Vicious Circle Principle and Logic Programs with Aggregates

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Abstract

The paper presents a knowledge representation language \texttt{a/log} which extends ASP with aggregates. The goal is to have a language based on simple syntax and clear intuitive and mathematical semantics. We give some properties of \texttt{a/log}, an algorithm for computing its answer sets, and comparison with other approaches.

KEYWORDS: Aggregates, Answer Set Programming

1 Introduction

The development of answer set semantics for logic programs (Gelfond and Lifschitz 1988; Gelfond and Lifschitz 1991) led to the creation of powerful knowledge representation language, Answer Set Prolog (ASP), capable of representing recursive definitions, defaults, effects of actions and other important phenomena of natural language. The design of algorithms for computing answer sets and their efficient implementations in systems called ASP solvers (Niemela et al. 2002; Leone et al. 2006; Gebser et al. 2007) allowed the language to become a powerful tool for building non-trivial knowledge intensive applications (Brewka et al. 2011; Erdem et al. 2012). There are a number of extensions of the ASP which also contributed to this success. This paper is about one such extension – logic programs with aggregates. By aggregates we mean functions defined on sets of objects of the domain. (For simplicity of exposition we limit our attention to aggregates defined on finite sets.) Here is a typical example.

Example 1 (Classes That Need Teaching Assistants)

Suppose that we have a complete list of students enrolled in a class \(c\) that is represented by the following collection of atoms:

\[
enrolled(c,mike).
enrolled(c,john).
\]

Suppose also that we would like to define a new relation \texttt{need}\(\texttt{fa}(C)\) that holds iff the class \(C\) needs a teaching assistant. In this particular school \texttt{need}\(\texttt{fa}(C)\) is true iff the number of students enrolled in the class is greater than 20. The definition can be given by a simple rule in the language of logic programs with aggregates:

\[
\texttt{need}\(\texttt{fa}(C)\) ← \texttt{card}\{X : \texttt{enrolled}(C,X)\} > 20
\]
where \textit{card} stands for the cardinality function. Let us call the resulting program $P_0$.

The program is simple, has a clear intuitive meaning, and can be run on some of the existing ASP solvers. However, the situation is more complex than that. Unfortunately, currently there is no the language of logic programs with aggregates. Instead there is a comparatively large collection of such languages with different syntax and, even more importantly, different semantics (Pelov et al. 2007; Niemela et al. 2002; Son and Pontelli 2007; Faber et al. 2011; Gelfond 2002; Kemp and Stuckey 1991). As an illustration consider the following example:

\textbf{Example 2}

Let $P_1$ consist of the following rule:

\[ p(a) \leftarrow \text{card}\{X : p(X)\} = 1. \]

Even for this seemingly simple program, there are different opinions about its meaning. According to (Faber et al. 2011) the program has one answer set $A = \{\}$; according to (Gelfond 2002; Kemp and Stuckey 1991) it has two answer sets: $A_1 = \{\}$ and $A_2 = \{p(a)\}$.

In our judgment this and other similar “clashes of intuition” cause a serious impediment to the use of aggregates for knowledge representation and reasoning. In this paper we aim at addressing this problem by suggesting yet another logic programming language with aggregates, called $\mathcal{A}log$, which is based on the following design principles:

\begin{itemize}
  \item the language should have a simple syntax and intuitive semantics based on understandable informal principles, and
  \item the informal semantics should have clear and elegant mathematics associated with it.
\end{itemize}

In our opinion existing extensions of ASP by aggregates often do not have clear intuitive principles underlying the semantics of the new constructs. Moreover, some of these languages violate such original foundational principles of ASP as the rationality principle. The problem is compounded by the fact that some of the semantics of aggregates use rather non-trivial mathematical constructions which makes it difficult to understand and explain their intuitive meaning.

The semantics of $\mathcal{A}log$ is based on Vicious Circle Principle (VCP): no object or property can be introduced by the definition referring to the totality of objects satisfying this property. According to Feferman (Feferman 2002) the principle was first formulated by Poincare (Poincare 1906) in his analysis of paradoxes of set theory. Similar ideas were already successfully used in a collection of logic programming definitions of stratification including that of stratified aggregates (see, for instance, (Faber et al. 2011). Unfortunately, limiting the language to stratified aggregates eliminates some of the useful forms of circles (see Example 9 below). In this paper we give a new form of VCP which goes beyond stratification: $p(a)$ cannot be introduced by the definition referring to a set of objects satisfying $p$ if this set can contain $a$. Technically, the principle is incorporated in our new definition of answer set (which coincides with the original definition for programs without aggregates). The definition is short and simple. We hope that, combined with a number of informal examples, it will be sufficient for developing an intuition necessary for the use of the language. The paper is organized as follows. In Section 2, we define the syntax and semantics of $\mathcal{A}log$. We give some properties of $\mathcal{A}log$ programs in Section 3 and present an algorithm for computing an answer set of an $\mathcal{A}log$ program in Section 4. A comparison with the existing work is done in Section 5, and we conclude the paper in Section 6.
2 Syntax and Semantics of $\mathcal{A}log$

We start with defining the syntax and intuitive semantics of the language.

2.1 Syntax

Let $\Sigma$ be a (possibly sorted) signature with a finite collection of predicate, function, and object constants and $\mathcal{A}$ be a finite collection of symbols used to denote functions from finite sets of terms of $\Sigma$ into integers. Terms and literals over signature $\Sigma$ are defined as usual and referred to as regular. Regular terms are called ground if they contain no variables and no occurrences of symbols for arithmetic functions. Similarly for literals. An aggregate term is an expression of the form

$$f\{\bar{X}:\text{cond}\}$$

where $f \in \mathcal{A}$, cond is a collection of regular literals, and $\bar{X}$ is a list of variables occurring in cond. We refer to an expression

$$\{\bar{X}:\text{cond}\}$$

as a set name. An occurrence of a variable from $\bar{X}$ in (2) is called bound within (2). If the condition from (2) contains no variables except those in $\bar{X}$ then it is read as the set of all objects of the program satisfying cond. If cond contains other variables, say $\bar{Y} = \langle Y_1, \ldots, Y_n \rangle$, then $\{\bar{X}:\text{cond}\}$ defines the function mapping possible values $\bar{c} = \langle c_1, \ldots, c_n \rangle$ of these variables into sets $\{\bar{X}:\text{cond}\}|_{\bar{c}}$ where $\text{cond}|_{\bar{c}}$ is the result of replacing $Y_1, \ldots, Y_n$ by $c_1, \ldots, c_n$.

