Numerical Analysis Preliminary Examination, August 2015

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Solve 8 out of 9 (eight out of nine) of the following problems. Clearly indicate which eight are to be graded.

1. Determine appropriate values of A_i and x_i so that the quadrature formula

$$\int_{-1}^{1} x^2 f(x)\mathrm{d}x \approx \sum_{i=0}^{1} A_i f(x_i)$$

will be exact when f is any polynomial of degree ≤ 3.

2. Find the least-squares solution of the system

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} .$$

3. Given any vector norm $\|\cdot\|_v$ in $\mathbb{R}^N$, give the definition of subordinate (or induced) matrix norm $\|\cdot\|_m$ to the given vector norm $\|\cdot\|_v$. Let A be a square matrix. Prove that, if $\|A\|_m < 1$,

$$\|(I - A)^{-1}\|_m \geq \frac{1}{1 + \|A\|_m} .$$

4. Given a square and nonsingular matrix A, consider an iterative method for the solution of a linear system $Ax = b$ given by

$$x^{(n+1)} = (I - Q^{-1}A)x^{(n)} + Q^{-1}b .$$

Prove that if A is diagonally dominant and Q is chosen as in the Jacobi method, then the iterative method converges to the solution of $Ax = b$ for any starting vector.

5. Consider a variation of Newton's method in which only one derivative is needed,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} .$$

If r is a root of f and $e_n = x_n - r$, find two constants C and s such that

$$e_{n+1} = Ce_n^s .$$

6. Given a normed linear space E and a subspace G in E, give the definition of best approximation of a function $f \in E$ from G. Prove that, if G is finite-dimensional, then each point of E possesses at least one best approximation in G.

If E is an inner product space, is the best approximation unique? If yes, why?

7. Milne's method approximates the solution of the initial value problem

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

using the step formula

$$x_n - x_{n-2} = h\left(\frac{1}{3}f_n + \frac{4}{3}f_{n-1} + \frac{1}{3}f_{n-2}\right),$$

where $f_n = f(t_n, x_n)$.

(a) Determine whether this method is convergent.

(b) Find the order of the local and global truncation errors.

8. Describe how to apply the shooting method for the numerical solution of the linear Boundary Value Problem

$$\begin{cases} x'' = u(t) + v(t)x + w(t)x' \\ x(a) = \alpha \\ x(b) = \beta\,. \end{cases}$$

9. Provide an upper bound for the error in a quadrature formula

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

that involves the supremum norm $\|\cdot\|_\infty$ of some derivative of f.