Numerical Analysis Preliminary Examination, August 2008

Department of Mathematics and Statistics

Do nine of the following ten problems. Clearly indicate which nine are to be graded. Calculators are not allowed.

1. Let A be the nonsingular 3×3 matrix $A = \begin{pmatrix} 2 & 4 & 3 \\ 2 & 1 & 5 \\ 1 & 3 & 5 \end{pmatrix}$.

(a) Consider the QR-factorization of A. Describe special properties of matrices Q and R.
(b) Explain how two Householder matrices H_1 and H_2 can be used to form the QR-factorization of A, i.e., $A = QR$. Also, express Q in terms of H_1 and H_2 and R in terms of H_1, H_2, and A.
(c) Suppose that $H_1 A = \begin{pmatrix} -3 & 1 & -7 \\ 0 & 3 & 1 \\ 0 & 4 & 3 \end{pmatrix}$. Find H_2 and find matrix R.
(Given $\vec{x} \in \mathbb{R}^n$, recall that if $\sigma = \text{sign}(x_1)||\vec{x}||_2, \vec{u} = \vec{x} + \sigma\vec{e}_1$, $\theta = \frac{1}{2}||\vec{u}||_2^2$, and $H = I - \vec{u}\vec{u}^T/\theta$, then $H\vec{x} = -\sigma\vec{e}_1$.)

2. Consider the linear system $\begin{cases} 4x_1 - x_2 & = & 2 \\ -x_1 + 4x_2 - x_3 & = & 6 \\ -x_2 + 4x_3 & = & 2. \end{cases}$

(a) Compute the Jacobi iteration matrix, J, for this system.
(b) Determine the spectral radius, $\rho(J)$, of matrix J.
(c) Determine or deduce that the spectral radius of the Gauss-Seidel matrix, L_1, satisfies $\rho(L_1) < \rho(J)$.

3. Let A be a nonsingular $n \times n$ matrix and suppose that C is an $n \times n$ matrix with $||I - AC|| \leq q < 1$. Let $X_{j+1} = X_j B + C$ for $j = 0, 1, 2, \ldots$ where $B = I - AC$. Prove that $||X_j - A^{-1}|| \leq \dfrac{q^j}{1-q}||X_1 - X_0||$.

4. Let $z = f(x, y) = 15x^3/2 + xy - 2x + 4y + y^2/2$ describe a surface where $(x, y) \in D = [0, \infty) \times (-\infty, \infty)$. The minimum point of $f(x, y)$ in D is $(0.5391, -4.5391)$.
(a) Describe the method of steepest descent for finding the minimum point.
(b) Let $(x_0, y_0) = (0, 0)$ be the initial point in the method of steepest descent. Apply one step of the method and calculate the point (x_1, y_1).

5. Let $x_i = 1/(i+1)$ for $i = 0, 1, \ldots, n$. Suppose that $f \in C^\infty[0, 1]$ and $||f^{(m)}||_\infty \leq 5^m$ for $m = 0, 1, 2, \ldots$. Let $p_n(x)$ be the unique polynomial of degree less than or equal to n such that $p_n(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$. Given $\epsilon > 0$, prove that there is an integer N such that $||p_n - f||_\infty < \epsilon$ when $n \geq N$.

6. Consider the numerical solution of the initial-value problem $y'(t) = f(t, y(t))$, $y(0) = y_0$.
(a) Find the interval of absolute stability of the Runge-Kutta method

$$y_{k+1} = y_k + \frac{h}{4}k_1 + \frac{3h}{4}k_2, \quad k_1 = f(t_k, y_k), \quad k_2 = f(t_{k+1}, y_k + hk_1).$$

(b) Suppose that the method is applied to $\vec{y}\,' = A\vec{y}$ where A is an $n \times n$ negative definite Hermitian matrix with spectral radius $\rho(A) = 100$. Determine the step width h that will guarantee absolute stability for this problem.

7. Let $f \in C[1,2]$ and let P^n be the set of polynomials of degree less than or equal to n. Define the inner product on $C[1,2]$ as $(f,g) = \int_1^2 x^2 f(x)g(x)\,dx$, with norm $\|f\| = (f,f)^{1/2}$. Let $\{\phi_k(x)\}_{k=0}^{\infty}$ be orthonormal polynomials with respect to this inner product. The least squares approximation $p_n \in P^n$ to $f \in C[1,2]$ is given by $p_n(x) = \sum_{k=0}^{n}(f,\phi_k)\phi_k(x)$.
(a) Prove that $(f - p_n, q_n) = 0$ for any $q_n \in P^n$.
(b) Prove that $\|p_n - f\|^2 < \|q_n - f\|^2$ for any $q_n \in P^n$.
(c) Prove that given $\epsilon > 0$, there is an N such that $\|p_n - f\| < \epsilon$ when $n \geq N$.
(d) Suppose that $(f, \phi_k) \leq 1/k^2$. Prove that $\|p_n - f\|^2 < c/n^2$ where $c = \sum_{k=1}^{\infty} 1/k^2 < \infty$.

8. Consider the iteration $\vec{x}^{(k+1)} = G(\vec{x}^{(k)}), \vec{x}^{(0)} = [0,0]^T$, where

$$G(\vec{x}) = \begin{pmatrix} \frac{1}{2}\cos(x_1) - \frac{1}{4}\sin(x_2) \\ \frac{1}{4}\cos(x_1) + \frac{1}{2}\sin(x_2) \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) Show that $\|G(\vec{x}) - G(\vec{y})\|_\infty \leq \alpha\|\vec{x} - \vec{y}\|_\infty$ where $\alpha = 3/4$.
(b) Prove that $\{\vec{x}^{(k)}\}_{k=0}^{\infty}$ converges to the unique vector $\vec{x}^* \in \mathbb{R}^2$ such that $\vec{x}^* = G(\vec{x}^*)$.

9. Consider the linear system $A\vec{x} = \vec{b}$ where A is nonsingular. Suppose that we compute $\vec{y}$ that solves $A\vec{y} = \vec{b} + \vec{p}$ with $\|\vec{p}\|$ small.
(a) Obtain an upper bound for $\|\vec{x} - \vec{y}\|/\|\vec{x}\|$ in terms of $\|\vec{p}\|/\|\vec{b}\|$ and $K(A) = \|A\|\|A^{-1}\|$.
(b) Obtain a lower bound for $\|\vec{x} - \vec{y}\|/\|\vec{x}\|$ in terms of $\|\vec{p}\|/\|\vec{b}\|$ and $K(A) = \|A\|\|A^{-1}\|$.

10. Let $F(h) = (f(x_0 + h) - 2f(x_0) + f(x_0 - h))/h^2$ be an approximation to $f''(x_0)$. Let $e(h) = f''(x_0) - F(h)$ be the error in the approximation. Assume that $f \in C^8[a,b]$ and that $x_0 - h, x_0, x_0 + h \in [a,b]$. Prove that the error, $e(h)$, has the form $e(h) = c_1h^2 + c_2h^4 + O(h^6)$ where c_1 and c_2 are independent of h.