By an aggregate atom we mean an expression of the form

$$(\text{aggregate term})(\text{arithmetic relation})(\text{arithmetic term})$$

where arithmetic relation is $>$, $\geq$, $<$, $\leq$, $=$ or $\neq$, and arithmetic term is constructed from variables and integers using arithmetic operations, $+$, $-$, $\times$, etc.

By e-literals we mean regular literals possibly preceded by default negation not. The latter (former) are called negative (positive) e-literals.

A rule of $\mathcal{A}log$ is an expression of the form

$$\text{head} \leftarrow \text{pos}, \text{neg}, \text{agg}$$

where head is a disjunction of regular literals, pos and neg are collections of regular literals and regular literals preceded by not respectively, and agg is a collection of aggregate atoms. All parts of the rule, including head, can be empty. An occurrence of a variable in (4) not bound within any set name in this rule is called free in (4). A rule of $\mathcal{A}log$ is called ground if it contains no occurrences of free variables and no occurrences of arithmetic functions.

A program of $\mathcal{A}log$ is a finite collection of $\mathcal{A}log$’s rules. A program is ground if its rules are ground.

As usual for ASP based languages, rules of $\mathcal{A}log$ program with variables are viewed as collections of their ground instantiations. A ground instantiation of a rule $r$ is the program obtained from $r$ by replacing free occurrences of variables in $r$ by ground terms of $\Sigma$ and evaluating all arithmetic functions. If the signature $\Sigma$ is sorted (as, for instance, in (Balai et al. 2013)) the substitutions should respect sort requirements for predicates and functions.

Clearly the grounding of an $\mathcal{A}log$ program is a ground program. The following examples illustrate the definition:
Example 3 (Grounding: all occurrences of the set variable are bound)

Consider a program $P_2$ with variables:

\[
q(Y) :- \text{card}\{X:p(X,Y)\} = 1, \ r(Y).
\]
\[
r(a). \ r(b). \ p(a,b).
\]

Here all occurrences of a set variable $X$ are bound; all occurrences of a variable $Y$ are free. The program’s grounding, ground($P_2$), is

\[
q(a) :- \text{card}\{X:p(X,a)\} = 1, \ r(a).
\]
\[
q(b) :- \text{card}\{X:p(X,b)\} = 1, \ r(b).
\]
\[
r(a). \ r(b). \ p(a,b).
\]

The next example deals with the case when some occurrences of the set variable in a rule are free and some are bound.

Example 4 (Grounding: some occurrences of a set variable are free)

Consider an $\mathcal{A}$log program $P_3$

\[
r :- \text{card}\{X:p(X)\} \geq 2, \ q(X).
\]
\[
p(a). \ p(b). \ q(a).
\]

Here the occurrence of $X$ in $q(X)$ is free. Hence the ground program ground($P_3$) is:

\[
r :- \text{card}\{X:p(X)\} \geq 2, \ q(a).
\]
\[
r :- \text{card}\{X:p(X)\} \geq 2, \ q(b).
\]
\[
p(a). \ p(b). \ q(a).
\]

Note that despite its apparent simplicity the syntax of $\mathcal{A}$log differs substantially from syntax of most other logic programming languages allowing aggregates (with the exception of that in (Gelfond 2002)). We illustrate the differences using the language presented in (Faber et al. 2011). (In what follows we refer to this language as $\mathcal{F}$log.) While syntactically programs of $\mathcal{A}$log can also be viewed as programs of $\mathcal{F}$log the opposite is not true. Among other things $\mathcal{F}$log allows parameters of aggregates to be substantially more complex than those of $\mathcal{A}$log. For instance, an expression $f\{a : p(a,a), b : p(b,a)\} = 1$ where $f$ is an aggregate atom of $\mathcal{F}$log but not of $\mathcal{A}$log. This construction which is different from a usual set-theoretic notation used in $\mathcal{A}$log is important for the $\mathcal{F}$log definition of grounding. For instance the grounding of the first rule of program $P_2$ from Example 3 understood as a program of $\mathcal{F}$log consists of $\mathcal{F}$log rules

\[
q(a) :- \text{card}\{a:p(a,a), b:p(b,a)\} = 1, \ r(a).
\]
\[
q(b) :- \text{card}\{a:p(a,b), b:p(b,b)\} = 1, \ r(b).
\]

which is not even a program of $\mathcal{A}$log. Another important difference between the grounding methods of these languages can be illustrated by the $\mathcal{F}$log grounding ground$_{\mathcal{F}}$(P$_3$) of program $P_3$ from Example 4 that looks as follows:

\[
r :- \text{card}\{a:p(a)\} \geq 2, \ q(a).
\]
\[
r :- \text{card}\{b:p(b)\} \geq 2, \ q(b).
\]
\[
p(a). \ p(b). \ q(a).
\]

Clearly this is substantially different from the $\mathcal{A}$log grounding of $P_3$ from Example 4. In Section 5 we show that this difference in grounding reflects substantial semantic differences between the two languages.
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2.2 Semantics

To define the semantics of $\mathcal{A}$log programs we expand the standard definition of answer set from (Gelfond and Lifschitz 1988). The resulting definition captures the rationality principle - believe nothing you are not forced to believe (Gelfond and Kahl 2014) - and avoids vicious circles. As usual the definition of answer set is given for ground programs. Some terminology: a ground aggregate atom $f\{X : p(X)\} \circ n$ (where $\circ$ is one of the arithmetic relations allowed in the language) is true in a set of ground regular literals $S$ if $f\{X : p(X) \in S\} \circ n$; otherwise the atom is false in $I$.

