OPTIMAL-FEEDBACK ACCELERATED PICARD ITERATION METHODS AND A FISH-SCALE GROWING METHOD FOR WIDE-RANGING AND MULTIPLE-REVOLUTION PERTURBED LAMBERT’S PROBLEMS

Xuechuan Wang,* Satya N. Atluri†

Wide-ranging and multiple-revolution perturbed Lambert’s problems are building blocks for practical missions such as development of cislunar space, inter-planetary navigation, orbital rendezvous, etc. However, it is of a great challenge to solve these problems both accurately and efficiently, considering the long transfer time and the complexity of high-fidelity modeling of space environment. For that, a methodology combining Optimal-Feedback Accelerated Picard Iteration methods and Fish-Scale Growing Method is demonstrated. The resulting iterative formulae are explicitly derived and applied to an Earth-Moon restricted three-body problem and multi-revolution earth rendezvous problem. The examples demonstrate the validity and high efficiency of the proposed methods.

INTRODUCTION

The Lambert’s problem has been studied extensively in literature. However, when perturbations and long transfer time are taken into account, this problem could still be very challenging for the most state-of-art algorithms of orbital design. The challenge comes from two aspects. First, the various perturbation factors in a high-fidelity model of orbital mechanics lead to extremely complex expressions of nonlinear terms and large-scale computations. Further, as the transfer time is prolonged, the terminal state of the orbit becomes very sensitive to the initial state and thus it could be very difficult to find a good initial approximation of the transfer orbit. To overcome these difficulties, it requires that the algorithms should possess the following merits: large convergence area, high approximating accuracy, low computational complexity, and ease of parallel processing.

Mathematically, the Lambert’s problem can be described as a two-point boundary value problem to be solved via numerical methods, optimization methods and the shooting methods [1, 2, 3]. Specifically, the collocation method [4] plays an important role in solving various Lambert’s problems in relative orbit transfer [5], optimal control of spacecraft formation flying [6], low-thruster transfer between earth and moon [7], etc. In the conventional collocation method, however, one needs to construct nonlinear algebraic equations and calculate the inverse of the Jacobian matrix, which could be very troublesome. Some simpler ideas without inverting matrices are in-
roduced by the modified Chebyshev-Picard iteration (MCPI) method proposed by Junkins et al. [8], and the Optimal-Feedback Accelerated Picard Iteration (OFAPI) methods by Wang and Atluri [9,10,11].

Further, directly solving a Lambert’s problem with long duration time and perturbations is often impossible due to the limited convergence area of the algorithms and the sensitivity of initial guess. For that, the so-called Fish-Scales-Growing Method (FSGM) [9] is used in this paper. In a conservative system, the solution of a two-point boundary value problem is determined by the principle of least action. It allows us to breakdown a Lambert’s problem with long transfer time into several smaller time intervals using the Fish-Scales-Growing Method. Theoretically, in a conservative system the present Fish-Scales-Growing Method will converge to the true solution for any arbitrary initial approximation. Moreover, the Fish-Scales-Growing Method breaks down the entire transfer time into several sub-intervals, thus it can be conveniently coded for parallel computation to enhance the computational efficiency.

The investigation of this paper focuses on developing a scheme for wide-ranging transfer capabilities and multiple-revolution trajectory planning by combining the OFAPI methods and the FSGM. An Earth-to-Moon orbit transfer and a multi-revolution earth orbit rendezvous problem will be used as examples to validate the designed scheme. These two problems are both accompanied with long transfer time and various perturbations including higher-order gravity force, atmospheric drag, three-body effect, etc. In particular, both the Earth-Moon orbit transfer and the multi-revolution problem admit many solutions, which lead to additional difficulties to the traditional solver algorithms.

PROBLEM DESCRIPTION

In the previous work [9], the potential of OFAPI method was revealed by solving the orbital propagation problem and the earth orbit transfer within one revolution, in conjunction with FSGM, where it is shown to have better performance than the conventional MCPI method. Herein, a computational scheme is proposed to extend these methods to solve practical orbit transfer problems in much larger space and time scales.

