Construction of pseudospectral meshless radial point interpolation for generalized biharmonic equation subject to simply supported and clamped boundary conditions

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Abstract

In this study, we develop an approximate formulation for a generalized form of the biharmonic problem based on pseudospectral meshless radial point interpolation (PSMRPI). The boundary conditions are considered as simply supported or clamped, with application to the theory of static analysis of thin-plates. The rigorous steps to analyze such problem are defining the high order derivatives, implementing multiple boundary conditions especially when the geometry of the domain of the problem is complex. In PSMRPI method the nodal points do not need to be regularly distributed and can even be quite arbitrary. It is easy to have high order derivatives of unknowns in terms of the values at nodal points by constructing operational matrices. Furthermore, it is observed that the multiple boundary conditions can be imposed by applying PSMRPI on nodal points near the boundaries of the domain. The main results on the generalized biharmonic problem are demonstrated by some examples to show the validity and trustworthiness of PSMRPI technique. Also, a comparison with the previously standard studied method for the biharmonic problem is done.

Key words: Spectral method; Pseudospectral method; Meshless technique; Radial
1 Preliminaries

The meshless (meshfree) methods are known as a strong competitor for meshbased methods like for example finite element method (FEM), the finite volume method (FVM), and the boundary element method (BEM) [1,2]. Generally, meshless (meshfree) methods can be divided to some types as follows:

- Those meshfree methods which are constructed as weak form like for example [3–12]: meshless local Petrov-Galerkin (MLPG), the method of approximate particular solutions (MAPS), element free Galerkin (EFG) and its modifications like for example improved element free Galerkin (IEFG), complex variable element free Galerkin (CVEFG) and improved complex variable element free Galerkin (ICVEFG).
- Those Meshfree techniques which benefit from collocation approach and usually applied in strong forms, like for example the Kansa’s method in the frame of radial basis functions (RBFs) [13–17].
- Those Meshless techniques benefit from weak forms and use collocation scheme at the same time [18–33].

The interested readers are referred also to [34–39] in order to find out more about the other methods like differential quadrature method especially various aspects of meshless methods. The formulation of the generalized biharmonic equation is given by

\[
\Delta\Delta u - \alpha\Delta u + \beta u = f(x), \quad x \in \Omega \subseteq \mathbb{R}^2,
\]

subject to simply supported boundary conditions

\[
u(x) = g_1(x), \quad x \in \partial \Omega, \quad \frac{\partial^2 u(x)}{\partial n^2} = g_2(x), \quad x \in \partial \Omega,
\]

or subject to clamped boundary conditions

\[
u(x) = g_1(x), \quad x \in \partial \Omega, \quad \frac{\partial u(x)}{\partial n} = g_2(x), \quad x \in \partial \Omega,
\]

where \( x = (x, y) \) and \( n \) is the outward unit normal vector on the boundary. Numerically solving this elliptic partial differential equation becomes more important due to dual

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boundary conditions.

The study of the existence and uniqueness of the solution of some kinds of biharmonic differential equations which are close to (1) has been done by many authors, see [40–42] and references therein. Arad et. al. [43] derived coefficients for a nine-point high-order accuracy discretization procedure for a biharmonic equation with both types of boundary conditions (2)-(3). Authors of [44] presented explicit block iterative method for the solution of the biharmonic equation. S. B. Sorokin [45] adopted a techniques of step by step inversion based on finite difference method for numerical solution of the Biharmonic equation in one dimension. In Ref. [46], authors have applied method of fundamental solutions for the numerical analysis of the biharmonic equation. Xin Li [47] presented radial basis approximation and applied it to biharmonic equation. Mai-Duy et. al. [48] studied a domain-type boundary-integral-equation approach for two-dimensional biharmonic Dirichlet problem. Shivanian [49] has studied both 2-D and 3-D inhomogeneous biharmonic BVPs (1)-(3) through a method based on combination of radial basis function collocation and spline interpolation. Also, very recently, some researchers have studied finite pointset method, which is a kind of meshless techniques, for biharmonic equation (1) with the boundary conditions (2)-(3) and obtained the solution with acceptable accuracy [50].