Definition 1 (Aggregate Reduct)
The aggregate reduct of a ground program $\Pi$ of $\mathcal{A}$log with respect to a set of ground regular literals $S$ is obtained from $\Pi$ by

1. removing from $\Pi$ all rules containing aggregate atoms false in $S$.
2. replacing every remaining aggregate atom $f\{X : p(X)\} \circ n$ by the set $\{p(t) : p(t) \in S\}$ (which is called the reduct of the aggregate with respect to $S$).

(Here $p(t)$ is the result of replacing variable $X$ by ground term $t$). The second clause of the definition reflects the principle of avoiding vicious circles – a rule with aggregate atom $f\{X : p(X)\} \circ n$ in the body can only be used if “the totality” of all objects satisfying $p$ has already being constructed. Attempting to apply this rule to define $p(t)$ will either lead to contradiction or to turning the rule into tautology (see Examples 7 and 9).

Definition 2 (Answer Set)
A set $S$ of ground regular literals over the signature of a ground program $\Pi$ of $\mathcal{A}$log is an answer set of $\Pi$ if it is an answer set of an aggregate reduct of $\Pi$ with respect to $S$.

We will illustrate this definition by a number of examples.

Example 5 (Example 3 Revisited)
Consider a program $P_2$ and its grounding from Example 3. It is easy to see that the aggregate reduct of the program with respect to any set $S$ not containing $p(a, b)$ consists of the program facts, and hence $S$ is not an answer set of $P_2$. However the program’s aggregate reduct with respect to $A = \{q(b), r(a), r(b), p(a, b)\}$ consists of the program’s facts and the rule

$q(b) :- p(a, b), r(b)$.

Hence $A$ is an answer set of $P_2$.

Example 6 (Example 4 Revisited)
Consider now the grounding

$r :- \text{card}\{X : p(X)\} \geq 2, q(a)$.
$r :- \text{card}\{X : p(X)\} \geq 2, q(b)$.
$p(a), p(b), q(a)$.

of program $P_3$ from Example 4. Any answer set $S$ of this program must contain its facts. Hence $\{X : p(X) \in S\} = \{a, b\}$. $S$ satisfies the body of the first rule and must also contain $r$. Indeed, the aggregate reduct of $P_3$ with respect to $S = \{p(a), p(b), q(a), r\}$ consists of the facts of $P_3$ and the rules
r :- p(a),p(b),q(a).
r :- p(a),p(b),q(b).

Hence $S$ is the answer set of $P_3$.

Neither of the two examples above required the application of VCP. The next example shows how this principle influences our definition of answer sets and hence our reasoning.

**Example 7 (Example 2 Revisited)**
Consider a program $P_1$ from Example 2. The program, consisting of a rule
\[ p(a) : - \text{card}\{X : p(X)\} = 1 \]
is grounded. It has two candidate answer sets, $S_1 = \{\}$ and $S_2 = \{p(a)\}$. The aggregate reduct of the program with respect to $S_1$ is the empty program. Hence, $S_1$ is an answer set of $P_1$. The program’s aggregate reduct with respect to $S_2$ however is
\[ p(a) : - p(a). \]
The answer set of this reduct is empty and hence $S_1$ is the only answer of $P_1$.

Example 7 shows how the attempt to define $p(a)$ in terms of totality of $p$ turns the defining rule into a tautology. The next example shows how it can lead to inconsistency of a program.

**Example 8 (Vicious Circles through Aggregates and Inconsistency)**
Consider a program $P_4$:
\[ p(a), p(b) : - \text{card}\{X:p(X)\} > 0. \]
Since every answer set of the program must contain $p(a)$, the program has two candidate answer sets: $S_1 = \{p(a)\}$ and $S_2 = \{p(a), p(b)\}$. The aggregate reduct of $P_4$ with respect to $S_1$ is
\[ p(a), p(b) : - p(a). \]
The answer set of the reduct is $\{p(a), p(b)\}$ and hence $S_1$ is not an answer set of $P_4$. The reduct of $P_4$ with respect to $S_2$ is
\[ p(a), p(b) : - p(a), p(b). \]
Again its answer set is not equal to $S_2$ and hence $P_4$ is inconsistent (i.e., has no answer sets). The inconsistency is the direct result of an attempt to violate the underlying principle of the semantics. Indeed, the definition of $p(b)$ refers to the set of objects satisfying $p$ that can contain $b$ which is prohibited by our version of VCP. One can, of course, argue that $S_2$ can be viewed as a reasonable collection of beliefs which can be formed by a rational reasoner associated with $P_4$. After all, we do not need the totality of $p$ to satisfy the body of the rule defining $p(b)$. It is sufficient to know that $p$ contains $a$. This is indeed true but this reasoning depends on the knowledge which is not directly incorporated in the definition of $p(b)$. If one were to replace $P_4$ by
\[ p(a), p(b) : - \text{card}\{X:p(X), X \neq b\} > 0. \]
then, as expected, the vicious circle principle will not be violated and the program will have unique answer set $\{p(a), p(b)\}$.

We end this section by a simple but practical example of a program which allows recursion through aggregates but avoids vicious circles.
Example 9 (Defining Digital Circuits)
Consider part of a logic program formalizing propagation of binary signals through simple digital circuits. We assume that the circuit does not have a feedback, i.e., a wire receiving a signal from a gate cannot be an input wire to this gate. The program may contain a simple rule

\[
\text{val}(W,0) :- \\
\text{gate}(G, \text{and}), \\
\text{output}(W, G), \\
\text{card}\{W: \text{val}(W,0), \text{input}(W, G)\} > 0.
\]

(partially) describing propagation of symbols through an \textit{and} gate. Here \text{val}(W,S) holds iff the digital signal on a wire \(W\) has value \(S\). Despite its recursive nature the definition of \text{val} avoids vicious circle. To define the signal on an output wire \(W\) of an \textit{and} gate \(G\) one needs to only construct a particular subset of input wires of \(G\). Since, due to absence of feedback in our circuit, \(W\) can not belong to the latter set our definition is reasonable. To illustrate that our definition of answer set produces the intended result let us consider program \(P_5\) consisting of the above rule and a collection of facts:

\[
\text{gate}(g, \text{and}). \\
\text{output}(w_0, g). \\
\text{input}(w_1, g). \\
\text{input}(w_2, g). \\
\text{val}(w_1,0).
\]

The grounding, \text{ground}(P_5), of \(P_5\) consists of the above facts and the three rules of the form

\[
\text{val}(w,0) :- \\
\text{gate}(g, \text{and}), \\
\text{output}(w, g), \\
\text{card}\{W: \text{val}(W,0), \text{input}(W, g)\} > 0.
\]

where \(w\) is \(w_0, w_1\), and \(w_2\).

