TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

August 2018

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. Prove that every cover of the interval $[0, 1]$ by open intervals has a finite subcover. (Do not just cite the Heine-Borel Theorem, but give it a proof in this particular case.)

2. State and prove the Tube Lemma.

3. a) Show that a topological space X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.
   b) Show that a topological space X is normal if and only if given a closed set $C \subset X$ and an open set U containing C, there is an open set V containing C such that $\overline{V} \subset U$.

4. State and prove the Tietze Extension Theorem.

5. State and prove the General Lifting Lemma. (You may use without proof the Path Lifting Lemma.)

6. Construct a topological space whose fundamental group is $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

7. Compute the homology groups with integer coefficients of the projective plane.

8. Compute the Euler characteristics of the 2-dimensional torus $S^1 \times S^1$, of the Klein bottle, and of the wedge of 3 circles.