In the current work, we develop PSMRPI method with the aim of applying it on an elliptic problem with multiple boundary conditions which is represented by (1)-(3). We conclude that it is accurate and reliable with less computational costs. Furthermore, observation of convergence in the numeric experiments revealed that PSMRPI approach possesses reliable convergency rate too.

2 Brief review of spectral methods

One of the powerful and highly accurate techniques to solve differential equations is pseudo-spectral (PS) methods. The essential idea [51] in pseudo-spectral (PS) methods is to represent the unknown function in terms of summation of some smooth basis functions $B_j$, $j = 1, ..., N$, like for example polynomials or other well known functions

$$\hat{u}(x) = \sum_{j=1}^{N} c_j B_j(x), x \in \mathbb{R}. \quad (4)$$

A consequential trait of pseudospectral methods is that the unknown function(s) are approximated at nodal collocation points $x_i, i = 1, ..., N$. Implementation of the pseudospectral method is usually done by the so-called differentiation matrices or operational matrices, i.e., we extract and establish a matrix $\mathcal{D}$ so that the following equation holds at
the collocation points $x_i$

$$u' = Du, \quad (5)$$

in where $(\cdot)'$ denotes the derivative, $u = [\hat{u}(x_1), \hat{u}(x_2), ..., \hat{u}(x_N)]^T$ is the vector of unknowns $\hat{u}$ at the collocation nodal points. Very often, orthogonal polynomials like for example Chebyshev polynomials are applied and played as basis functions, and the corresponding nodal collocation knots are chosen as Chebyshev points. In what follows, we describe how to establish a differentiation matrix via pseudospectral methods. Take into account the expansion (4) and assume $B_{j,i} = 1$, $j = 1, ..., N$, be a set of arbitrary linearly independent functions being convenient enough as the basis for the space of approximations. Setting $x = x_i, i = 1, ..., N$, in Eq. (4) we obtain

$$\hat{u}(x_i) = \sum_{j=1}^{N} c_j B_j(x_i), \quad i = 1, ..., N, \quad (6)$$

or equivalently in matrix-form

$$u = Ac, \quad (7)$$

where $c = [c_1, ..., c_N]$ denotes the vector of coefficient, the matrix $A$ is defined by its entries $A_{ij} = B_j(x_i)$, and $u$ is as defined previously. Taking advantage of the linearity of the expansion (4), we can easily get the derivative of $\hat{u}$ through differentiating individually the basis functions and then get

$$\frac{d}{dx} \hat{u}(x) = \sum_{j=1}^{N} c_j \frac{d}{dx} B_j(x), \quad i = 1, ..., N, \quad (8)$$

evaluation of the above expression again at the collocation nodal points $x_i$, we turn into

$$u' = A_x c, \quad (9)$$

in where $u$ and $c$ are as denoted previously in Eq. (7), and the entries of the matrix $A_x$ are expressed by $\frac{d}{dx} B_j(x_i)$. Turning into obtaining the operational matrix $D$ it is sufficient to ensure invertibility of the matrix of estimation $A$, if so, then

$$u' = A_x A^{-1} u, \quad (10)$$

therefore the operational differentiating matrix $D$ matched to Eq. (5) is represented via

$$D = A_x A^{-1}. \quad (11)$$
3 PSMRPI method

Employing above pseudospectral procedure, a local radial point interpolation technique has been adopted to extract shape functions in order to be applied as basis functions in the frame of previous section method. This local-RBF method which is called pseudospectral meshless radial point interpolation (PSMRPI) method, has been extensively applied on some important problems [52–61]. In fact, it is replaced basis functions $B_j$ by the local radial basis functions and then the matrix of estimation $A$ will be non-singular because of strictly (conditionally) positive definite property of applied radial basis function.