Let \(S = \{\text{gate}(g, \text{and}), \text{val}(w_1,0), \text{val}(w_0,0), \text{output}(w_0, g), \text{input}(w_1, g), \text{input}(w_2, g)\}\). The aggregate reduct of \text{ground}(P_5) with respect to \(S\) is the collection of facts and the rules

\[
\text{val}(w,0) :- \\
\text{gate}(g, \text{and}), \\
\text{output}(w, g), \\
\text{input}(w_1, g), \\
\text{val}(w_1, 0).
\]

where \(w\) is \(w_0, w_1\), and \(w_2\).

The answer set of the reduct is \(S\) and hence \(S\) is an answer set of \(P_5\). As expected it is the only answer set. (Indeed it is easy to see that other candidates do not satisfy our definition.)

3 Properties of \(\mathcal{A}\text{log}\) programs
In this section we give some basic properties of \(\mathcal{A}\text{log}\) programs. Propositions 1 and 2 ensure that, as in regular ASP, answer sets of \(\mathcal{A}\text{log}\) program are formed using the program rules together with the rationality principle. Proposition 3 is the \(\mathcal{A}\text{log}\) version of the basic technical tool used
in theoretical investigations of ASP and its extensions. Proposition 4 shows that complexity of entailment in $\mathcal{A}log$ is the same as that in regular ASP.

We will use the following terminology: e-literals $p$ and $\neg p$ are called contrary; not $l$ denotes a literal contrary to e-literal $l$; a partial interpretation $I$ over signature $\Sigma$ is a consistent set of e-literals of this signature; an e-literal $l$ is true in $I$ if $l \in I$; it is false if $\neg l \in I$; otherwise $l$ is undefined in $I$. An aggregate atom $f\{X : q(X)\} \circ n$ is true in $I$ if $f\{t : q(t) \in I\} \circ n$ is true, i.e., the value of $f$ on the set $\{t : q(t) \in I\}$ and the number $n$ satisfy property $\circ$. Otherwise, the atom is false in $I$. The head of a rule is satisfied by $I$ if at least one of its literals is true in $I$; the body of a rule is satisfied by $I$ if all of its aggregate atoms and e-literals are true in $I$. A rule is satisfied by $I$ if its head is satisfied by $I$ or its body is not satisfied by $I$.

**Proposition 1 (Rule Satisfaction and Supportedness)**

Let $A$ be an answer set of a ground $\mathcal{A}log$ program $\Pi$. Then

1. $A$ satisfies every rule $r$ of $\Pi$.
2. If $p \in A$ then there is a rule $r$ from $\Pi$ such that the body of $r$ is satisfied by $A$ and $p$ is the only atom in the head of $r$ which is true in $A$. (It is often said that rule $r$ supports atom $p$.)

**Proposition 2 (Anti-chain Property)**

Let $A_1$ be an answer set of an $\mathcal{A}log$ program $\Pi$. Then there is no answer set $A_2$ of $\Pi$ such that $A_1$ is a proper subset of $A_2$.

**Proposition 3 (Splitting Set Theorem)**

Let $\Pi_1$ and $\Pi_2$ be programs of $\mathcal{A}log$ such that no atom occurring in $\Pi_1$ is a head atom of $\Pi_2$. Let $S$ be a set of atoms containing all head atoms of $\Pi_1$ but no head atoms of $\Pi_2$. A set $A$ of atoms is an answer set of $\Pi_1 \cup \Pi_2$ iff $A \cap S$ is an answer set of $\Pi_1$ and $A$ is an answer set of $(A \cap S) \cup \Pi_2$.

**Proposition 4 (Complexity)**

The problem of checking if a ground atom $a$ belongs to all answer sets of an $\mathcal{A}log$ program is $\Pi_2^c$ complete.

### 4 An Algorithm for Computing Answer Sets

In this section we briefly outline an algorithm, called $\mathcal{A}solver$, for computing answer sets of $\mathcal{A}log$ programs. We follow the tradition and limit our attention to programs without classical negation. Hence, in this section we consider only programs of this type. By an atom we mean an e-atom or an aggregate atom.

**Definition 3 (Strong Satisfiability and Refutability)**

- An atom is strongly satisfied (strongly refuted) by a partial interpretation $I$ if it is true (false) in every partial interpretation containing $I$; an atom which is neither strongly satisfied nor strongly refuted by $I$ is undecided by $I$.
- A set $S$ of atoms is strongly satisfied by $I$ if all atoms in $S$ are strongly satisfied by $I$;
- $S$ is strongly refuted by $I$ if for every partial interpretation $I'$ containing $I$, some atom of $S$ is false in $I'$.  

For instance, an e-atom is strongly satisfied (refuted) by \( I \) iff it is true (false) in \( I \); an atom \( \text{card}\{X : p(X)\} > n \) which is true in \( I \) is strongly satisfied by \( I \); an atom \( \text{card}\{X : p(X)\} < n \) which is false in \( I \) is strongly refuted by \( I \); and a set \( \{f\{X : p(X)\} > 5, f\{X : p(X)\} < 3\} \) is strongly refuted by any partial interpretation.