Earth-Moon Orbit Transfer

The circular restricted three-body problem (CR3BP) in the Earth-Moon system is used to formulate the dynamics herein. Denoting the primary body (Earth), the secondary body (Moon), and the third body (Spacecraft) as P1, P2, P3 respectively, the motion of P3 is governed by the following dimensionless equations.

\[
\begin{align*}
\ddot{x} - 2\dot{y} - x &= \frac{(1 - \mu)(x + \mu)}{d^3} - \frac{\mu}{r^3}(x - 1 + \mu) \\
\ddot{y} + 2\dot{x} - y &= \frac{(1 - \mu)}{d^3} y - \frac{\mu}{r^3} y \\
\ddot{z} &= \frac{(1 - \mu)}{d^3} z - \frac{\mu}{r^3} z
\end{align*}
\]

where \(d = \sqrt{(x + \mu)^2 + y^2 + z^2} \), \(r = \sqrt{(x-1+\mu)^2 + y^2 + z^2} \) are distances from P3 to P1 and P2 respectively. The mass fraction is \(\mu = m_1/(m_1 + m_2)\), where \(m_1\) and \(m_2\) are masses of P1 and P2 respectively. The positions and velocities compose the state vector \([x, y, z, \dot{x}, \dot{y}, \dot{z}]\).
The Earth-Moon orbit transfer procedure is usually composed of two stages. The first stage is cataloging and choosing an unstable manifold in space that departs Earth and fly to Moon. It could be an orbit around the libration point, or an unstable resonant orbit. The second stage is determining a transfer orbit from a certain point on the manifold and the final Moon orbit. Herein, we are interested in the second stage where the transfer orbit is calculated.

**Multi-revolution Earth Orbit Transfer**

The dynamical model of an Earth orbit can be simply described by

$$\ddot{r} = -\frac{\mu}{r^3} r + a_p,$$

where $\mu$ is the gravitational parameters, $r = [x, y, z]^T$ is the position vector in the inertial frame, $r = \sqrt{x^2 + y^2 + z^2}$, and $a_p$ is the perturbation acceleration.

Although the governing equations take a simple form herein, it can become extremely complex as full gravity force of Earth and other perturbations are included. Moreover, for a multi-revolution problem, the maximum number of revolutions is

$$N_{\text{max}} = \text{floor}\left(\frac{\Delta t}{2\pi \sqrt{\frac{\mu}{a_m^3}}}\right),$$

where $\Delta t$ is the transfer time and $a_m$ is related with the initial and final positions. In that case, the number of possible transfer orbits are $2N_{\text{max}} + 1$.

In this paper, a high-fidelity model incorporating higher order gravity force will be considered to examine the proposed computational scheme.

**METHODOLOGY**

**The Optimal-Feedback Accelerated Picard Iteration (OFAPI) Method**

Generally, for solving a system of first order differential equations

$$\dot{x} = g(x, \tau), \quad \tau \in [t_0, t],$$

the OFAPI method approximates the solution at any time $t$ with an initial approximation $x_0(\tau)$ and the correctional iterative formula as

$$x_{n+1}(t) = x_n(t)|_{\tau=t} + \int_{t_0}^{t} \lambda(\tau)\{\dot{x}_n(\tau) - g[x_n(\tau), \tau]\}d\tau,$$

where $\lambda(\tau)$ is a matrix of Lagrange multipliers yet to be determined. Eq. (2) indicates that the $(n+1)$th correction to the analytical solution $x_{n+1}$ involves the addition of $x_n$ and a feedback weighted optimal error in the solution $x_n$ up to the current time $t$. $\lambda(\tau)$ can be optimally determined by making the right-hand side of Eq. (2) stationary about $\delta x_n(\tau)$, thus resulting the constraints.
\[
\begin{align*}
\left[ \delta x_n(t) \right]_{\tau=t} : I + \lambda(\tau) &= 0, \\
\left[ \delta x_n(t) : \dot{\lambda}(\tau) + \lambda(\tau) J \right] &= 0, \quad t_0 \leq \tau \leq t,
\end{align*}
\]  
(3)

where \( J(\tau) = \partial g(x_n, \tau)/\partial x_n \).