**Definition 1** We call a continuous function $\varphi : \mathbb{R}^d \to \mathbb{C}$ as a conditionally positive semi-definite of order $m$ if all pairwise separate nodal points $y_1, \ldots, y_N \in \mathbb{R}^d$, $N \in \mathbb{N}$, and all $d = [d_1, \ldots, d_N]^T \in \mathbb{C}^N$ justifying

$$\sum_{j=1}^{N} d_j p(y_j) = 0,$$

for any complex-valued polynomials of degree downwards from $m$, the quadratic form

$$\sum_{j=1}^{N} \sum_{k=1}^{N} d_j d_k \varphi(y_j - y_k)$$

be nonnegative.

**Theorem 1** *(Micchelli)* Taking into account $\varphi \in C[0, \infty) \cap C^\infty(0, \infty)$, the expression $(-1)^m \varphi^{(m)}$ is completely monotone on $(0, \infty)$ if and only if the function $\psi = \varphi(||.||^2)$ is strictly conditionally positive definite of order $m \in \mathbb{N}_0$ on $\mathbb{R}^d$.

Theorem 1 is very helpful to find strictly conditionally positive definite functions such as surface (thin-plate) splines $\varphi(r) = (-1)^{k+1} r^{2k} \log(r)$ on $\mathbb{R}^d$ in which its order is $m = k + 1$. In fact, this RBF will be used as core basis function to generate shape (basis) functions in the frame of PSMRPI through interpolating procedure.

Consider a continuous function $u(x)$ defined on a domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, covering by collocation nodal points. Let $u(x)$ at a node of interest $x$ be approximated by

$$u(x) = \sum_{i=1}^{n} R_i(x) a_i + \sum_{j=1}^{np} P_j(x) b_j = R^{tr}(x)a + P^{tr}(x)b,$$

(12)

in where $R_i(x)$ denotes radial basis function (RBF), $n$ is the count of used RBFs, $P_j(x)$ is monomial in the space coordinate $x$, and $np$ is the count of polynomials. We notify here that the monomials are necessary because we use thin-plate splines (TPS) as basis function. They are not needed in the case of applying strictly positive definite RBF. In
this way, we ensure that the singularity is avoided in both cases. To determine coefficients $a_i$ and $b_j$ in Eq. (12), we surround the node of interest $\mathbf{x}$ by a support domain which is covered by the number of $n$ field nodes (support domain can be for example a disk with radius $r_s$). Then coefficients $a_i$ and $b_j$ can easily be found by imposing Eq. (12) at these $n$ field nodes about $\mathbf{x}$ in the support domain. Hence, via the interpolation idea Eq. (12) turns to the following form

$$u(\mathbf{x}) = \Psi^{tr}(\mathbf{x}) \mathbf{U}_s = \sum_{i=1}^{n} \psi_i(\mathbf{x})u_i. \quad (13)$$

It is more important to mention that the basis (shape) functions $\psi_i(\mathbf{x})$ satisfies the Kronecker delta characteristic i.e.

$$\psi_i(\mathbf{x}_j) = \begin{cases} 1, & i = j, \ j = 1, 2, \ldots, n, \\ 0, & i \neq j, \ i, j = 1, 2, \ldots, n, \end{cases} \quad (14)$$

this property can be verified easily and it goes back to the fact that how they are constructed. Furthermore, another important property that they satisfy is the partitions of unity, in other words:

$$\sum_{i=1}^{n} \psi_i(\mathbf{x}) = 1. \quad (15)$$

The interest readers are referred to [53] to disclose the way they are built and how they are presented. In the following step, we reveal how to globalize local-made shape functions to construct differentiation matrices which play important role in the current technique. Differentiation matrices obviously help us to apply the method on high order partial differential equations. Suppose that the total number of scattered nodal collocation points covering the domain of the problem i.e. $\bar{\Omega} = \Omega \cup \partial \Omega$, denotes by $N$. From the other point of view, It is know that $n$ is dependent upon the point of interest $\mathbf{x}$ (hence, afterwards we call it $n_\mathbf{x}$) in Eq. (13) that is the number of points contained in the local support domain $\Omega_\mathbf{x}$ with respect to the node of interest $\mathbf{x}$ (for instance $\Omega_\mathbf{x}$ is supposed to be a disk placed about $\mathbf{x}$ with radius $r_s$). Hence, it holds $n_\mathbf{x} \leq N$, and consequently Eq. (13) can be globalized as