\( \alpha \) solver consists of three functions: Solver, Cons, and IsAnswerSet. The main function, Solver, is similar to that used in standard ASP algorithms (See, for instance, Solver1 from (Gelfond and Kahl 2014)). But unlike these functions which normally have two parameters - partial interpretation \( I \) and program \( \Pi \) - Solver has two additional parameters, \( TA \) and \( FA \) containing aggregate atoms that must be true and false respectively in the answer set under construction. Solver returns \( (I, \text{true}) \) where \( I \) is an answer set of \( \Pi \) compatible with its parameters and \textit{false} if no such answer set exists. The Solver’s description will be omitted due to space limitations. The second function, Cons, computes the consequences of its parameters - a program \( \Pi \), a partial interpretation \( I \), and two above described sets \( TA \) and \( FA \) of aggregates atoms. Due to the presence of aggregates the function is sufficiently different from a typical Cons function of ASP solvers so we describe it in some detail. The new value of \( I \), containing the desired consequences is computed by application of the following \textit{inference rules}:

1. If the body of a rule \( r \) is strongly satisfied by \( I \) and all atoms in the head of \( r \) except \( p \) are false in \( I \) then \( p \) must be in \( I \).
2. If an atom \( p \in I \) belongs to the head of exactly one rule \( r \) of \( \Pi \) then every other atom from the head of \( r \) must have its complement in \( I \), the e-atoms from the body of \( r \) must be in \( I \) and its aggregate atoms must be in \( TA \).
3. If every atom of the head of a rule \( r \) is false in \( I \), and \( l \) is the only premise of \( r \) which is either an undefined e-atom or an aggregate atom not in \( FA \), and the rest of the body is strongly satisfied by \( I \), then
   
   (a) if \( l \) is an e-atom, then the complement of \( l \) must be in \( I \),
   
   (b) if \( l \) is an aggregate atom, then it must be in \( FA \).
4. If the body of every rule with \( p \) in the head is strongly refuted by \( I \), then \( \{\not p\} \) must be in \( I \).

Given an interpretation \( I \), a program \( \Pi \), inference rule \( i \in [1..4] \) and \( r \in \Pi \), let function \( i\text{Cons}(i, I, \Pi, r) \) return \( \langle \delta I, \delta TA, \delta FA \rangle \) where \( \delta I, \delta TA \) and \( \delta FA \) are the results of applying inference rule \( i \) to \( r \). (Note, that inference rule 4 does not really use \( r \).) We also need the following terminology. We say that \( I \) is \textit{compatible} with \( TA \) if \( TA \) is not strongly refuted by \( I \); \( I \) is \textit{compatible} with \( FA \) if no atom from \( FA \) is strongly satisfied by \( I \). A set \( A \) of regular atoms is \textit{compatible} with \( TA \) and \( FA \) if the set \( \text{compl}(A) = \{p : p \in A\} \cup \{\not a : a \notin A\} \) is \textit{compatible} with \( TA \) and \( FA \); \( A \) is compatible with \( I \) if \( I \subseteq \text{compl}(A) \). The algorithm Cons is listed below.

function Cons  
\textbf{input:} partial interpretation \( h_0 \), sets \( TA_0 \) and \( FA_0 \) of aggregate atoms compatible with \( h_0 \), and program \( \Pi_0 \) with signature \( \Sigma_0 \);

\textbf{output:} \( (\Pi, I, TA, FA, \text{true}) \) where \( I \) is a partial interpretation such that \( h_0 \subseteq I \).
\( TA \) and \( FA \) are sets of aggregate atoms such that \( TA_0 \subseteq TA \) and \( FA_0 \subseteq FA \).
\( I \) is compatible with \( TA \) and \( FA \), and \( \Pi \) is a program with signature \( \Sigma_0 \) such that for every \( \Lambda \),
A is an answer set of $\Pi_0$ that is compatible with $I_0$ iff $A$ is an answer set of $\Pi$ that is compatible with $I$.

$(\Pi_0,I_0,TA_0,FA_0,\text{false})$ if there is no answer set of $\Pi_0$ compatible with $I_0$;

var $I,T$: set of e-atoms; $TA,FA$: set of aggregate atoms; $\Pi$: program;
1. Initialize $I, \Pi, TA$ and $FA$ to be $I_0, \Pi_0, TA_0$ and $FA_0$ respectively;
2. repeat
3. $T := I$;
4. Remove from $\Pi$ all the rules whose bodies are strongly falsified by $I$;
5. Remove from the bodies of rules of $\Pi$ all negative e-atoms true in $I$ and aggregate atoms strongly satisfied by $I$;
6. Non-deterministically select an inference rule $i$ from (1)–(4);
7. for every $r \in \Pi$
8. $< \delta I, \delta TA, \delta FA > := iCons(I,\Pi,i,r)$;
9. $I := I \cup \delta I, TA := TA \cup \delta TA$,
10. $FA := FA \cup \delta FA$;
11. until $I = T$;
12. if $I$ is consistent, $TA$ and $FA$ are compatible with $I$ then
13. return $< \Pi,I,TA,FA,\text{true}>$;
14. else return $< \Pi_0,I_0,TA_0,FA_0,\text{false}>$;

The third function, $\text{IsAnswerSet}$ of our solver $\text{Solver}$ checks if interpretation $I$ is an answer set of a program $\Pi$. It computes the aggregate reduct of $\Pi$ with respect to $I$ and applies usual checking algorithm (see, for instance, (Koch et al. 2003)).

**Proposition 5 (Correctness of the Solver)**

If, given a program $\Pi_0$, a partial interpretation $I_0$, and sets $TA_0$ and $FA_0$ of aggregate atoms $\text{Solver}(I_0,TA_0,FA_0,\Pi_0)$ returns $(I,\text{true})$ then $I$ is an answer set of $\Pi_0$ compatible with $I_0$, $TA_0$ and $FA_0$. If there is no such answer set, the solver returns $false$.

