Problem 1. In each part, give the requested example. Clear pictures and descriptions of the functions are acceptable in place of explicit formulas. Be sure to give the justification that your example works.

A. Give an example of a function $f \in L^1(\mathbb{R})$ such that $f(x) \not\to 0$ as $x \to \infty$.

B. Give an example of a sequence $\{ f_n \}_{n=1}^{\infty}$ of functions in $L^1(\mathbb{R})$ and a function $f \in L^1(\mathbb{R})$ such that $f_n \to f$ pointwise almost everywhere, but $\{ f_n \}$ does not converge to f in the L^1-norm.

C. Give an example of a sequence $\{ f_n \} \subseteq L^1(\mathbb{R})$ and a function $f \in L^1(\mathbb{R})$ such that $f_n \to f$ in the L^1-norm, but $\{ f_n \}$ does not converge to f pointwise almost everywhere.

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let ρ be a nonnegative function in $L^1(\mu)$. Define a set function ν on the σ-algebra $\mathcal{M}$ by

$$\nu(E) = \int_E \rho \, d\mu.$$

A. Show that ν is a measure.

B. Show that for all functions $f \in L^1(\nu)$,

$$\int_E f \, d\nu = \int_E f \rho \, d\mu,$$

for all $E \in \mathcal{M}$.

Problem 3. Prove that

$$\lim_{n \to \infty} \int_0^1 e^{inx} f(x) \, dx = 0$$

for all $f \in C([0, 1])$. Show this also holds if $f \in L^1([0, 1])$. Possible hint: the Stone-Weierstrass theorem.

Problem 4. Suppose that $g \in L^1([a, b])$. Show that

$$f(x) = \int_a^x g(t) \, dt$$

is a function of bounded variation on $[a, b]$.

Problem 5. Show that $L^p(\mathbb{R})$ is not a Hilbert space unless $p = 2$, i.e., the L^p norm is not induced by any inner product. Hint: parallelogram law.

Problem 6. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $T\colon \mathcal{X} \to \mathcal{Y}$ be a linear map (not assumed to be continuous). Suppose that for all $f \in \mathcal{Y}^*$, $f \circ T$ is in $\mathcal{X}^*$. Show that T is continuous (equivalently, bounded). Hint: let $\mathcal{A} = \{\, f \circ T \mid \|f\| = 1 \,\}$ and apply the Principle of Uniform Boundedness.

Problem 7. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $1 < r, s < \infty$ and suppose that $\frac{1}{r} + \frac{1}{s} \leq 1$. Show that if $f \in L^r(\mu)$ and $g \in L^s(\mu)$, then fg is in $L^t(\mu)$ where

$$t = \frac{rs}{r+s}.$$

Hint: Hölder's inequality.

Problem 8. Suppose that $f, g \in L^1(\mathbb{R})$. Assume without proof that $(x, t) \mapsto f(x-t)g(t)$ is measurable. Show that

$$(f * g)(x) = \int f(x-t)g(t)\, dt$$

defines an L^1 function on $\mathbb{R}$ and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Problem 9. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Suppose that $K \in L^2(X \times Y)$. Show that if $f \in L^2(Y)$ then the formula

$$(Tf)(x) = \int K(x, y) f(y)\, d\nu(y)$$

defines a function $Tf \in L^2(X)$. Show that $T\colon L^2(X) \to L^2(Y)$ is a bounded linear operator satisfying

$$\|Tf\|_2 \leq \|K\|_2 \|f\|_2.$$

Problem 10. Let F be defined by

$$F(t) = \int_0^\infty e^{-x^2} \cos(2xt)\, dx.$$

Show that

$$F(t) = \frac{\sqrt{\pi}}{2} e^{-t^2}.$$

Justify your steps. Hint: Find a differential equation satisfied by F. You may use the fact that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

without proof.

EXAM

Preliminary Examination in Real Analysis

Spring 2002

- Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.
- There are 10 problems. **The best 7 scores will be added for your grade.**

Good luck!