# Vehicular Traffic Capacity at Unsignalized Crosswalks with Probabilistic Yielding Behavior 

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#### Abstract

Motorists' probabilistic yielding behavior is often observed at unsignalized crosswalks, but its impedance on traffic capacity has not been thoroughly examined. The uniqueness of this problem, which is also the challenge, is that neither pedestrians nor motorists hold absolute priority because of uncertainty in yielding behavior. Therefore, from the perspective of queuing theory, understanding the service mechanism is key to determining system performance. In this study, based on explicit analyses of complete yielding scenarios, the distribution of service time for queuing vehicles is derived. Traffic capacity is then determined on the basis of mean service time for queuers. In the special case of drivers fully respecting the priority of pedestrian flow, the capacity model reduces to the classic formula with absolute priority. The solutions from the proposed capacity model precisely match the results from stochastic simulations. To facilitate practical applications, an approximation is also developed that greatly simplifies the capacity formula but still gives very close estimations to the exact solutions. This simplified formula is recommended for practical applications. The proposed capacity formula, as well as the service time distribution, can also be applied to develop performance measures such as traffic delay and queue length through use of sophisticated queuing formulas.


Motorists are legally required to yield to pedestrians under most circumstances at unsignalized crosswalks, including both midblock crossings and crosswalks at unsignalized intersections. However, actual yielding behavior varies considerably (1). Oftentimes, a yielding rate for a certain type of crossing treatment is reported to reflect the percentage of yielding drivers. In the Highway Capacity Manual 2010 (HCM 2010) (2), examples of yielding rates are documented for different types of crossing treatments, as shown in Table $1(3,4)$.

Such yielding behavior apparently impacts traffic flow directly and may result in a considerable drop in traffic capacity given a high yielding rate. But, unfortunately, this impedance has not been examined and quantified in existing literature. Quantitative methods to determine traffic capacity with probabilistic yielding behavior taken into account are lacking. This situation is emphasized in the HCM 2010 as a limitation in evaluating the major street through traffic at two-way stop-controlled intersections and unsignalized crosswalks (2).

[^0]Conventionally, studies of the interaction between two conflicting flows have mostly assumed that one direction holds the absolute priority over the other (5). Two methods are often applied to derive capacity: the gap acceptance approach and the queuing approach. In the gap acceptance technique, the capacity of the low-priority stream is determined by combining the gap distribution of the high-priority flow with the gap acceptance function (5). Owing to the assumption of absolute priority, the flow of the high-priority stream is not interrupted, and therefore, the gap distribution is always the same as the headway distribution of the high-priority stream. This approach was extensively studied and generalized by many researchers to accommodate various geometries and arrival patterns (6, 7). Notably, an interesting extension was made by Troutbeck in developing the limited-priority theory, in which the gap distribution was derived by considering both the headway distribution of the major stream and the interruptions caused by minor-stream vehicles ( 6 ).

The other approach for deriving capacity is based on queuing theory. The service time distribution for queuing vehicles is determined first, and then the capacity is obtained by letting the traffic intensity equal one $(8-10)$. The advantage of the queuing model is that, once the service time distribution is found, basic performance measures such as the traffic delay and queue length can be easily obtained by using sophisticated queuing equations, for example, the M/G2/1 (11) and M/G/1 formulas (12, 13). Using the M/G2/1 model, Heidemann and Wegmann offered a thorough analysis of the performance measures for absolute-priority based unsignalized intersections (10).

Recently, Wei et al. proposed an estimation model for computing the vehicular traffic delay at unsignalized crosswalks with probabilistic yielding behavior $(14,15)$. The method decomposes the vehicular stream into stochastic free-flow and queuing traffic. The vehicular delay is then derived by the analogy of the traffic signal by using McNeil's equation (16). However, the model was based on a critical assumption that only those drivers in the free-flow traffic would yield to pedestrians. During the queue dispersion period, drivers do not yield at all. As a consequence, the capacity always equals the reciprocal of the move-up time, and the capacity drop cannot be determined from the model.