To illustrate the algorithm consider a program $\Pi$

$$:- p(a).$$

$$p(a) :- \text{card}\{X: q(X)\} > 0, q(a) \text{ or } p(b).$$

and trace $\text{Solver}(\Pi,I,TA,FA)$ where $I$, $TA$, and $FA$ are empty. $\text{Solver}$ starts by calling $\text{Cons}$ which computes the consequence $not p(a)$ (from the first rule of the program), $FA = \{\text{card}\{X: q(X)\} > 0\}$ (from the second rule of the program) and $not q(b)$ (from the fourth inference rule), and returns $true$, $I = \{not q(b), not p(a)\}$ and new $FA$; $TA$ is unchanged. $\text{Solver}$ then guesses $q(a)$ to be true, i.e., $I = \{not q(b), not p(a), q(a)\}$, and calls $\text{Cons}$ again. $\text{Cons}$ does not produce any new consequences but finds that $FA$ is not compatible with $I$ (line 12 of the algorithm). So, it returns $false$, which causes $\text{Solver}$ to set $q(a)$ to be false, i.e., $I = \{not q(b), not p(a), not q(a)\}$. $\text{Solver}$ then calls $\text{Cons}$ again which returns $I = \{not q(b), not p(a), not q(a), p(b)\}$. $\text{Solver}$ finds that $I$ is complete and calls $\text{IsAnswerSet}$ which returns true. Finally, $\text{Solver}$ returns $I$ as an answer set of the program.

5 **Comparison with Other Approaches**

There are a large number of approaches to the syntax and semantics of extensions of ASP by aggregates. In this section we concentrate on languages from (Son and Pontelli 2007) and (Faber...
et al. 2011) which we refer to as $\mathcal{I}log$ and $\mathcal{F}log$ respectively. Due to multiple equivalence results discussed in these papers this is sufficient to cover most of the approaches. The main difference between the syntax of aggregates in $\mathcal{s}\mathcal{l}og$ and $\mathcal{F}log$ is in treatment of variables occurring in aggregate terms. $\mathcal{s}\mathcal{l}og$ uses usual logical concept of bound and free occurrence of a variable (the occurrence of $X$ within $S = \{ X : p(X,Y) \}$ is bound while the occurrence of $Y$ is free). $\mathcal{F}log$ uses very different concepts of global and local variable of a rule. A variable is local in rule $r$ if it occurs solely in an aggregate term of $r$; otherwise, the variable is global. As the result, in $\mathcal{s}\mathcal{l}og$, every aggregate term $\{ X : p(X) \}$ can be replaced by a term $\{ Y : p(Y) \}$ while it is not the case in $\mathcal{F}log$. In our opinion the approach of $\mathcal{F}log$ (and many other languages and systems which adopted this syntax) makes declarative reading of aggregate terms substantially more difficult. To see the semantic ramifications of the $\mathcal{F}log$ treatment of variables consider the following example:

Example 10 (Variables in Aggregate Terms: Global versus Bound)
Consider program $P_3$ from Example 4. According to $\mathcal{F}log$ the meaning of an occurrence of an expression $\{ X : p(X) \}$ in the body of the program’s first rule changes if $X$ is replaced by a different variable. In $\mathcal{s}\mathcal{l}og$, where $X$ is understood as bound this is not the case. This leads to substantial difference in grounding and in the semantics of the program. In $\mathcal{s}\mathcal{l}og$ $P_3$ has one answer set, $\{ p(a), p(b), q(a), r \}$. In $\mathcal{F}log$ answer sets of $P_3$ are those of $\text{ground}_1(P_3)$. The answer set of the latter is $\{ p(a), p(b), q(a) \}$.

Other semantic differences are due to the multiplicity of informal (and not necessarily clearly spelled out) principles underlying various semantics.

Example 11 (Vicious Circles in $\mathcal{F}log$)
Consider the following program, $P_6$, adopted from (Son and Pontelli 2007):

\[
\begin{align*}
p(1) & :- p(0). \\
p(0) & :- p(1). \\
p(1) & :- \text{count}\{ X : p(X) \} \neq 1.
\end{align*}
\]

which, if viewed as $\mathcal{F}log$ program, has one answer set $A = \{ p(0), p(1) \}$. Informal argument justifying this result goes something like this: Clearly, $A$ satisfies the rules of the program. To satisfy the minimality principle no proper subset of $A$ should be able to do that, which is easily checked to be true. Faber et al use so called black box principle: “when checking stability they [aggregate literals] are either present in their entirety or missing altogether”, i.e., the semantics of $\mathcal{F}log$ does not consider the process of derivation of elements of the aggregate parameter. Note however, that the program’s definition of $p(1)$ is given in terms of fully defined term $\{ X : p(X) \}$, i.e., the definition contains a vicious circle. This explains why $A$ is not an answer set of $P_6$ in $\mathcal{s}\mathcal{l}og$. In this particular example we are in agreement with $\mathcal{F}log$ which requires that the value of an aggregate atom can be computed before the rule with this atom in the body can be used in the construction of an answer set.

The absence of answer set of $P_6$ in $\mathcal{F}log$ may suggest that it adheres to our formalization of the VCP. The next example shows that it is not the case.

$^1$ The other difference in reading of $S$ is related to the treatment of variable $Y$. In $\mathcal{F}log$ the variable is bound by an unseen existential quantifier. If all the variables are local then $S = \{ X : p(X,Y) \}$ is really $S_1 = \{ X : \exists Y\ p(X,Y) \}$. In $\mathcal{s}\mathcal{l}og$ $Y$ is free. Both approaches are reasonable but we prefer to deal with the different possible readings by introducing an explicit existential quantifier as in Prolog. It is easy semantically and we do not discuss it in the paper.
Example 12 (VCP and Constructive Semantics of aggregates)
Let us consider a program $P_7$.

\[
\begin{align*}
    p(a) & \leftarrow \text{count}\{X:p(X)\} > 0. \\
    p(b) & \leftarrow \text{not } q. \\
    q & \leftarrow \text{not } p(b).
\end{align*}
\]

As shown in (Son and Pontelli 2007) the program has two $\mathcal{A}$log answer sets, $A = \{q\}$ and $B = \{p(a), p(b)\}$. If viewed as a program of $\mathcal{A}$log, $P_7$ will have one answer set, $A$. This happens because the $\mathcal{A}$log construction of $B$ uses knowledge about properties of the aggregate atom of the first rule; the semantics of $\mathcal{A}$log only takes into account the meaning of the parameter of the aggregate term. Both approaches can, probably, be successfully defended but, in our opinion, the constructive semantics has a disadvantage of being less general (it is only applicable to non-disjunctive programs), and more complex mathematically.

