

### 2.3 Solution Algorithm

To solve the incremental tangent stiffness equations, we employ a Newton Homotopy method [33, 34]. One of the most important reasons that we use newly developed scalar Homotopy methods is that this approach does not need to invert the Jacobian matrix (the tangent stiffness matrix) for solving NAEs. In the case of complex problems, such as elastic-plastic analysis of large
deformations and near the limit-load points in post-buckling analyses of geometrically nonlinear frames, where the Jacobian matrix may be singular the iterative Newton’s methods become problematic and necessitate the use of Arc-Length methods as in software such as ABAQUS.

One of the other reasons behind the advantages of recent developed Homotopy methods is showing much better performance than the Newton–Raphson method, when the Jacobian matrix is nearly singular or is severely ill-conditioned. For instance, when we considered the problem discussed in Section 3.1 (Three-Member Rigid-Knee Frame) using the Newton-Raphson algorithm, the provided code couldn’t converge to capture the critical load while it converged rapidly after switching to the Homotopy algorithm. Moreover, we discovered that while the Newton-type algorithm fails to converge, the Newton Homotopy method provides convergent solutions in the presence of the plasticity and buckling in a large number of members of the micro-lattice. As another benefits of the employed algorithm, our developed CELLS/LIDS code is not sensitive to the initial guess of the solution vector, unlike the Newton–Raphson method.

The Homotopy method was firstly introduced by Davidenko [35] to enhance the convergence rate from a local convergence to a global one for the solution of the NAEs of $F(X) = 0, X \in \mathbb{R}^n$ is the solution vector. This methodology was based on the employment of a vector Homotopy function, $H(X, t)$ to continuously transform a function $G(X)$ into $F(X)$. The variable $t$ ($0 \leq t \leq 1$) was the Homotopy parameter, treating as a time-like fictitious variable, and the Homotopy function was any continuous function in such a way that $H(X, 0) = 0 \iff G(X) = 0$ and $H(X, 1) = 0 \iff F(X) = 0$. More details on the vector Homotopy functions are given in Appendix D. To improve the vector Homotopy method, Liu et al. [33] proposed a scalar Homotopy function, $h(X, t)$ in such a way that $h(X, 0) = 0 \iff \|G(X)\| = 0$ and $h(X, 1) = 0 \iff \|F(X)\| = 0$. They [33] introduced the following scalar Fixed-point Homotopy function

$$h(X, t) = \frac{1}{2} (t \|F(X)\|^2 - (1 - t) \|X - X_0\|^2), \quad 0 \leq t \leq 1. \quad (74)$$

Later, Dai et al. [34] suggested more convenient scalar Homotopy functions which hold for $t \in [0, \infty)$ instead of $t \in [0, 1]$. We consider the following scalar Newton Homotopy function to solve the system of equations $F(X) = 0$,

$$h_n(X, t) = \frac{1}{2} \|F(X)\|^2 + \frac{1}{2Q(t)} \|F(X_0)\|^2, \quad t \geq 0, \quad (75)$$
resulting in,

\[ \dot{X} = -\frac{1}{2} \frac{\dot{Q} \| F \|^2}{Q \| B F \|^2} B^T F, \quad t \geq 0, \]

(76)

where, \( B \) is the Jacobian (tangent stiffness) matrix evaluated with \( B = \frac{\partial F}{\partial X} \), and \( Q(t) \) is a positive and monotonically increasing function to enhance the convergence speed. Various possible choices of \( Q(t) \) can be found in [34]. Finally, the solution vector, \( X \) can be obtained by numerically integrating of Eq. (76) or using iterative Newton Homotopy methods discussed in Appendix D.

3. Representative Approach and its Validation

This section is devoted to consider the validity of our proposed methodology. To this end, three different problems are analyzed and compared with results from other methods given in the literature. The critical load of three-member rigid-knee frame is computed in section 3.1. Section 3.2 examines the classical Williams’ toggle problem. Section 3.3 is devoted to consider the accuracy and efficiency of the calculated stiffness matrix in the presence of plasticity by solving the problem of right-angle bent.

3.1 Three-Member Rigid-Knee Frame

The geometry of the three-member rigid-knee frame and the cross section of elements are shown in Fig. 2. Using the CELLS/LIDS [CELLular Structures/Large Inelastic DeformationS] code, the longer element is divided into 6 elements and shorter elements are divided into 3 elements. A transverse perturbation loading, 0.001\( P \) is also applied at the midpoint of the longer member. Load versus displacement at the location of point load is plotted in Fig. 3 and compared with the corresponding results presented by Shi and Atluri [28]. As it is observed, there is a good agreement between present calculated results and those obtained by Ref. [28]. Please note that, they [28] have also mentioned that their computed critical load is a little higher than that obtained by Mallet and Berke [18].
Fig. 2. The geometry of three-member rigid-knee frame and the cross section of elements.

Fig. 3. Load versus displacement at the location of point load.
3.2 Classical Williams’ Toggle Problem

Williams [36] developed a theory to study the behavior of the members of a rigid jointed plane framework and applied it to the case of the rigid jointed toggle. Classical toggle problem is exhibited in Fig. 4, consisting of two rigidly jointed elements with equal lengths $L$ and anchored at their remote ends. The angle between the element and the horizontal axis, $b$ is related to the length of the elements via the relation $L \sin(b) = 0.32$. The characteristics of the cross section of elements are also included in Fig. 4. The structure is subjected to an external load $W$ along $z$-direction at the apex, as illustrated in Fig. 4. The deflection of the apex versus the applied load is calculated and compared with results given by Williams [36] in Fig. 5. As it is seen, good correspondence is obtained.