$$u(\mathbf{x}) = \Psi^{tr}(\mathbf{x}) \mathbf{U}_s = \sum_{j=1}^{N} \psi_j(\mathbf{x})u_j. \quad (16)$$

As a matter of fact, corresponding to the node $\mathbf{x}_j$, there is a shape function $\psi_j(\mathbf{x}), j = 1, 2, \ldots, N$, and $\Omega^{c}_\mathbf{x} = \{ \mathbf{x}_j : \mathbf{x}_j \notin \Omega_\mathbf{x} \}$, such that from Eq. (14)

$$\forall \mathbf{x}_j \in \Omega^{c}_\mathbf{x} : \psi_j(\mathbf{x}) = 0. \quad (17)$$
The derivatives of \( u(x) \) are computed without difficulty as

\[
\frac{\partial u(x)}{\partial x} = \sum_{j=1}^{N} \frac{\partial \psi_j(x)}{\partial x} u_j, \quad \frac{\partial u(x)}{\partial y} = \sum_{j=1}^{N} \frac{\partial \psi_j(x)}{\partial y} u_j, \quad \frac{\partial u(x)}{\partial z} = \sum_{j=1}^{N} \frac{\partial \psi_j(x)}{\partial z} u_j, \tag{18}
\]

as well as for higher order differentiating of \( u(x) \)

\[
\frac{\partial^s u(x)}{\partial x^s} = \sum_{j=1}^{N} \frac{\partial^s \psi_j(x)}{\partial x^s} u_j, \quad \frac{\partial^s u(x)}{\partial y^s} = \sum_{j=1}^{N} \frac{\partial^s \psi_j(x)}{\partial y^s} u_j, \quad \frac{\partial^s u(x)}{\partial z^s} = \sum_{j=1}^{N} \frac{\partial^s \psi_j(x)}{\partial z^s} u_j. \tag{19}
\]

Denoting \( u^{(s)}_x(.) = \frac{\partial^s(.)}{\partial x^s} \), \( u^{(s)}_y(.) = \frac{\partial^s(.)}{\partial y^s} \), \( u^{(s)}_z(.) = \frac{\partial^s(.)}{\partial z^s} \), and setting \( x = x_i \) in Eq. (19), we reach to the matrix-vector form multiplications as follows:

\[
U^{(s)}_x = D^{(s)}_x U, \quad U^{(s)}_y = D^{(s)}_y U, \quad U^{(s)}_z = D^{(s)}_z U, \tag{20}
\]

where

\[
U^{(s)}_x = (u^{(s)}_{x_1}, ..., u^{(s)}_{x_N})^T, U^{(s)}_y = (u^{(s)}_{y_1}, ..., u^{(s)}_{y_N})^T, U^{(s)}_z = (u^{(s)}_{z_1}, ..., u^{(s)}_{z_N})^T, \tag{21}
\]

\[
D^{(s)}_{x_{ij}} = \frac{\partial^s \psi_j(x_i)}{\partial x^s}, \quad D^{(s)}_{y_{ij}} = \frac{\partial^s \psi_j(x_i)}{\partial y^s}, \quad D^{(s)}_{z_{ij}} = \frac{\partial^s \psi_j(x_i)}{\partial z^s}, \tag{22}
\]

\[
U = (u_1, u_2, ..., u_N)^T. \tag{23}
\]