A key difference between our algorithm and those in the existing work (Faber et al. 2008; Gebser et al. 2009) is that the other work needs rather involved methods to ground the aggregates while our algorithm does not need to ground the aggregate atoms. As a result, the ground program used by our algorithm may be smaller, and our algorithm is simpler.

There is also a close connection between the above semantics of aggregates all of which are based on some notion of a reduct or a fixpoint computation and approaches in which aggregates are represented as special cases of more general constructs, such as propositional formulas (Ferraris 2005; Harrison et al. 2013) and abstract constraint atoms (Marek et al. 2004; Liu et al. 2010; Wang et al. 2012) (Our semantics can be easily extended to the latter). Some of the existing equivalence results allow us to establish the relationship between these approaches and $\mathcal{A}$log. Others require further investigation.

6 Conclusion and Future Work

We presented an extension, $\mathcal{A}$log, of ASP which allows for the representation of and reasoning with aggregates. We believe that the language satisfies design criteria of simplicity of syntax and formal and informal semantics. There are many ways in which this work can be continued. The first, and simplest, step is to expand $\mathcal{A}$log by allowing choice rules similar to those of (Niemela et al. 2002). This can be done in a natural way by combining ideas from this paper and that from (Gelfond 2002). We also plan to investigate mapping of $\mathcal{A}$log into logic programs with arbitrary propositional formulas. There are many interesting and, we believe, important questions related to optimization of the $\mathcal{A}$log solver from Section 4. After clarity is reached in this area one will, of course, try to address the questions of implementation.

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References


8 Appendix

In this appendix, given an $\text{A}log$ program $\Pi$, a set $A$ of literals and a rule $r \in \Pi$, we use $\alpha(r,A)$ to denote the rule obtained from $r$ in the aggregate reduct of $\Pi$ with respect to $A$. $\alpha(r,A)$ is nil, called an empty rule, if $r$ is discarded in the aggregate reduct. We use $\alpha(\Pi,A)$ to denote the aggregate reduct of $\Pi$, i.e., $\{\alpha(r,A) : r \in \Pi \text{ and } \alpha(r,A) \neq \text{nil}\}$.

**Proposition 1 (Rule Satisfaction and Supportedness)**

Let $A$ be an answer set of a ground $\text{A}log$ program $\Pi$. Then

1. $A$ satisfies every rule $r$ of $\Pi$.
2. If $p \in A$ then there is a rule $r$ from $\Pi$ such that the body of $r$ is satisfied by $A$ and $p$ is the only atom in the head of $r$ which is true in $A$. (It is often said that rule $r$ supports atom $p$.)

Proof: Let

(1) $A$ be an answer set of $\Pi$.

We first prove $A$ satisfies every rule $r$ of $\Pi$. Let $r$ be a rule of $\Pi$ such that

(2) $A$ satisfies the body of $r$.

Statement (2) implies that every aggregate atom, if there is any, of the body of $r$ is satisfied by $A$. By the definition of the aggregate reduct, there must be a non-empty rule $r' \in \alpha(\Pi,A)$ such that

(3) $r' = \alpha(r,A)$.

By the definition of aggregate reduct, $A$ satisfies the body of $r$ iff it satisfies that of $r'$. Therefore, (2) and (3) imply that

(4) $A$ satisfies the body of $r'$.

By the definition of answer set of $\text{A}log$, (1) implies that

(5) $A$ is an answer set of $\alpha(\Pi,A)$.

Since $\alpha(\Pi,A)$ is an ASP program, (3) and (5) imply that

(6) $A$ satisfies $r'$.

Statements (4) and (6) imply $A$ satisfies the head of $r'$ and thus the head of $r$ because $r$ and $r'$ have the same head.

Therefore $r$ is satisfied by $A$, which concludes our proof of the first part of the proposition.

We next prove the second part of the proposition. Consider $p \in A$. (1) implies that $A$ is an answer set of $\alpha(\Pi,A)$. By the supportedness Lemma for ASP programs (Gelfond and Kahl 2014), there is a rule $r' \in \alpha(\Pi,A)$ such that

(7) $r'$ supports $p$. 

Let $r \in \Pi$ be a rule such that $r' = \alpha(r,A)$. By the definition of aggregate reduct,

(8) $A$ satisfies the body of $r$ iff $A$ satisfies that of $r'$.

Since $r$ and $r'$ have the same heads, (7) and (8) imply that rule $r$ of $\Pi$ supports $p$ in $A$, which concludes the proof of the second part of the proposition.

**Proposition 2 (Anti-chain Property)**

Let $A_1$ be an answer set of an $\mathcal{AS}\mathcal{L}$ program $\Pi$. Then there is no answer set $A_2$ of $\Pi$ such that $A_1$ is a proper subset of $A_2$.

**Proof:** Let us assume that there are $A_1$ and $A_2$ such that

(1) $A_1 \subseteq A_2$ and

(2) $A_1$ and $A_2$ are answer sets of $\Pi$

and show that $A_1 = A_2$.

Let $R_1$ and $R_2$ be the aggregate reducts of $\Pi$ with respect to $A_1$ and $A_2$ respectively. Let us first show that $A_1$ satisfies the rules of $R_2$. Consider

(3) $r_2 \in R_2$.

By the definition of aggregate reduct there is $r \in \Pi$ such that

(4) $r_2 = \alpha(r,A_2)$.

Consider

(5) $r_1 = \alpha(r,A_1)$.

If $r$ contains no aggregate atoms then

(6) $r_1 = r_2$.