Cross section properties

Solid square

$L = 12.94 \text{ in}$

$EI = 9.27 \times 10^3 \text{ lb.in}^2$

$EA = 1.885 \times 10^6 \text{ lb}$

$L \sin(b) = 0.32 \text{ in}$

Fig. 4. Classical toggle problem.
3.3 Elastic-Plastic Right-Angle Bent

Throughout this section, the accuracy and efficiency of our methodology to consider the effect of plasticity is investigated. To this end, the problem of right-angle bent is calculated and compared with the results from other works. Two equal members of length $l$ with square cross sections are located in the $xy$-plane and are subjected to an external load, $F$ along the $z$-direction at the midpoint of one element, as shown in Fig. 6. Both members are anchored at their remote ends. Therefore, they are under both bending, $M$ and twisting, $T$. The yielding condition for such a perfectly plastic material subjected to bending and twisting is $\left(\frac{M}{M_0}\right)^2 + \left(\frac{T}{T_0}\right)^2 = 1$, in which $M_0$ and $T_0$ are, respectively, fully plastic bending and twisting moments. Employing our CELLS/LIDS code, each member is simulated by four elements. The formation of plastic hinges via the increase of external load is presented in Fig. 6 and the calculated amounts of $Fl/M_0$ at the onset of plastic hinges are compared with the results given by Shi and Atluri [28]. The variation of $\delta \times E1/M_0 l^2$ with respect to $Fl/M_0$ is also plotted in Fig. 7. Here, $\delta$ is the displacement of the tip of the right-angle bent along the $z$-direction, and $E$ is the Young’s modulus. The results also show good agreement with those in Hodge [25].
Fig. 6. Progressive development of plastic hinges in the right-angle bent.

Fig. 7. The variation of normalized load with respect to the normalized displacement at the tip of right-angle bent.
4. Low-Mass Metallic Systems with Architected Cellular Microstructures

This section is devoted to study computationally the large elastic-plastic deformations of nickel-based cellular micro-lattices fabricated at HRL Laboratories [7, 8]. To mimic the fabricated cellular microstructures, we model repetitive RVEs constructed by the strut members with the same geometry and dimension as the experiment. Each member of the actual cellular microstructure undergoing large elastic-plastic deformations is modeled by a single spatial beam finite element with 12 DOF, providing the capability to decrease considerably the number of DOF in comparison with the same simulation using commercial FE software. The strut members are connected in such a way that the topology of the fabricated cellular material is achieved. In the following, more details on the formation of RVE mimicking the actual microstructural samples are given. The properties of nickel as the parent material of the architected material is introduced within the CELLS/LIDS code by the Young’s modulus, $E^s = 200$ Gpa and the yield stress, $\sigma^s_y = 450$ MPa. The considered RVE is a Bravais lattice formed by repeating octahedral unit cells without any lattice members in the basal plane, as shown in Fig. 8. The lattice constant parameter of the unit cell is $a$, Fig. 8. The RVE is constructed by a node-strut representation and includes the nodes coordinate and the nodes connectivity, which determines the length of the members as well as the topology of the micro-lattice. Furthermore, the present RVE approach accurately captures the microstructural length scale by introducing the area, the first and the second moments of inertia, and the polar moment of inertia of the symmetrical/unsymmetrical cross section of the hollow tube member within the formulation. Periodic boundary conditions (PBCs) are considered along the x- and y-directions of RVE, which are the directions perpendicular to the depth of the thin film micro-lattice. PBCs are involved by $^a\mathbf{a}|_{x=0} = ^a\mathbf{a}|_{x=na}$ along x −direction and by $^k\mathbf{a}|_{y=0} = ^l\mathbf{a}|_{y=ma}$ along y −direction, in which $^\alpha\mathbf{a}$ $(\alpha = i,j,k,l)$ is the displacement vector of the $\alpha$th node (Eq. 39) on the boundary of RVE, and $n$ or $m$ is determined based on the size of the RVE along the x- or y-direction, respectively. For example, for the $Na \times Ma \times Ka$ RVE, $n = N$ and $m = M$. The depth of RVE is modeled to be equal to the thickness of the thin film. Section 4.1 studies the $1a \times 1a \times 2a$ RVE including 20 nodes and 32 strut members, and Section 4.2 examines both the $2a \times 2a \times 2a$ RVE with 60 nodes and 128 strut members and the $1a \times 1a \times 4a$ RVE with 36 nodes and 64 members. We study the mechanical behavior of the thin film cellular micro-lattice under tension, compression, and shear loadings. To this end, nodes on both top and bottom faces of RVE
are loaded accordingly. Micro-lattice members are cylindrical hollow tubes, the dimensions of which are also included in Fig. 8. Torrents et al. [8] tested samples with the strut member length of \( L = 1 - 4 \, \text{mm} \), strut member diameter of \( D = 100 - 500 \, \mu \text{m} \), the wall thickness of \( t = 100 - 500 \, \text{nm} \), and the inclination angle of \( \theta = 60^\circ \). In Sections 4.1 and 4.2, we analyze the mechanical behavior of two different fabricated cellular micro-lattices in which the geometry of their strut members \((L, D, t, \text{and } \theta)\) are explained respectively. Since nonlinear coupling of axial, torsional, and bidirectional-bending deformations is considered for each member, the plasticity condition is determined by the following relation

\[
f(N_{11}, M_{22}, M_{33}, T) = \frac{1}{M_0} \left\{ M_{22}^2 + M_{33}^2 + T^2 \right\}^{1/2} + \frac{N_{11}^2}{N_0^2} - 1 = 0,
\]

where, \( M_0 \) and \( N_0 \) are fully plastic bending moment and fully plastic axial force, respectively.