4 Application to the biharmonic problem

The current section is devoted to implement PSMRPI, presented in earlier section, on the problem (1)-(3). Replacing approximation formula (16), into the Eq. (1), we are conducted to

\[
\sum_{j=1}^{N} \frac{\partial^4 \psi_j(x)}{\partial x^4} u_j + 2 \sum_{j=1}^{N} \frac{\partial^4 \psi_j(x)}{\partial x^2 \partial y^2} u_j + \sum_{j=1}^{N} \frac{\partial^4 \psi_j(x)}{\partial y^4} u_j
\]

\[
- \alpha \left( \sum_{j=1}^{N} \frac{\partial^2 \psi_j(x)}{\partial x^2} u_j + \sum_{j=1}^{N} \frac{\partial^2 \psi_j(x)}{\partial y^2} u_j \right) + \beta \sum_{j=1}^{N} \psi_j(x) u_j = f(x). \tag{24}
\]

By setting \( x = x_i, \ i = 1, 2, ..., N \), in the above equation, we obtain

\[
\sum_{j=1}^{N} \frac{\partial^4 \psi_j(x_i)}{\partial x^4} u_j + 2 \sum_{j=1}^{N} \frac{\partial^4 \psi_j(x_i)}{\partial x^2 \partial y^2} u_j + \sum_{j=1}^{N} \frac{\partial^4 \psi_j(x_i)}{\partial y^4} u_j
\]

\[
- \alpha \left( \sum_{j=1}^{N} \frac{\partial^2 \psi_j(x_i)}{\partial x^2} u_j + \sum_{j=1}^{N} \frac{\partial^2 \psi_j(x_i)}{\partial y^2} u_j \right) + \beta \sum_{j=1}^{N} \psi_j(x_i) u_j = f(x_i). \tag{25}
\]
Eqs. (25), indeed, presents $N \times N$ system of linear algebraic equations. In the matrix format, it can be presented as

$$(A + \beta I)U = b$$

(26)

where

$$b_i = f(x_i), \ i = 1, 2, \ldots, N, \ U = \left(u_1, u_2, \ldots, u_N\right)^T,$$

and

$$A_{ij} = \frac{\partial^4 \psi_j(x_i)}{\partial x^4} + 2 \frac{\partial^4 \psi_j(x_i)}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi_j(x_i)}{\partial y^4} - \alpha \left(\frac{\partial^2 \psi_j(x_i)}{\partial x^2} + \frac{\partial^2 \psi_j(x_i)}{\partial y^2}\right).$$

(28)

5 Imposing multiple boundary conditions

In order to describe in a simple way how to impose multiple boundary conditions, we consider the domain shown in Fig. 1. In this figure, the left hand is regularly distributed nodal points while the right hand is irregularly distributed Halton points. Let us focus on boundary conditions (2), the boundary conditions (3) will be treated in the similar way. Now, in order to impose boundary conditions (2), for those nodes located on $\partial \Omega_1$ with the total number $N_{\partial \Omega_1}$, we have

$$u(x_i) = g_1(x_i), \ i = 1, 2, \ldots, N_{\partial \Omega_1},$$

(29)

$$n_1(x_i) \frac{\partial^2 u(x_i)}{\partial x^2} + n_2(x_i) \frac{\partial^2 u(x_i)}{\partial y^2} = g_2(x_i), \ i = 1, 2, \ldots, N_{\partial \Omega_1},$$

(30)

where $n = n_1(x_i)\textbf{i} + n_2(x_i)\textbf{j}$, is the unit external perpendicular vector on the boundary $\partial \Omega_1$ at $x_i \in \partial \Omega_1$. Eq. (30) can be rewritten as

$$n_1(x_i) \sum_{j=1}^{N} \frac{\partial^2 \psi_j(x_i)}{\partial x^2} u_j + n_2(x_i) \sum_{j=1}^{N} \frac{\partial^2 \psi_j(x_i)}{\partial y^2} u_j = g_2(x_i), \ i = 1, 2, \ldots, N_{\partial \Omega_1}. \quad (31)$$

