Logarithmic capacity and rational lemniscates

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S. Pouliasis and T. Ransford, *On the harmonic measure and the capacity of rational lemniscates*, Potential Analysis, Vol. 44 (2016), Issue 2, 249–261.

K will be a compact subset of $\mathbb{C}$,
μ (positive) Borel measure, $\mathrm{supp}(\mu) \subset K$, $\mu(K) = 1$,

Definition (Logarithmic capacity)

$$cap(K) = \exp\left[-\inf_{\mu} \iint \log \frac{1}{|z-w|} d\mu(z) d\mu(w) \right]$$

Examples

- $cap(\overline{D(z,r)}) = r$,
- $cap([a,b]) = \frac{b-a}{4}$,
- $cap(Cantor\ set) \geq \frac{1}{9}$.

Definition

K is removable for bounded harmonic functions if, for every open neighborhood U of K, each bounded harmonic function on $U \setminus K$ extends across K to be harmonic on U.

Theorem

K is removable for bounded harmonic functions if and only if $cap(K) = 0$.

$D = \hat{\mathbb{C}} \setminus K$,
$w \in D$,

Definition (Green function of D with pole at w)

$G_D(\cdot, w) : D \mapsto (0, +\infty]$

- *harmonic on* $D \setminus \{w\}$,
- $z \mapsto G_D(z, w) - \log \frac{1}{|z-w|}$ *is harmonic on* D,
- $\lim_{z \to \zeta} G_D(z, w) = 0$, $\zeta \in \partial D$ *except on a set of zero logarithmic capacity.*

Theorem

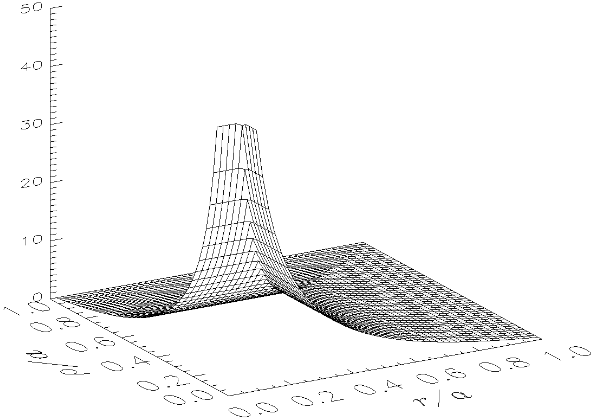
D has a Green function if and only if $\mathrm{cap}(\partial D) > 0$.

Examples

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$,

$$G_{\mathbb{D}}(z, w) = \log \left| \frac{1 - z\bar{w}}{z - w} \right|, \qquad z, w \in \mathbb{D}.$$

Green function of a rectangle with pole at 0.



Definition (Analytic capacity)

$$\gamma(K) = \sup\{|f'(\infty)| : f \in Hol(\hat{\mathbb{C}} \setminus K), ||f||_\infty \leq 1\},$$

where

$$f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)).$$

Definition (Ahlfors function)

If $\gamma(K) > 0$, $\exists$ *unique* g *s.t.*

$$\gamma(K) = g'(\infty).$$

Also,

$$g(\infty) = 0.$$

Examples

- $\gamma(\overline{D(z,r)}) = r$,
- *if* $K \subset \mathbb{R}$, $\gamma(K) = \frac{m(K)}{4}$,
- $\gamma(K) \leq cap(K)$,
- *if* K *is connected,* $\gamma(K) = cap(K)$,
- *if* $D = \hat{\mathbb{C}} \setminus K$ *is bounded by* n *Jordan curves, then the Ahlfors function* g *is an* n *to* 1 *proper holomorphic function from* D *to* $\mathbb{D}$ *that maps every component of* ∂D *homeomorphically onto* $\partial\mathbb{D}$ *and, if* $a_1, \ldots, a_{n-1}$ *are the finite zeros of* g*, then*

$$\gamma(K) = cap(K) \cdot \exp\left[-\sum_{i=1}^{n-1} G_D(a_i, \infty) \right].$$

Analytic capacity introduced by L. V. Ahlfors,

Bounded analytic functions, Duke Math. J. 14, (1947), 1-11.

to study removable sets for bounded holomorphic functions (give a "geometric" characterization of them, Painlevé problem).

Theorem (Ahlfors)

K is removable for bounded holomorphic functions if and only if $\gamma(K) = 0$.

Painlevé problem solved by X. Tolsa

Painlevé's problem and the semi-additivity of analytic capacity, Acta Math. 190 (2003), no. 1, 105-149.

Bilipschitz maps, analytic capacity, and the Cauchy integral, Ann. of Math. (2) 162 (2005), no. 3, 1243-1304.

Theorem (Tolsa, semi-additivity of analytic capacity, conjectured by A. G. Vituškin 1967)

$\exists\ C \geq 1$ *s.t.*

$$\gamma(K_1 \cup K_2) \leq C(\gamma(K_1) + \gamma(K_2))$$

for all compact subsets K_1 *and* K_2 *of* $\mathbb{C}$.

It is an open problem if we can actually take $C = 1$,

Open problem (Subadditivity problem)

Is it true that

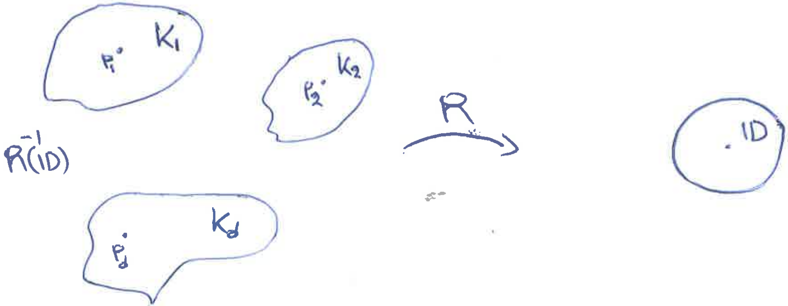
$$\gamma(K_1 \cup K_2) \leq \gamma(K_1) + \gamma(K_2),$$

for all compact subsets K_1 *and* K_2 *of* $\mathbb{C}$?

