# Real Analysis Preliminary Examination

May, 2016

**Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.**

1. Let $\mathcal{M}$ be a $\sigma$-algebra and $\mu : \mathcal{M} \mapsto [0, \infty]$ be a set function with the following properties:

   i) there is a set $E \in \mathcal{M}$ such that $\mu(E) < \infty$;

   ii) for any disjoint sets $\{E_1, \ldots, E_n\} \subset \mathcal{M}$, $\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i)$;

   iii) for any increasing sequence $E_1 \subset E_2 \subset \cdots$ in $\mathcal{M}$, $\mu\left(\bigcup_{i=1}^\infty E_i\right) \leq \liminf_{i\to\infty} \mu(E_i)$.

   Prove that $\mu$ is a measure on $\mathcal{M}$.

2. Let $E$ be a bounded subset of $\mathbb{R}$ and $E_r := \{x + r : x \in E\}$ be a translation of $E$. Assume that the sets $\{E_r : r \in [0,1] \cap \mathbb{Q}\}$ are disjoint, and $\bigcup_{r\in[0,1]\cap\mathbb{Q}} E_r$ has nonempty interior. Prove that $E$ is not Lebesgue measurable.

3. Let $\mu$ be the counting measure on the $\sigma$-algebra $P(\mathbb{R})$ of all subsets of $\mathbb{R}$. Prove that if $f$ is $\mu$-integrable, then $\{x \in \mathbb{R} : f(x) \neq 0\}$ is countable.

4. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function, and let $\mu_F$ be the associated Lebesgue-Stieltjes measure. Find $\mu_F(\{a\})$, $\mu_F([a, b))$, $\mu([a, b])$, and $\mu_F((a, b))$. Justify your answers.

5. Let $f, g$ be Lebesgue integrable functions on $[0, 1]$ and $F(x) = \int_0^x f(t)\, dt$, $G(x) = \int_0^x g(t)\, dt$. Prove that

$$\int_0^1 F(x)g(x)\, dm(x) = F(1)G(1) - \int_0^1 f(x)G(x)\, dm(x).$$

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Evaluate $\lim_{n\to\infty} \int_{\mathbb{R}} f(x+n) \frac{x^2}{1+x^2}\, dx$. Justify your answer.

7. Let $\mu$ be a $\sigma$-finite positive measure and $\nu$ be a $\sigma$-finite signed measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$. Prove that

$$|\nu| \ll \mu \quad \text{and} \quad \frac{d|\nu|}{d\mu} = \left|\frac{d\nu}{d\mu}\right|.$$

8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Prove that for all $p > 1$ and for any measurable function $f$,

$$\|f\|_1 \leq \mu(X)^{1-\frac{1}{p}} \|f\|_p.$$

9. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Prove that for any $\epsilon > 0$, the closed and bounded set $B_\epsilon(0) = \{x \in \mathcal{H} : \|x\| \leq \epsilon\}$ is not compact.