The uniqueness of the interaction between the pedestrian and vehicular flows with the uncertain yielding behavior is that none of the streams holds absolute priority. It depends on the individual decision made by the driver. If the driver decides not to yield, the pedestrian has to wait on the side of the crosswalk. From the perspective of drivers, the gaps in the pedestrian flow are not consistent with their original arrival patterns due to the because of the interruption by nonyielding drivers. Therefore, determining the interrupted (modified) gap distribution in the pedestrian flow, which is key to applying the gap acceptance technique, is difficult, if not impossible. Nevertheless, the queuing model does not require an explicit specification of the interrupted-gap distribution.

TABLE 1 Examples of Yielding Rates from Field Observations (3, 4)

| Crossing Treatment | Number <br> of Sites | Mean <br> Yielding <br> Rate |
| :--- | :---: | :---: |
| Overhead flashing beacon (push button) | 4 | 0.49 |
| Overhead flashing beacon (passive activation) | 3 | 0.67 |
| Pedestrian crossing flag | 4 | 0.74 |
| High-visibility sign and marking | 2 | 0.20 |
| Rectangular rapid-flash beacon | 17 | 0.81 |

Instead, it focuses on the service time of an individual vehicle at the crosswalk. The essence is to depict the service mechanism through analyzing the probabilistic interaction between pedestrians and drivers.

This work attempts to develop a capacity model for unsignalized crosswalks with probabilistic priority by using queuing theory. Yielding scenarios are thoroughly analyzed on the basis of the interaction between pedestrians and drivers. The distribution of the service time of queuing vehicles is derived. Its expectation is then applied to determine the traffic capacity. Stochastic simulations are performed to validate the capacity formula. A simplified formula is also developed for ease of practical applications.

The remainder of this paper is organized as follows. First, probabilistic yielding behavior and major assumptions are introduced. Then, service time distribution for queuers and capacity formula are presented in separate sections. Simulation validation and discussions are presented next. The simplified formula is presented, and its accuracy is also examined. And, last, conclusions are offered.

## PROBABILISTIC YIELDING BEHAVIOR AND MAJOR ASSUMPTIONS

In this section, probabilistic yielding behavior is introduced. Assumptions are made for mathematically describing the interaction between pedestrians and drivers.

When a pedestrian seeks to cross a road, that person observes successive gaps in vehicular traffic flow. If the gap is less than the safety gap, the pedestrian has to wait, unless the driver decides to yield. The probability that a driver is willing to yield to pedestrians is termed the "yielding rate." Figure 1 shows a hypothetical yielding scenario. Suppose that, when pedestrian $A$ arrives, the gap to the coming vehicle is less than the safety gap. But the driver decides to yield, and therefore, pedestrian $A$ is able to cross. While pedestrian $A$ is crossing, other pedestrians, $B$ and $B_{1}$ in Figure 1, may arrive. These pedestrians are also able to cross because the driver has already decided to yield. Therefore, the yielding driver has to wait for a gap in the pedestrian flow that is larger than the crossing time. That larger gap implies that, once the driver decides to yield, he or she cannot start again until the gap between successive arrivals of pedestrians is larger than the crossing time. Yielding drivers need to go through a typical gap-scanning process. If a gap that is larger than the crossing time emerges (the gap between $B_{n}$ and $B_{n+1}$ ), the vehicle is able to leave after the last pedestrian ( $B_{n}$ in Figure 1) has reached the other side of the crosswalk.
Assumptions about the yielding and pedestrian crossing behavior are needed so that the capacity can be analytically derived. First, one assumes that the driver makes the yielding decision only after the preceding vehicle passes the crosswalk. This assumption is reasonable because the driver may not be able to observe the pedestrian while that driver is following a vehicle. Even if the driver observes a pedestrian and decides to yield, the pedestrian still cannot cross because the gap is blocked by the leading vehicle.

The arrival of pedestrians is assumed to be a Poisson process, and its headway distribution is specified as
$f_{p}\left(t_{p}\right)= \begin{cases}\lambda_{p} e^{-e_{p} t_{p}} & t_{p} \geq 0 \\ 0 & t_{p}<0\end{cases}$
where $\lambda_{p}$ is the arrival rate of pedestrians and other variables are defined in Table 2.
The capacity formula is independent of the arrival process of the traffic flow. The model proposed here deals only with scenarios for


FIGURE 1 Illustration of yielding scenario at unsignalized crosswalk.