4.1 Architected Material with More Flexibility as Compared to Parent Material

An RVE including 20 nodes and 32 members with PBCs along \( x \) – and \( y \) –directions is employed to model a cellular thin film with the thickness of \( 2a \), Fig. 9. This figure shows the application of the external compressive loading, which changes according to tensile as well as shear loads. The dimensions of each member in the micro-lattice is as follows: \( L = 1050 \, \mu \text{m}, D = 150 \, \mu \text{m}, \text{and } t = 500 \, \text{nm} \).
The engineering stress as a function of the engineering strain is presented in Fig. 10 for the nickel cellular micro-lattice under compressive, tensile, and shear loads. The stress-strain curves corresponding to the tensile and compressive loads result in the overall yield stress of the RVE, \( \sigma_y = 15.117 \, kPa \) and the Young’s modulus, \( E = 2.291 \, MPa \). Torrents et al. [8] measured the respective values \( \sigma_y = 14.2 \pm 2.5 \, kPa \) and \( E = 1.0 \pm 0.15 \, MPa \) for their tested micro-lattice labeled with G \( (L = 1050 \pm 32 \, \mu m, D = 160 \pm 24 \, \mu m, t = 0.55 \pm 0.06 \, \mu m) \). The results calculated from our computational methodology agree excellently with those obtained from experiment by Torrents et al. [8]. It is found that this architected material shows a yield stress much smaller than the parent material, which offers more flexibility in tailoring the response to impulsive loads. In addition, we are able to calculate the shear modulus of the cellular micro-lattice from our obtained stress-strain curve corresponding to the shear load, resulting in \( G = 1.773 \, MPa \).
Fig. 10. Stress-strain curve of the cellular micro-lattice subjected to tension, compression, and shear.

The progressive development of plastic hinges as the tensile and compressive loads increase is shown in Fig. 11. The total deformation of RVE considering the effect of plasticity corresponding to the step B of compressive loading, step F of tensile loading, and step G of shear loading is also given in Figs. 12(a) through (c), respectively. Since plastic deformation can absorb energy, this architected material will be appropriate for protection from impacts and shockwaves in applications varying from helmets to vehicles and sporting gear [1].
(a) Plastic hinges formed at steps A and C shown in Fig. 10.  
(b) Plastic hinges formed at steps B and D shown in Fig. 10.  
(c) Plastic hinges formed at step E shown in Fig. 10.

Fig. 11. Progressive development of plastic hinges in cellular micro-lattice under tension and compression.
(a) At the step B of compressive loading
(b) At the step F of tensile loading
4.2 Architected Material with Further Increased Relative Density

In this case, the fabricated sample is computationally modeled using an RVE consisting of 60 nodes and 128 members with PBCs along the $x$- and $y$-directions, Fig. 13. The strut member dimensions are $L = 1200 \, \mu m$, $D = 175 \, \mu m$, and $t = 26 \, \mu m$. The wall thickness of the member in this case is 52 times greater than that of the previous case in section 4.1. The RVE is subjected to both tensile and compressive loading to study the mechanical properties of the architected material. The engineering stress-engineering strain curve is plotted in Fig. 14. Stress analysis shows bilinear elastic moduli for this cellular micro-lattice subjected to both tension and compression. Elastic
modulus for the first phase is calculated as 0.619 GPa under both tensile and compressive loading. For the second phase it is calculated to be 0.284 GPa under tension and 0.364 GPa under compression. The yield stress is obtained as 7.2222 MPa and 6.8519 MPa subjected to tensile and compressive loading, respectively. Plastic-hinges emanate at the stress level 6.6667 MPa when the micro-lattice is under tension and originate at the stress level 6.8519 MPa when the micro-lattice is subjected to compression. It is found that both Young’s modulus and the yield stress of the cellular micro-lattice increase significantly by increasing the strut thickness. It is well-known that the elastic modulus and the yield strength of the cellular materials increase with the increase of their relative density [9]. Relative density is calculated as $\rho/\rho_s$ where, $\rho$ is the mass of the lattice divided by the total bounding volume ($v$), and $\rho_s$ is the mass of the lattice divided by only the volume of the constituent solid material ($v_s$). Therefore, $\rho/\rho_s = (m/v)/(m/v_s) = v_s/v$ in which $v_s = \# of Members \times \pi[(D/2 + t)^2 - (D/2)^2] \times L$ and $v = 8a^3$ or $2a^3$ for $2a \times 2a \times 2a$ RVE or $1a \times 1a \times 2a$ RVE, respectively. We calculate the relative densities of the cellular micro-lattices examined through this section (Section 4.2) and Section 4.1 as 0.03511 and 0.00066, respectively. Torrents et al. [8] also extracted experimentally the strain-stress curve of this micro-lattice (labeled A) under compression. They measured the Young’s modulus, $E = 0.58 \pm 0.003$ GPa and the yield stress, $\sigma_y = 8.510 \pm 0.025$ MPa for the tested micro-lattice with the strut diameter $D = 175 \pm 26 \mu m$, strut length $L = 1200 \pm 36 \mu m$, and wall thickness $t = 26.00 \pm 2.6 \mu m$. As it is seen there is a very good correspondence between our calculated mechanical properties of the sample under compressive loading and those measured experimentally by Torrents et al. [8].