Eq. (31) equally, via notifications (22), is simplified into

$$n_1(x_i) \sum_{j=1}^{N} D_{x_{ij}}^{(2)} u_j + n_2(x_i) \sum_{j=1}^{N} D_{y_{ij}}^{(2)} u_j = g_2(x_i), \ i = 1, 2, \ldots, N_{\partial \Omega_1}, \quad (32)$$

or

$$\sum_{j=1}^{N} \left(n_1(x_i) D_{x_{ij}}^{(2)} + n_2(x_i) D_{y_{ij}}^{(2)}\right) u_j = g_2(x_i), \ i = 1, 2, \ldots, N_{\partial \Omega_1}. \quad (33)$$
Therefore, considering both Eqs. (29) and (33), we have in sum \(2N_{\partial \Omega_1}\) equations which should be imposed.

Now, let us have a look into the block matrix of the left hand of (26) based on all collocation points \(x_i\), and then replace those rows of blocks \(A + \beta I\) corresponding to collocation at boundary nodes with unit vectors that have a one in the position corresponding to the diagonals of \(A + \beta I\). Also, we consider in right hand of (26) and replace those components of the vector \(b\) matched to collocation points located on boundary knots with \(g_1(x_i)\). We treat the same way for (33), but in this case, use those rows of \(A + \beta I\) corresponding to those nodes which are close to the boundary with total number of \(N_{\partial \Omega_1}\) again. Therefore, the requirement (33) will be explicitly enforced immediately when we have set the right-hand side with respect to the value of \(g_2(x_i)\).

Through reordering the columns and rows of the obtained matrix, it is gained a block matrix in the following format

\[
A'_{N \times N} = \begin{pmatrix}
M & P & Q \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix},
\]

where the non-zero square blocks \(M\) is of size \((N - 2N_{\partial \Omega_1}) \times (N - 2N_{\partial \Omega_1})\), the non-zero square blocks \(P\) and \(Q\) are of size \(N_{\partial \Omega_1} \times N_{\partial \Omega_1}\) and, \(I\) is the identity matrix of order \(N_{\partial \Omega_1} \times N_{\partial \Omega_1}\). Besides, parallel to the right hand side, we would have the following replacement

\[
b'_{N \times 1} = \begin{pmatrix}
b'' \\
g_1 \\
g_2
\end{pmatrix}
\]

where the blocks \(b''\), \(g_1\) and \(g_2\) are of size \((N - 2N_{\partial \Omega_1}) \times 1\), \(N_{\partial \Omega_1} \times 1\) and \(N_{\partial \Omega_1} \times 1\), respectively. Moreover, the components of unknowns in Eqs. (26) are also reordered accordingly which is denoted by \textit{prime}. Therefore, the algebraic system of Eqs. (26) is converted to

\[
A'_{N \times N} U' = b'_{N \times 1}.
\]

Solving the system of Eqs. (36) and reordering indices back into the original we obtain approximate solution. If the domain \(\Omega_2\) in Fig. 1 be considered then we treat similar approach above and more, to impose the boundary conditions (2)-(3) one way is to cover...
the boundary of the domain with too much nodal points and then simply to impose the boundary condition (2) for half of boundary nodes and the boundary condition (3) for the other half of them.

Fig. 1. The considered domain for the biharmonic problem (1)-(3) covering by regularly and irregularly scattered nodal points.