Definition (M. Fortier Bourque and M. Younsi, 2013)

A rational function R is called d-good ($d \in \mathbb{N}$) if

- *the degree of R is d,*
- $R(\infty) = 0$
- *the open set $\Omega := R^{-1}(\mathbb{D})$ is connected and bounded by d disjoint analytic Jordan curves γ_i, $i = 1, \ldots, d$.*


K1
K2
Kd
R
D

$$R(z) = \sum_{i=1}^{d} \frac{a_i}{z - p_i}$$

and the set

$$\begin{aligned} K &:= \cup_{i=1}^{d} K_i \\ &= \{z \in \mathbb{C} : |R(z)| \geq 1\}, \end{aligned}$$

is called the lemniscate of R.

Theorem (P. Mattila and M. Melnikov, 1994)

$\exists\ C \geq 1$ *s.t. for every good rational function,*

$$\gamma(K) \leq C \sum_i |a_i|.$$

Problem: compare $\gamma(K_i)$ with $|a_i|$.

Note: Since K_i is connected, $\gamma(K_i) = cap(K_i)$.

Theorem (with T. Ransford, 2015)

For every d-good rational function,

$$\operatorname{cap}(K_i) \geq |a_i|, \qquad i = 1, \ldots, d,$$

and

$$\operatorname{cap}(K) \geq \Big[\prod_{\substack{i,j=1 \\ i \neq j}}^{d} |p_i - p_j| \prod_{i=1}^{d} |a_i| \Big]^{\frac{1}{d^2}}.$$

If V_i is a neighborhood of K_i, R is injective on V_i and

$$\{z \in \hat{\mathbb{C}} : |z| \geq \frac{1}{r}\} \subset R(V_i), \qquad r > 1,$$

then

$$\operatorname{cap}(K_i) \leq \frac{r^6}{(r^2-1)(r-1)^4} |a_i|.$$

Corrolary

Let R be a good rational function, let $p \in \Omega = R^{-1}(\mathbb{D})$ and let

$$R_\epsilon(z) := R(z) + \frac{\epsilon}{z-p}, \qquad \epsilon > 0,\ z \in \hat{\mathbb{C}}.$$

If K_ϵ is the component of the lemniscate $\{z \in \hat{\mathbb{C}} : |R_\epsilon(z)| \geq 1\}$ of R_ϵ that contains p, then

$$\mathrm{cap}(K_\epsilon) = \mathcal{O}(\epsilon), \qquad \textit{as } \epsilon \to 0.$$

Question

Given $d \geq 2$, does there exist a constant $C(d) > 0$ with the following property: if $R(z) := \sum_{i=1}^{d}(a_i/(z-p_i))$ is a d-good rational function, then

$$\mathrm{cap}(K_i) \leq C(d)|a_i|,$$

where K_i is the component of the lemniscate $K := \{z \in \mathbb{C} : |R(z)| \geq 1\}$ containing p_i?

Question

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Answer: No.

Theorem (with T. Ransford, 2015)

Let $a > 0$ *and* $\eta \in (\frac{2}{3}, 1)$. *For* $p > 1$ *define*

$$R_p(z) := \frac{a}{z-p} + \frac{p-p^\eta}{z-ip} + \frac{p-p^\eta}{z+ip}.$$

Then there exists $p_0 := p_0(a,\eta)$ *such that, for all* $p > p_0$*,*

1. R_p *is a 3-good rational function,*
2. *the component of the lemniscate* $\{z \in \hat{\mathbb{C}} : |R_p(z)| \geq 1\}$ *containing* p *has logarithmic capacity at least* $ap^{1-\eta}/8$*.*

Definition (Harmonic measure of Ω with respect to z_0)

Let $\Omega \subset \hat{\mathbb{C}}$ be an open set, $E \subset \partial\Omega$ and $z_0 \in \Omega$. Let h be the harmonic function on Ω with boundary limits 1 on E and 0 on $\partial\Omega \setminus E$. Then

$$\omega_{z_0}^{\Omega}(E) = h_E(z_0).$$

Theorem (A reflection principle for harmonic measure)

Let R be a rational function of degree d, let $\zeta_1, ..., \zeta_d$ be the zeros and $p_1, ..., p_d$ be the poles of R and let $\Omega := R^{-1}(\mathbb{D})$. Then

$$\sum_{j=1}^{d} \omega_{\zeta_j}^{\Omega}(E) = \sum_{i=1}^{d} \omega_{p_i}^{\hat{\mathbb{C}}\setminus\overline{\Omega}}(E),$$

for every Borel set $E \subset \partial\Omega$.

Theorem

Let Ω be a finitely connected domain bounded by d disjoint analytic Jordan curves $\gamma_1, \dots, \gamma_d$, with $\infty \in \Omega$. Let f be a proper holomorphic function of degree d from Ω to $\mathbb{D}$ and let $\zeta_1, \dots, \zeta_d$ be its zeros. Suppose further that, for every $i = 1, \dots, d$ there exists p_i in the interior of γ_i such that

$$\sum_{j=1}^{d} \omega_{\zeta_j}^{\Omega}(E) = \sum_{i=1}^{d} \omega_{p_i}^{\hat{\mathbb{C}} \setminus \overline{\Omega}}(E)$$

for every Borel set $E \subset \partial\Omega$. Then f is a rational function.

Thank you!