To investigate the effect of the size of the RVE on the macroscale response of the cellular micro-lattice, the depth of the $2a \times 2a \times 2a$ RVE (Fig. 13) is increased by a factor of two. Due to the PBCs along x- and y-directions, the size of the RVE along these directions is considered to be $1a$. Therefore, a $1a \times 1a \times 4a$ RVE consisting of 36 nodes and 64 strut members is modeled, Fig. 15, and the corresponding stress-strain curve under compression is included in Fig. 14. The stress analysis of this $1a \times 1a \times 4a$ RVE also exhibits bilinear elastic behavior with the elastic moduli of 0.7841 GPa for the first linear phase and 0.4721 GPa for the second linear phase. The yield stress is calculated to be 7.3704 MPa which becomes closer to the corresponding experimental value, $\sigma_y = 8.510 \pm 0.025$ MPa, in comparison with 6.8519 MPa calculated for $2a \times 2a \times 2a$
RVE. Figs. 16(a-b) show the elastic-plastic deformation of the $2a \times 2a \times 2a$ RVE under tension and compression, respectively.

Fig. 13. $2a \times 2a \times 2a$ RVE including 60 nodes and 128 strut members.
Fig. 4. Stress-strain curve of the cellular microlattice subjected to tension and compression.
Fig. 15. $1a \times 1a \times 4a$ RVE consisting of 36 nodes and 64 strut members.

(a) At the step A of tensile loading
At the step B of compressive loading

Fig. 16. Elastic-plastic deformation of the cellular microstructure in red color at different steps of loading shown in Fig. 14. Initial unloaded state is also presented by black dashed line.

5. Conclusion

We presented a computational approach for the large elastic-plastic deformation analysis of low-mass metallic systems with architected cellular microstructures. Studies on this class of materials are of interest since they can be optimized for specific loading conditions by changing the base material as well as the topology of the architecture. The repetitive RVE approach is utilized to mimic the fabricated cellular micro-lattices. The RVE is generated by a node-strut representation
consisting of the coordinate of nodes and the connectivity of nodes. Therefore, we can easily study
the effect of the change of topology on the overall mechanical response of the cellular material by
changing both coordinate and connectivity of nodes. Moreover, the microstructural length scale of
the cellular material is accurately captured by introducing the area, the first and the second
moments of inertia, and the polar moment of inertia of the symmetrical/unsymmetrical cross
section of the strut member within the formulation. In the current methodology, each member of
the actual micro-lattice undergoing large elastic-plastic deformations is modeled by a single FE
with 12 DOF, which enables to study the static and dynamic behavior of the macrostructure
directly and high efficiently by using arbitrarily large number of members. We study the nonlinear
coupling of axial, torsional, and bidirectional-bending deformations for each 3D spatial beam
element. The effect of plasticity is included employing plastic-hinge method, and the tangent
stiffness matrix is explicitly derived for each member, utilizing the mixed variational principle in
the updated Lagrangian co-rotational reference frame. In order to avoid the inversion of the
Jacobian matrix, we employ Homotopy methods to solve the incremental tangent stiffness
equations. The proposed methodology is validated by comparing the results of the elastic and
elastic-plastic large deformation analyses of some problems with the corresponding results given
in the literature. Moreover, two fabricated cellular micro-lattices with different dimensional
parameters including the unit cell size and the strut thickness are modeled using different RVEs.
We study their mechanical behaviors under all tensile, compressive, and shear loading. The
comparison of the calculated mechanical properties utilizing the present methodology with the
corresponding experimental measurements available in the literature reveals a very good
agreement. Using this developed computational approach, we can homogenize any cellular
structure easily, and we can design the topology of microstructure for any designated properties.

Appendix A

\[ \int_{l} \vec{N}_{11} \vec{e}_{11} \, dl = \int_{l} \vec{N}_{11} \vec{u}_{10} \, dl = - \int_{l} \vec{N}_{11} \vec{u}_{10} \, dl + \vec{N}_{11} \vec{u}_{10} |_{0}^{l}, \]  

(A.1)

\[ \int_{l} \vec{M}_{22} \vec{\pi}_{22} \, dl = - \int_{l} \vec{M}_{22} \vec{u}_{20} \, dl = - \int_{l} \vec{M}_{22} \vec{u}_{20} \, dl + \vec{M}_{22} \vec{u}_{20} |_{0}^{l} + \vec{M}_{22} \vec{u}_{20} |_{0}^{l}, \]  

(A.2)

\[ \int_{l} \vec{M}_{33} \vec{\pi}_{33} \, dl = - \int_{l} \vec{M}_{33} \vec{u}_{30} \, dl = - \int_{l} \vec{M}_{33} \vec{u}_{30} \, dl + \vec{M}_{33} \vec{u}_{30} |_{0}^{l} + \vec{M}_{33} \vec{u}_{30} |_{0}^{l}, \]  

(A.3)
\[ \int_t \mathcal{T} \theta dl = \int_t \mathcal{T} \hat{\theta}_1 dl = - \int_t \mathcal{T}_1 \hat{\theta} dl + \mathcal{T}_1 |_{0}^{t}, \] \hspace{1cm} (A.4)