6 Numerical Results

To implement the PSMRPI method, the radius of support domain (that is a circle), to construct basis functions is considered \( r_s = \frac{5}{\sqrt{N}} \). In Eq. (12), we set \( np = 21 \), i.e. the quintic basis functions are used. With the aim of showing the accuracy and the convergence of the approach, we apply the following defined absolute and relative error which are defined by

\[
\text{AbsoluteError}(\mathbf{x}) = |u_{\text{exact}}(\mathbf{x}) - u_{\text{approx}}(\mathbf{x})|,
\]

and

\[
\text{RelativeError}(\mathbf{x}) = \frac{|u_{\text{exact}}(\mathbf{x}) - u_{\text{approx}}(\mathbf{x})|}{\|u_{\text{exact}}(\mathbf{x})\|_\infty}.
\]
Also, define the root mean square error as

\[ \text{RMS_{error}} = \frac{\|u_{\text{exact}}(x) - u_{\text{approx}}(x)\|_{L^2}}{N}, \quad \|u(x)\|_{L^2} = \left(\sum_{i=1}^{N} u^2(x_i)\right)^{\frac{1}{2}}, \tag{39} \]

doing the root mean square error for the approximated solution. Furthermore, the convergence order with respect to the number of nodal points is calculated by the formula

\[ C - \text{order} = \frac{\ln\left(\frac{e_{R_1}}{e_{R_2}}\right)}{\ln\left(\frac{N_2}{N_1}\right)}, \]

where \(e_{R_{i}}, i = 1, 2\) is relative error value corresponding to \(N_{i}, i = 1, 2,\) respectively.

**Example 1** Consider the equation \(\Delta \Delta u = 4 \sin(\pi x) \sin(\pi y)\) with simply supported boundary conditions (2) in \(\Omega_1\) as shown in Fig. 1. The exact solution is given by

\[ u(x) = \frac{1}{\pi^4} \sin(\pi x) \sin(\pi y) \]

and, \(g_1(x)\) and \(g_2(x)\) are obtained accordingly. Figs. 2, 3 and 4 give the approximate PSMRPI solution and its derivative with respect to \(x\) and \(y\), respectively. The AbsoluteError of these approximations has been plotted in Figs. 5, 6 and 7, respectively. Furthermore, the relative error \(e_R\) for different nodal points is observed in Fig. 8 and corresponding \(C - \text{order}\) has been given in Table 1 and more we have compared the obtained results with those obtained in Ref. [49] in Table 2. As it is seen the accuracy is acceptable and the convergence is inspected.
Table 1
C-order of numerical SMRPI solution for Example 1.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$e_R$</th>
<th>$C - order$</th>
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<tr>
<td>81</td>
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<tr>
<td>3721</td>
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<td>0.1884</td>
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</tbody>
</table>

Fig. 3. Derivative of PSMRPI solution with respect to $x$ for Example 1 with $N=1681$. 

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Fig. 4. Derivative of PSMRPI solution with respect to $y$ for Example 1 with $N=1681$.

Fig. 5. The $\text{AbsoluteError}$ of the PSMRPI solution for Example 1 with $N=1681$. 
Fig. 6. The *Absolute Error* of the derivative of PSMRPI solution with respect to $x$ for Example 1 with $N=1681$.

Fig. 7. The *Absolute Error* of the derivative of PSMRPI solution with respect to $y$ for Example 1 with $N=1681$. 
Example 2 Consider another equation as

\[ \Delta \Delta u - 12 \Delta u - 72u = -8 \left( -11 + 24y^2 + 9x^4y^4 + 6x^2(4 - 9y^2) \right) \]

with clamped boundary conditions (3) in \( \Omega_2 \) as shown in Fig. 1. The exact solution is given by

\[ u(\mathbf{x}) = (1 - x^2)^2(1 - y^2)^2 \]

and, \( g_1(\mathbf{x}) \) and \( g_2(\mathbf{x}) \) are extracted accordingly. In this example, the domain is covering by Halton (irregularly distributed) points. Figs. 9, 10 and 11 give the approximate PSMRPI solution and its derivative with respect to \( x \) and \( y \), respectively. Notice that the derivatives with respect to both \( x \) and \( y \) have been approximated on those nodal points of Halton points near the line \( y = 0 \). The AbsoluteError of these approximations has been plotted in Figs. 12, 13 and 14, respectively. Moreover, in this example for the case of regular distribution of nodal points, a compassion has been made with the results of the method of Ref. [49] in Table 2.
Fig. 9. The PSMRPI solution for Example 2 with N=780.

