2013 August ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problem 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Let $A(t)$ be an $n \times n$ matrix whose entries are continuous functions of t.

   a) Prove that the solution space of the system $\dot{x} = A(t)x$, $x \in \mathbb{R}^n$, is an n-dimensional linear space.

   b) Prove that for time-independent A, the solution for the initial value problem $\dot{x} = Ax + f(t)$, $x(0) = x_0$ is

   $$x(t) = e^{At} \left(x_0 + \int_0^t e^{-A\tau} f(\tau) \, d\tau \right),$$

   where e^{At} is defined by

   $$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots.$$

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a local Lipschitz function, $\phi(t)$ be the solution of $\dot{x} = f(x)$, $x(0) = x_0$, and (α, β) be the maximal interval of existence of $\phi(t)$. Prove that if the set $\{\phi(t) : t \in [0, \beta)\}$ is bounded, then $\beta = \infty$.

3. Determine the stability of the origin for the system

   $$\begin{aligned}
   \dot{x}_1 &= -x_2^2 \\
   \dot{x}_2 &= -x_2 + x_1^2 + x_1 x_2
   \end{aligned}$$

   by using the center manifold theorem.

4. Consider the system

   $$\begin{aligned}
   \dot{x}_1 &= x_1 - x_2 - x_1^3, \\
   \dot{x}_2 &= x_1 + x_2 - x_2^3.
   \end{aligned}$$

   Accept the fact without proof that the origin is the only equilibrium point of the system. Prove that the system has a periodic orbit in the annular region

   $$A = \{x \in \mathbb{R}^2 : 1 \le x_1^2 + x_2^2 \le 2\}.$$

   State any theorem that you invoke.

   (*Hint:* $x_1^2 + x_2^2 - x_1^4 - x_2^4 = (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2 + 2x_1^2 x_2^2 = r^2 - r^4 + \frac{r^4}{2}\sin^2(2\theta)$ *where* $x_1 = r\cos\theta$, $x_2 = r\sin\theta$.)

Partial Differential Equations

Do three out of four problems below. Write in the following boxes the three problems

that are to be graded:

Failing to clearly indicate three problems will result in Problems 1, 2 and 3 being graded.

1. Let c be a constant and consider the equation

$$\Delta u - 2cu_{x_1} = -c^2 u.$$

State and prove the Mean Value Formula for a solution of the above equation.

Hint: you may use substitution $v(x) = f(x_1)u$ for an appropriate function $f(x_1)$ to show that $\Delta v = 0$.

2. Let $U \subset \mathbb{R}^2$ be an unbounded domain (open, connected set) in $(0, \infty) \times (0, 1)$. Suppose $u(x_1, x_2)$ is a function in $C^2(U) \cap C(\bar{U})$ that satisfies

$$\begin{cases} \Delta u = 0 & \text{in } U, \\ |u| \leq f(x_1, x_2) & \text{on } \partial U, \end{cases}$$

Assume that $|u(x_1, x_2)| \leq C$ for all $(x_1, x_2) \in \bar{U}$ and $f(x_1, x_2) \leq N$ for all $(x_1, x_2) \in \partial U$, where C and N are some constants.

Prove that $|u(x_1, x_2)| \leq N$ for all $(x_1, x_2) \in U$.

3. Let $U \subset \mathbb{R}^2$ be a bounded domain, which means that there exist constants a and b such that

$U \subset \{(x_1, x_2) : a \leq x_1, \leq b; a \leq x_2, \leq b\}.$

Let $D = U \times (0, \infty)$.

Suppose function $u \in C^2(D) \cap C(\bar{D})$ satisfies

$$u_t = a\Delta u \qquad \text{in } D,$$
$$u(x, t) = 0 \qquad \text{on } \partial U \times (0, \infty),$$
$$|u(x, 0)| \leq M \qquad \text{in } U.$$

Here, a and M are positive constants.

Prove that there exist constants $A > 0$ and $C > 0$ such that

$$|u(x, t)| \leq Ce^{-At} \quad \text{for all } (x, t) \in U. \tag{1}$$

4. Let U be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and let $D = U \times (0, \infty)$.

Suppose $u = u(x, t) \in C^2(\bar{D})$ is a solution of the following problem

$$\begin{cases} a\Delta u - u_{tt} = 0 & \text{in } D, \\ u(x, 0) = g_0(x), \quad u_t(x, 0) = g_1(x) & \text{in } U, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial U \times [0, \infty), \end{cases}$$

where $a > 0$ is a constant, g_0, g_1 are given functions in $C^1(\bar{U})$, and n is the unit outward normal vector on the boundary ∂U.

Prove that

$$\int_U a|\nabla u(x, t)|^2 + |u_t(x, t)|^2 dx \quad = \quad \int_U a|\nabla g_0(x)|^2 + |g_1(x)|^2 dx \quad \text{for all } t \geq 0.$$