**Appendix B**

\[ K_N = \frac{l}{6 \sigma_1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \text{Symm.} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 \\ 0 \end{bmatrix} \] \hspace{1cm} (B.1)

\[ K_L = \begin{bmatrix} K_{L1}^{11} & K_{L1}^{12} \\ K_{L1}^{12} & K_{L1}^{22} \end{bmatrix} \] \hspace{1cm} (B.2)

in which,

\[ K_{L1}^{11} = \frac{E}{lA} \begin{bmatrix} A^2 & 0 & 0 & 0 & \frac{Al_3}{l} & -Al_2 \\ \frac{12(-l_2^2+Al_{22})}{l^2} & \frac{12(-l_2l_3+Al_{23})}{l^2} & 0 & \frac{6l_3l_2-Al_{23}}{l} & \frac{6(-l_2^2+Al_{22})}{l} \\ \frac{12(-l_2^2+Al_{22})}{l^2} & 0 & \frac{6l_2l_3-Al_{23}}{l} & \frac{6(-l_2^2+Al_{22})}{l} \\ \text{Symm.} & 0 & 0 & \left(3l_2^2 - 4Al_{33}\right) & \left(3l_2l_3 - 4Al_{23}\right) \\ \left(-3l_2^2 + 4Al_{33}\right) & \left(3l_2l_3 - 4Al_{23}\right) & \left(-3l_2^2 + 4Al_{22}\right) \end{bmatrix} \] \hspace{1cm} (B.3)

\[ K_{L1}^{12} = \frac{E}{lA} \begin{bmatrix} -A^2 & 0 & 0 & 0 & -Al_3 & Al_2 \\ \frac{12(l_2^2-Al_{22})}{l^2} & \frac{12(l_2l_3-Al_{23})}{l^2} & 0 & \frac{6l_3l_2-Al_{23}}{l} & \frac{6(-l_2^2+Al_{22})}{l} \\ \frac{12(l_2^2-Al_{22})}{l^2} & 0 & \frac{6l_2l_3-Al_{23}}{l} & \frac{6(-l_2^2+Al_{22})}{l} \\ \text{Symm.} & -\frac{Al_{11r}}{E} & 0 & 0 & \left(-3l_2^2 + 2Al_{33}\right) & \left(3l_2l_3 - 2Al_{23}\right) \\ \left(-3l_2^2 + 2Al_{33}\right) & \left(3l_2l_3 - 2Al_{23}\right) & \left(-3l_2^2 + 2Al_{22}\right) \end{bmatrix} \] \hspace{1cm} (B.4)
\[
K_P^\text{11} = \begin{bmatrix}
K_P^\text{11} & K_P^\text{12} \\
K_P^\text{12} & K_P^\text{22}
\end{bmatrix}
\]
\[
22\mathbf{K}_p^{12} = \begin{bmatrix}
\frac{\text{Symm.}}{
\begin{array}{c}
\frac{4}{D_1} \\
\frac{E_1 r r M_1 N_0^2 \gamma v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 \gamma v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 \gamma v}{D_1}
\end{array}
\end{bmatrix}
\begin{array}{c}
\frac{D_1}{E_1 N_4 N_7} \\
\frac{E_1 N_4 N_7}{A l D_1} \\
\frac{E_1 N_4 N_7}{A l D_1} \\
\frac{E_1 N_4 N_7}{A l D_1}
\end{array}
\begin{array}{c}
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1}
\end{array}
\begin{array}{c}
\frac{D_1}{E_1 N_10 N_4} \\
\frac{E_1 N_10 N_4}{A l D_1} \\
\frac{E_1 N_10 N_4}{A l D_1} \\
\frac{E_1 N_10 N_4}{A l D_1}
\end{array}
\end{bmatrix},
\]
(B.11)

\[
K_p^{22} = \begin{bmatrix}
11K_p^{22} & 12K_p^{22} \\
\text{Symm.} & 22K_p^{22}
\end{bmatrix},
\]
(B.12)

in which,

\[
11K_p^{22} = \begin{bmatrix}
\frac{\text{Symm.}}{
\begin{array}{c}
\frac{4}{D_1} \\
\frac{E_1 N_1^2}{D_1} \\
\frac{E_1 N_1^2}{D_1} \\
\frac{E_1 N_1^2}{D_1}
\end{array}
\end{bmatrix}
\begin{array}{c}
\frac{6 E_1 (l-2l_p) N_2 N_0^2 N_1}{l D_1} \\
\frac{36 E_1 (l-2l_p) N_2 N_0^4}{A l^3 D_1} \\
\frac{36 E_1 (l-2l_p) N_2 N_0^4}{A l^3 D_1} \\
\frac{36 E_1 (l-2l_p) N_2 N_0^4}{A l^3 D_1}
\end{array}
\begin{array}{c}
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1}
\end{array}
\begin{array}{c}
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1}
\end{array}
\begin{array}{c}
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1} \\
\frac{E_1 N_1^2 N_1}{D_1}
\end{array}
\end{bmatrix},
\]
(B.13)