Fig. 10. Derivative of PSMRPI solution with respect to $x$ near the line $y = 0$ for Example 2 with N=780.
Fig. 11. Derivative of PSMRPI solution with respect to \( y \) near the line \( y = 0 \) for Example 2 with \( N=780 \).

Fig. 12. The AbsoluteError of the PSMRPI solution for Example 2 with \( N=780 \).
Fig. 13. The AbsoluteError of the derivative of PSMRPI solution with respect to $x$ near the line $y = 0$ for Example 2 with $N=780$.

Fig. 14. The AbsoluteError of the derivative of PSMRPI solution with respect to $y$ near the line $y = 0$ for Example 2 with $N=780$.

**Example 3** As a last test problem, focus on biharmonic equation

$$\Delta \Delta u + 64 \Delta u + u = -2048 + 27x (x - 1)^2 - 4 (2y - 1)^2$$

subject to clamped boundary conditions (3) in $\Omega$ as shown in Fig. 15. The exact solution is given by

$$u(x) = 27x (x - 1)^2 - 4 (2y - 1)^2$$

and, $g_1(x)$ and $g_2(x)$ are obtained accordingly. We have shown the numerical SMRPI solution on the domain $\Omega$ and its boundary in Fig. 6. Also, in this figure, the RelativeError
Table 2
Comparison of the obtained results by the present method and Ref. [49] for Examples 1 and 2 with regular distribution of nodal points.

<table>
<thead>
<tr>
<th></th>
<th>Example 1</th>
<th>Example 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>81</td>
<td>$9.237 \times 10^{-1}$</td>
<td>$1.257 \times 10^{-1}$</td>
</tr>
<tr>
<td>121</td>
<td>$2.125 \times 10^{-1}$</td>
<td>$5.010 \times 10^{-2}$</td>
</tr>
<tr>
<td>441</td>
<td>$6.350 \times 10^{-2}$</td>
<td>$8.096 \times 10^{-3}$</td>
</tr>
<tr>
<td>1081</td>
<td>$1.030 \times 10^{-2}$</td>
<td>$7.387 \times 10^{-3}$</td>
</tr>
<tr>
<td>1681</td>
<td>$8.097 \times 10^{-3}$</td>
<td>$5.620 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 3
C-order of numerical SMRPI solution for Example 3.

<table>
<thead>
<tr>
<th>N</th>
<th>$e_R$</th>
<th>$C - order$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>0.0062</td>
<td>-- --</td>
</tr>
<tr>
<td>88</td>
<td>0.0055</td>
<td>0.3128</td>
</tr>
<tr>
<td>314</td>
<td>0.0014</td>
<td>1.0756</td>
</tr>
<tr>
<td>766</td>
<td>7.8414e-04</td>
<td>0.6500</td>
</tr>
<tr>
<td>1188</td>
<td>5.6943e-04</td>
<td>0.7291</td>
</tr>
</tbody>
</table>

is seen for both the domain and its boundary. Moreover, we have provided $C - order$ of numerical SMRPI solution in Table 3.

Fig. 15. Domain of the problem for Example 3 with N=1188 nodal points.
Fig. 16. SMRPI solution to the Example 3 and corresponding Relative Error with N=1188.
7 Conclusions

In this paper, we have developed pseudospectral meshless radial point interpolation (PSMRPI) in order to apply it to a generalization of the well known elliptic partial differential equation namely the biharmonic problem subject to multiple boundary conditions. There are many options to impose simply supported or clamped boundary conditions in the framework of PSMRPI. There are no concerns to approximate the high order derivatives of unknowns. Since the method is truly meshless technique we do not worry about the complexity of the domain which is easily covered by scattered nodal points. It is seen that PSMRPI has less computational costs. Therefore, we can apply the PSMRPI approach to some other types of more complicated problems with reliable accuracy.

References