First order Taylor series approximation of \( \lambda(\tau) \) can be readily obtained from the Eq. (3), in the following form

\[
\lambda(\tau) \approx T_0[\lambda] + T_1[\lambda](\tau-t),
\]  
(4)

where \( T_k[\lambda] \) is the \( k \) th order differential transformation of \( \lambda(\tau) \), i.e.

\[
T_k[\lambda] = \frac{1}{k!} \frac{d^k \lambda(\tau)}{d\tau^k}.
\]  
(5)

Using Eq. (3), \( T_k[\lambda] \) can be determined in an iterative way:

\[
T_0[\lambda] = \text{diag}[-1,-1,\ldots], \quad T_{k+1}[\lambda] = -\frac{T_k[\lambda J]}{k+1}, \quad 0 \leq k \leq K+1.
\]  
(6)

Let \( G = \dot{x}_n(\tau) - g[x_n(\tau), \tau] \), by substituting Eq. (6) into Eq. (2), we have

\[
x_{n+1}(t) = x_n(t) + \int_{t_0}^{t} \{ T_0[\lambda] + T_1[\lambda](\tau-t)\} G d\tau,
\]  
(7)

Two other ways of approximating \( \lambda(\tau) \) have been proposed by the authors leading to 3 distinct OFAPI algorithms [11].

Suppose \( x(t) \) is approximated by a vector of trial functions \( u \). Let each element \( u_e \) of the trial function be the linear combinations of basis functions \( \phi_{e,\alpha}(t) \)

\[
u_e = \sum_{n=1}^{N} \alpha_{e,\alpha} \phi_{e,\alpha}(t) = \Phi_e(t)A_e.
\]  
(8)

Through collocating points in time domain, from Eq. (7) we have

\[
U_{n+1} = U_n + (T_0 - [T_1 \cdot \Phi])E G + T_1 E [\Phi \cdot G],
\]  
(9)

which can be used to iteratively solve for the values of \( x(t) \) at collocated time points. For higher computational efficiency and accuracy, we set the basis functions as orthogonal polynomials. Herein, the first kind of Chebyshev polynomials are adopted, and the collocation points in each time interval are selected as Chebyshev-Gauss-Lobatto (CGL) nodes. For further details of Eq. (9) please refer to [10].

A flow chart of OFAPI method is provided herein
The Fish Scales Growing Method

The iterative procedure of this method is as follows. An initial approximation is provided by a reference trajectory, which could be a nominal solution obtained from the linearized problem or unperturbed problem. Then divide the domain \((t_0,t_f)\) of the two-point boundary value problem (TPBVP) into multiple isometric intervals \([t_i,t_{i+1}]\), \(0\leq i < N\), \(t_{i+1} - t_i = (t_f - t_0)/N\). For each interval, the points \(x(t_i)\) and \(x(t_{i+1})\) on the reference trajectory are set as boundaries. In each iteration, we need to solve the \(N\) TPBVPs defined by the boundaries \(x(t_i)\) and \(x(t_{i+1})\) in the corresponding interval \([t_i,t_{i+1}]\). The points \(\hat{x}(t_j)\), \(t_j = (t_i + t_{i+1})/2\) on the solutions of these \(N+1\) TPBVPs are collected for the next step. Then we solve the \(N-1\) TPBVPs defined by the boundaries...
\( \tilde{x}(t_j) \) and \( \tilde{x}(t_{j+1}) \), \( 0 \leq j < N-1 \) in the corresponding interval \([t_j, t_{j+1}]\). After that, the points \( \tilde{x}(t_j) \) on the solutions of these \( N-1 \) TPBVPs are used to replace \( x(t_j) \) and \( x(t_{j+1}) \). If \( \|\tilde{x}(t_j) - x(t_j)\| \leq \varepsilon \), the iteration ends. Otherwise one should replace \( x(t_j) \) with \( \tilde{x}(t_j) \) and restart the iteration.