\[
12K_p^{22} = \begin{bmatrix}
\frac{\text{Symm.}}{
\begin{array}{c}
\frac{4}{D_1} \\
\frac{E_1 N_1^2}{D_1} \\
\frac{E_1 N_1^2}{D_1} \\
\frac{E_1 N_1^2}{D_1}
\end{array}
\end{bmatrix}
\begin{array}{c}
\frac{6 E_1 (l-2l_p) M_1 N_2 N_0^4}{l D_1} \\
\frac{6 E_1 (l-2l_p) M_1 N_2 N_0^4}{l D_1} \\
\frac{6 E_1 (l-2l_p) M_1 N_2 N_0^4}{l D_1} \\
\frac{6 E_1 (l-2l_p) M_1 N_2 N_0^4}{l D_1}
\end{array}
\begin{array}{c}
\frac{E_1 N_1 N_7}{D_1} \\
\frac{E_1 N_1 N_7}{D_1} \\
\frac{E_1 N_1 N_7}{D_1} \\
\frac{E_1 N_1 N_7}{D_1}
\end{array}
\begin{array}{c}
\frac{E_1 N_1 N_6}{D_1} \\
\frac{E_1 N_1 N_6}{D_1} \\
\frac{E_1 N_1 N_6}{D_1} \\
\frac{E_1 N_1 N_6}{D_1}
\end{array}
\begin{array}{c}
\frac{E_1 N_1 N_6}{D_1} \\
\frac{E_1 N_1 N_6}{D_1} \\
\frac{E_1 N_1 N_6}{D_1} \\
\frac{E_1 N_1 N_6}{D_1}
\end{array}
\end{bmatrix},
\]
(B.14)

\[
22K_p^{22} = \begin{bmatrix}
\frac{\text{Symm.}}{
\begin{array}{c}
\frac{4}{D_1} \\
\frac{E_1 r r M_1 N_0^2 \gamma v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 \gamma v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 \gamma v}{D_1}
\end{array}
\end{bmatrix}
\begin{array}{c}
\frac{(E N_7)^2}{A l D_1} \\
\frac{(E N_7)^2}{A l D_1} \\
\frac{(E N_7)^2}{A l D_1} \\
\frac{(E N_7)^2}{A l D_1}
\end{array}
\begin{array}{c}
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1}
\end{array}
\begin{array}{c}
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1}
\end{array}
\begin{array}{c}
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1} \\
\frac{E_1 r r M_1 N_0^2 N_6 v}{D_1}
\end{array}
\end{bmatrix},
\]
(B.15)

and,

\[
D_1 = E(-3l_3^2 (l-2l_p)^2 M_2^2 N_0^4 - 6l_2 l_3 (l-2l_p)^2 M_2 M_3 N_0^4 + (-3l_2^2 (l-2l_p)^2 M_3^2 + 4A(l^2 - 3l_3^2 + 6l_2 l_3 (l_3 M_2^2 + 2l_2 M_3 + l_2 M_3^2)) N_0^4 + 4Al_3 l^2 M_2 N N_0^2 S + 4Al_3 l^2 M_3 N N_0^2 S + 4Al_3 l^2 N^2 S^2 + A l r r l^2 M_1^2 N_0^4 v),
\]
(B.16)

\[
N_1 = l_3 M_2 N_0^2 + l_2 M_3 N_0^2 + 2A N S,
\]
(B.17)

\[
N_2 = -Al_3 M_2 + l_2 l_3 M_2 + l_2 M_3 - Al_2 M_3,
\]
(B.18)

\[
N_3 = l_3 M_2 - Al_3 M_2 - Al_3 M_2 + l_2 l_3 M_3,
\]
(B.19)
\[ N_4 = (-3l_3^2(l - 2l_p)M_2 - 3l_2l_3(l - 2l_p)M_3 + 2A(2l - 3l_p)(I_{33}M_2 + I_{23}M_3))N_0^2 + 2Al_3INs, \]  
\[ \text{(B.20)} \]

\[ N_5 = (-3l_2(l - 2l_p)(l_3M_2 + l_2M_3) + 2A(l - 3l_p)(l_{23}M_2 + l_{12}M_3))N_0^2 + 2Al_2INs, \]  
\[ \text{(B.21)} \]

\[ N_6 = (-3l_2(l - 2l_p)(l_3M_2 + l_2M_3) + 2A(l - 3l_p)(l_{23}M_2 + l_{12}M_3))N_0^2 - 2Al_2INs, \]  
\[ \text{(B.22)} \]

\[ N_7 = (3l_3^2(l - 2l_p)M_2 + 3l_2l_3(l - 2l_p)M_3 - 2A(l - 3l_p)(l_{33}M_2 + l_{32}M_3))N_0^2 + 2Al_3INs, \]  
\[ \text{(B.23)} \]

where,

\[ S = M_0 \{ M_1^2 + M_2^2 + M_3^2 \}^{1/2}. \]  
\[ \text{(B.24)} \]

\( N_0 \) and \( M_0 \) are the fully plastic axial force and the fully plastic bending moment, respectively.

**Appendix C**

Referring to Fig. 1, \( \bar{x_i} \) and \( \bar{e_i} \) are the global coordinates and the corresponding orthonormal basis vectors, respectively. \( \bar{x_i} \) and \( \bar{e_i} \) are the local coordinates and the corresponding basis vectors of the undeformed state and \( x_i \) and \( e_i \) are those of the deformed state. Herein, transformation matrices relating local coordinates corresponding to the deformed and undeformed states to the global coordinates are discussed. If \( ^aX_i \) denote the global coordinates of the \( \alpha \)th node of the element in the undeformed state, then the local orthonormal basis vectors of the undeformed state can be described with respect to those of the global coordinates as

\[ \bar{e_1} = (\Delta \bar{X}_1 \bar{e_1} + \Delta \bar{X}_2 \bar{e_2} + \Delta \bar{X}_3 \bar{e_3})/\bar{L}, \]  
\[ \text{(C.1)} \]

\[ \bar{e_2} = (\bar{e_3} \times \bar{e_1})/|\bar{e_3} \times \bar{e_1}|, \]  
\[ \text{(C.2)} \]

\[ \bar{e_3} = \bar{e_1} \times \bar{e_2}, \]  
\[ \text{(C.3)} \]

in which,

\[ \Delta \bar{X}_i = ^2X_i - ^1X_i, \quad i = 1, 2, 3, \]  
\[ \text{(C.4)} \]

\[ \bar{L} = \left\{ (\Delta \bar{X}_1)^2 + (\Delta \bar{X}_2)^2 + (\Delta \bar{X}_3)^2 \right\}^{1/2}. \]  
\[ \text{(C.5)} \]