By (5) and (6), $r_2 \in R_1$ and hence, by (2) $A_1$ satisfies $r_2$.

Assume now that $r$ contains one aggregate term, $f\{X : p(X)\}$, i.e. $r$ is of the form

(7) $h \leftarrow B, C[f\{X : p(X)\}]$

where $C$ is some property of the aggregate.

Then $r_2$ has the form

(8) $h \leftarrow B, P_2$

where
(9) \( P_2 = \{ p(t) : p(t) \in A_2 \} \) and \( f(P_2) \) satisfies condition \( C \).

Let

(10) \( P_1 = \{ p(t) : p(t) \in A_1 \} \)

and consider two cases:

(11a) \( \alpha(r,A_1) = \emptyset \).

In this case \( C(f(P_1)) \) does not hold. Hence, \( P_1 \neq P_2 \). Since \( A_1 \subseteq A_2 \) we have that \( P_1 \subseteq P_2 \), the body of rule (8) is not satisfied by \( A_1 \), and hence the rule (8) is.

(11b) \( \alpha(r,A_1) \neq \emptyset \).

Then \( r_1 \) has the form

(12) \( h \leftarrow B, P_1 \)

where

(13) \( P_1 = \{ p(t) : p(t) \in A_1 \} \) and \( f(P_1) \) satisfies condition \( C \).

Assume that \( A_1 \) satisfies the body, \( B, P_2 \), of rule (8). Then

(14) \( P_2 \subseteq A_1 \)

This, together with (9) and (10) implies

(15) \( P_2 \subseteq P_1 \).

From (1), (9), and (10) we have \( P_1 \subseteq P_2 \). Hence

(16) \( P_1 = P_2 \).

This means that \( A_1 \) satisfies the body of \( r_1 \) and hence it satisfies \( h \) and, therefore, \( r_2 \).

Similar argument works for rules containing multiple aggregate atoms and, therefore, \( A_1 \) satisfies \( R_2 \).

Since \( A_2 \) is a minimal set satisfying \( R_2 \) and \( A_1 \) satisfies \( R_2 \) and \( A_1 \subseteq A_2 \) we have that \( A_1 = A_2 \).

This completes our proof. \( \Box \)
Proposition 3 (Splitting Set Theorem)

Let

1. \( \Pi_1 \) and \( \Pi_2 \) be ground programs of \( \mathcal{A}log \) such that no atom occurring in \( \Pi_1 \) is unifiable with any atom occurring in the heads of \( \Pi_2 \).
2. \( S \) be a set of ground literals containing all head literals of \( \Pi_1 \) but no head literals of \( \Pi_2 \).

Then

(3) \( A \) is an answer set of \( \Pi_1 \cup \Pi_2 \)

iff

(4a) \( A \cap S \) is an answer set of \( \Pi_1 \) and

(4b) \( A \) is an answer set of \( \{ A \cap S \} \cup \Pi_2 \).

Proof. By the definitions of answer set and aggregate reduct

(3) holds iff

(5) \( A \) is an answer set of \( \alpha(\Pi_1, A) \cup \alpha(\Pi_2, A) \)

It is easy to see that conditions (1), (2), and the definition of \( \alpha \) imply that \( \alpha(\Pi_1, A) \), \( \alpha(\Pi_2, A) \), and \( S \) satisfy condition of the splitting set theorem for ASP (Lifschitz and Turner 1994). Hence

(5) holds iff

(6a) \( A \cap S \) is an answer set of \( \alpha(\Pi_1, A) \)

and

(6b) \( A \) is an answer set of \( \{ A \cap S \} \cup \alpha(\Pi_2, A) \).

To complete the proof it suffices to show that

(7) Statements (6a) and (6b) hold iff (4a) and (4b) hold.

By definition of \( \alpha \),

(8) \( \{ A \cap S \} \cup \alpha(\Pi_2, A) = \alpha((A \cap S) \cup \Pi_2, A) \)

and hence, by the definition of answer set we have

(9) (6b) iff (4b).

Now notice that from (4b), clause 2 of Proposition 1, and conditions (1) and (2) of our theorem we have that for any ground instance \( p(t) \) of a literal occurring in an aggregate atom of \( \Pi_1 \)
(10) \( p(t) \in A \text{ iff } p(t) \in A \cap S \)

and, hence

(11) \( \alpha(\Pi_1,A) = \alpha(\Pi_1,A \cap S) \).

From (9), (11), and the definition of answer set we have that

(12) (6a) iff (4a)

which completes the proof of our theorem.

Lemma 1
Checking whether a set \( M \) of literals is an answer set of \( P \), a program with aggregates, is in co-NP.

Proof: To prove that \( M \) is not an answer set of \( P \), we first check if \( M \) is not a model of the aggregate reduct of \( P \), which is in polynomial time. If \( M \) is not a model, \( M \) is not an answer set of \( P \). Otherwise, we guess a set \( M' \) of \( P \), and check if \( M' \) is a model of the aggregate reduct of \( P \) and \( M' \subset M \). This checking is also in polynomial time. Therefore, the problem of checking whether a set \( M \) of literals is an answer set of \( P \) is in co-NP.

Proposition 4 (Complexity)
The problem of checking if a ground atom \( a \) belongs to all answer sets of an \( 
slolg \) program is \( \Pi^p_2 \) complete.

Proof: First we show that the cautious reasoning problem is in \( \Pi^p_2 \). We verify that a ground atom \( a \) is not a cautious consequence of a program \( P \) as follows: Guess a set \( M \) of literals and check that (1) \( M \) is an answer set for \( P \), and (2) \( a \) is not true wrt \( M \). Task (2) is clearly polynomial, while (1) is in co-NP by virtue of Lemma 1. The problem therefore lies in \( \Pi^p_2 \).

Next, cautious reasoning over programs without aggregates is \( \Pi^p_2 \) hard by (Dantsin et al. 2001). Therefore, cautious reasoning over programs with aggregates is \( \Pi^p_2 \) hard too.

In summary, cautious reasoning over programs with aggregates is \( \Pi^p_2 \) complete.