![Figure 2 Illustration of the FSGM](image)

A schematic of the Fish-Scales-Growing Method is illustrated in Fig. 2. The domain \([t_0, t_f]\) is divided into two smaller intervals \([t_0, t_1]\) and \([t_1, t_f]\), where \( t_1 = (t_0 + t_f)/2 \). In each iteration, three boundary value problems are defined and need to be solved so as to correct the position values \( x_i \) at the time instant \( t_i \). As the iteration goes on, \( x_i \) approaches to the real value, thus the piecewise trajectory evolves to the true solution of the original TPBVP. A flow chart of this method is provided in Fig. 3.
PRELIMINARY SIMULATION RESULTS

Example 1: Earth to Moon Orbit Transfer

Using the dimensionless model of CR3BP, an Earth-Moon orbit transfer problem is solved using the OFAPI method in conjunction with the present Fish-Scales-Growing Method. Figure 4 shows an example of direct transfer from 185-km Earth orbit to 100-km Lunar orbit. Using Fish-Scale Growing Method (FSGM), the transfer problem is broken down into several subproblems, which
are then solved independently using OFAPI method. Fig. 4 (a) demonstrates the intriguing evolution of the piecewise solution from a straight line (initial guess) to an accurate transfer orbit (final solution). Using the initial velocity provided by the final solution of OFAPI-FSGM scheme, the ode45 (MATLAB) is initialized and generates a testing orbit shown in Fig. 4 (b). The comparison between the final solution of OFAPI-FSGM and the testing orbit shows that they match very well. Note that ode45 itself cannot solve boundary value problems, herein it is just used to integrate the orbit trajectory with the initial condition provided by the proposed scheme. As is aforementioned, the FSGM is not bothered by the selection of initial guess, because it is absolutely and globally convergent in conservative systems. This statement is verified by the results presented in Fig. 4.

![Figure 4 Earth to Moon transfer orbit obtained using OFAPI method and FSGM](image)

**Example 2: Multi-revolution Earth Orbit Transfer**

A multi-revolution problem with the transfer time being \([0, 180,000]\) is solved by OFPI method in conjunction with FSGM.

![Figure 5 Test of FSGM in multi-revolution lambert’s problem](image)

The whole transfer time is divided into 10 segments, while in each segment the solution is approximated using 13 collocation points. The Keplerian orbit is used as an initial approximation. To evaluate the improvement of the solution obtained using the proposed method, the discrepan-
cies $\delta r$ of the position and $\delta v$ of the velocity between the exact solution and the iteratively corrected solution is recorded and plotted in Fig. 5. It is shown that the proposed method further improves the Keplerian solution and the discrepancies decreases monotonically.

CONCLUSION

A Scheme for solving perturbed Lambert’s problem, i.e. the Optimal-Feedback Accelerated Picard Iteration (OFAPI) method in conjunction with the Fish Scales Growing Method (FSGM), is presented in this paper. This scheme enables one to break down a long-duration orbit transfer problem into many much easier subproblems and bypasses the dilemma of choosing initial approximating orbit. In the meantime, it also provides very accurate numerical solution through using the OFAPI method. Compared with the conventional MCPI method and other solvers of perturbed Lambert’s problem in literature, the proposed scheme can be applied to problems with relatively longer transfer time and still achieves high accuracy and efficiency.

This scheme is used to solve a wide-ranging orbit transfer problem (Earth to Moon) and a multiple-revolution perturbed Lambert’s problem. Simulation results demonstrate that the OFAPI-FSGM scheme is insensitive to the initial guess and converges to very accurate results. At last, since this scheme possesses the merit that the subproblems can be solved independently, much computational time can be saved using parallel processing.

REFERENCES