Thus, \( \bar{e}_i \) and \( e_i \) (\( i = 1, 2, 3 \)) are related via the following equation, [32]
\[
\begin{bmatrix}
\bar{e}_1 \\
\bar{e}_2 \\
\bar{e}_3
\end{bmatrix} = \begin{bmatrix}
\frac{\Delta \bar{X}_1}{L} & \frac{\Delta \bar{X}_2}{L} & \frac{\Delta \bar{X}_3}{L} \\
\frac{-\Delta \bar{X}_2}{S} & \frac{\Delta \bar{X}_1}{S} & 0 \\
\frac{-\Delta \bar{X}_1 \Delta \bar{X}_3}{LS} & \frac{-\Delta \bar{X}_2 \Delta \bar{X}_3}{LS} & \frac{S}{L}
\end{bmatrix} \begin{bmatrix}
\bar{e}_1 \\
\bar{e}_2 \\
\bar{e}_3
\end{bmatrix},
\] (C.6)

where,

\[
\tilde{S} = \left\{ (\Delta \bar{X}_1)^2 + (\Delta \bar{X}_2)^2 \right\}^{1/2}.
\] (C.7)

Therefore, the matrix transforming global coordinates to the local coordinates of the undeformed state is obtained

\[
\bar{f} = \begin{bmatrix}
\frac{\Delta \bar{X}_1}{L} & \frac{\Delta \bar{X}_2}{L} & \frac{\Delta \bar{X}_3}{L} \\
\frac{\Delta \bar{X}_2}{S} & \frac{\Delta \bar{X}_1}{S} & 0 \\
\frac{\Delta \bar{X}_1 \Delta \bar{X}_3}{LS} & \frac{\Delta \bar{X}_2 \Delta \bar{X}_3}{LS} & \frac{S}{L}
\end{bmatrix}.
\] (C.8)

Note that, for the case when the element is parallel to the \( \bar{x}_3 \)-axis, the local coordinates for the undeformed state are determined by

\[
\bar{e}_1 = \bar{e}_3,
\]

\[
\bar{e}_2 = \bar{e}_2,
\] (C.9)

\[
\bar{e}_3 = -\bar{e}_1.
\]

Similarly, the transformation matrix relating local coordinates of the deformed state to the global coordinates can be obtained [37]. For this case, \( ^{\alpha}X'_i \) is introduced to describe the global coordinates of the \( \alpha \)th node of the element in the deformed state. Therefore, orthonormal basis vectors in the co-rotational reference coordinate system, \( \bar{e}_i \) can be chosen as below

\[
e_1 = (\Delta X_1 \bar{e}_1 + \Delta X_2 \bar{e}_2 + \Delta X_3 \bar{e}_3)/L,
\] (C.10)

\[
e_2 = (\bar{e}_3 \times e_1)/|\bar{e}_3 \times e_1|,
\] (C.11)

\[
e_3 = e_1 \times e_2,
\] (C.12)

where, \( \Delta X_i = ^{\alpha}X'_i - ^1X'_i \), and \( L = \{(\Delta X_1)^2 + (\Delta X_2)^2 + (\Delta X_3)^2\}^{1/2} \).

By replacing \( \bar{e}_3 \) from Eq. (C.6) into the Eq. (C.11),
\[
\begin{bmatrix}
\mathbf{e}_1 \\
\mathbf{e}_2 \\
\mathbf{e}_3
\end{bmatrix} = \\
\begin{bmatrix}
\frac{\Delta X_1}{L} & 0 & 0 \\
\frac{\Delta X_2}{L} & \frac{\Delta X_2 S}{L^2} & 0 \\
\frac{\Delta X_3}{L} & 0 & \frac{\Delta X_3 S}{L^2}
\end{bmatrix} + \\
\begin{bmatrix}
\frac{\Delta X_1^2}{L^2} & \frac{\Delta X_1 X_2}{L^2} & \frac{\Delta X_1 X_3}{L^2} \\
\frac{\Delta X_2 X_1}{L^2} & \frac{\Delta X_2 X_2}{L^2} & \frac{\Delta X_2 X_3}{L^2} \\
\frac{\Delta X_3 X_1}{L^2} & \frac{\Delta X_3 X_2}{L^2} & \frac{\Delta X_3 X_3}{L^2}
\end{bmatrix}
\begin{bmatrix}
\Delta X_1 \\
\Delta X_2 \\
\Delta X_3
\end{bmatrix}
\]
\[
(C.13)
\]

where,
\[
\mathcal{L} = \left\{ \left( -\frac{\Delta X_2 \Delta X_3 \Delta X_3}{L S L} - \frac{\Delta X_2 S}{L L} \right)^2 + \left( \frac{\Delta X_2 \Delta X_3 \Delta X_3}{L S L} + \frac{\Delta X_2 S}{L L} \right)^2 + \left( -\frac{\Delta X_2 \Delta X_3 \Delta X_3}{L S L} + \frac{\Delta X_2 S}{L L} \right)^2 \right\}^{1/2},
\]
\[
(C.14)
\]

Thus, the transformation matrix is obtained as
\[
\mathbf{T} = \\
\begin{bmatrix}
\frac{\Delta X_1}{L} & 0 & 0 \\
\frac{\Delta X_2}{L} & \frac{\Delta X_2 S}{L^2} & 0 \\
\frac{\Delta X_3}{L} & 0 & \frac{\Delta X_3 S}{L^2}
\end{bmatrix} + \\
\begin{bmatrix}
\frac{\Delta X_1^2}{L^2} & \frac{\Delta X_1 X_2}{L^2} & \frac{\Delta X_1 X_3}{L^2} \\
\frac{\Delta X_2 X_1}{L^2} & \frac{\Delta X_2 X_2}{L^2} & \frac{\Delta X_2 X_3}{L^2} \\
\frac{\Delta X_3 X_1}{L^2} & \frac{\Delta X_3 X_2}{L^2} & \frac{\Delta X_3 X_3}{L^2}
\end{bmatrix}
\begin{bmatrix}
\Delta X_1 \\
\Delta X_2 \\
\Delta X_3
\end{bmatrix}
\]
\[
(C.15)
\]

Finally, the transformation matrix for changing the generalized element coordinates consisting of 12 components in the global reference frame to the corresponding coordinates in the co-rotational reference frame is given by
\[
\mathbf{Q} = \begin{bmatrix}
\mathbf{T} & 0 \\
\mathbf{T} & \mathbf{T}
\end{bmatrix}.
\]
\[
(C.16)
\]

Then, components of the second order tensors, such as tangent stiffness matrix, as well as first order tensors, like the generalized nodal displacements and the generalized nodal forces, are transformed to the global coordinates system based on quotient rule using presented transformation matrices.
Appendix D

Two of the extensively used vector Homotopy functions are the Fixed-point Homotopy function and the Newton Homotopy function defined, respectively, as

\[ H_F(X,t) = tF(X) + (1 - t)(X - X_0) = 0, \quad 0 \leq t \leq 1, \]  
\[ H_N(X,t) = tF(X) + (1 - t)(F(X) - F(X_0)) = 0, \quad 0 \leq t \leq 1, \]  

here, \( X_0 \) represents the initial guess of the solution. Using the vector Homotopy method, the solution of \( F(X) = 0 \) can be obtained by numerically integration of the following relation

\[ \dot{X} = -\left(\frac{\partial H}{\partial X}\right)^{-1} \frac{\partial H}{\partial t}, \quad 0 \leq t \leq 1, \]  

which requires the inversion of the matrix \( \frac{\partial H}{\partial X} \) at each iteration.

A series of iterative Newton Homotopy methods has also been developed, where \( Q(t) \) is not needed to be determined [34]. Considering \( \dot{X} = \lambda u \), the general form of the scalar Newton Homotopy function becomes

\[ \dot{X} = -\frac{Q(t)\|F(X)\|^2}{2Q(t)F^TBu}u. \]  

Using the forward Euler method, Eq. (D.4) is discretized and the general form of the iterative Newton Homotopy methods is obtained as

\[ X(t + \Delta t) = X(t) - (1 - \gamma)\frac{F^TBu}{\|Bu\|^2}u, \]  

where, \(-1 < \gamma < 1\).

The reason why the homotopy methods converge with the required accuracy in the case of complex problems (in the presence of the plasticity and buckling in a large number of members of the micro-lattice) is raising the position of the driving vector, \( u \) in \( \dot{X} = \lambda u \) such that it introduces the best descent direction in searching the solution vector, \( X \). In the so-called continuous Newton method \( u = B^{-1}F \), which results in suffering from the accuracy of inverting the Jacobian matrix when it is singular or severely ill-conditioned and, thus, showing the oscillatory non-convergent behavior.
While \( u = B^T F (\lambda = -\frac{1}{2} \frac{\dot{Q}}{Q} \|B^T F\|^2) \) in Eq. (76) and can be extended to be constructed by two vectors such as \( F \) and \( B^T F \). The hyper-surface formulated in Eq. (75) defines a future cone in the Minkowski space \( \mathbb{M}^{n+1} \) in terms of the residual vector \( F \) and a positive and monotonically increasing function \( Q(t) \) as below

\[
X^T g X = 0, \tag{D.6}
\]

where,

\[
X = \begin{bmatrix} 
\frac{F(X)}{\|F(X_0)\|} \\
\frac{1}{\sqrt{Q(t)}} 
\end{bmatrix}, \tag{D.7}
\]

\[
g = \begin{bmatrix} 
I_n & 0_{n \times 1} \\
0_{1 \times n} & -1 
\end{bmatrix}, \tag{D.8}
\]

and \( I_n \) is the \( n \times n \) identity matrix. Then, the solution vector, \( X \) is searched along the path kept on the manifold defined by the following equation

\[
\|F(X)\|^2 = \frac{\|F(X_0)\|^2}{Q(t)}. \tag{D.9}
\]

Therefore, an absolutely convergent property is achieved by guaranteeing \( Q(t) \) as a monotonically increasing function of \( t \). In fact, Eq. (D.9) enforces the residual error \( \|F(X)\| \) to vanish when \( t \) is large.

References