TABLE 2 List of Notations

| Notation | Description |
| :--- | :--- |
| $M$ | Yielding rate |
| $\lambda_{p}$ | Arrival rate of pedestrian flow (pedestrians per second) <br> $\lambda_{v}$ |
| $t_{p}$ | Traffic flow rate (vehicles per second) <br> Headway of pedestrian flow (s) |
| $f_{p}$ | Probability density function of pedestrian arrival headway |
| $F_{p}$ | Cumulative distribution function of pedestrian arrival headway <br> $N_{p}(t)$ |
| $N_{v}(t)$ | Number of arrivals in pedestrian flow for a period $t$ |
|  | Number of departures of vehicles from an infinite queue for a |
| $t_{f}$ | period $t$ |

one-lane traffic. Notations used throughout the paper are listed in Table 2.

## SERVICE TIME FOR QUEUERS

The service time distribution for queuers, as well as the expectation, is derived in this section, which directly leads to the derivation of the traffic capacity, discussed in a later section. Consider a vehicle arriving at the crosswalk and joining a queue, which here is called a "queuer." Denote $t_{\mathrm{sq}}$ as the service time for queuers. The following concepts are used to derive the service time as illustrated in Figure 2:

- Move-up (follow-up) time $t_{f}$. Time spent moving from the second to the first position in the queue,
- Scanning time $t_{w}$. Time spent waiting for a gap between arrivals of pedestrians that is larger than the crossing time, and
- Lag time $t_{l}$. Time elapsed before the first gap in the scanning period starts.

First, the scenario in which drivers incur a service time that equals only the move-up time is analyzed. In this scenario, drivers could pass the crosswalk immediately after moving up to the first position of the queue, which further includes the following two scenarios:

Case $1 a$. During the move-up time, no pedestrian is waiting or arriving at the side of the crosswalk. The driver does not need to make a yielding decision.


FIGURE 2 Illustration of yielding scenario at unsignalized crosswalk for queuing vehicles yielding to pedestrians: (a) unable to cross in previous gaps and $(b)$ arriving during move-up time.

Case $1 b$. The driver chooses not to yield even though pedestrians are waiting or arriving during the move-up time.

In these two cases, the service time for queuers equals the moveup time. Conversely, if the driver yields to pedestrians who are waiting at the side of the crosswalk or arrive during the move-up time, the service time $t_{\mathrm{sq}}$ is then larger than the move-up time and includes three parts: lag time $t_{l}$, scanning time $t_{w}$, and time spent waiting for the last pedestrian to cross $\delta$. This yielding scenario can be further divided into the following two cases:

Case $2 a$. Pedestrians waiting when the preceding vehicle passes the crosswalk were unable to cross because the preceding vehicle
chose not to yield. In this scenario, the lag time $t_{l}$ is zero, as illustrated in Figure $2 a$.

Case $2 b$. No pedestrians are waiting when the preceding vehicle departs. But at least one pedestrian arrives during the move-up time, and the driver decides to yield. In this case, the lag time $t_{l}$ starts from the beginning of the move-up period and ends at the instant the pedestrian arrives, as illustrated in Figure 2b. The lag time has to be distinguished from the gap in the scanning period because its distribution is conditional on the lag time being smaller than the move-up time.

Combining the above yielding and nonyielding cases, the service time $t_{\text {sq }}$ can be specified as
$t_{\mathrm{sq}}= \begin{cases}t_{f} & \text { No yielding, Cases } 1 a \text { and } 1 b \\ \delta+t_{w} & \begin{array}{l}\text { Yielding to pedestrians who were } \\ \text { left from previous gap, Case } 2 a\end{array} \\ t_{l}+\delta+t_{w} & \begin{array}{l}\text { Yielding to pedestrians who arrive } \\ \text { during move-up time, Case } 2 b\end{array}\end{cases}$
Before the distribution of service time is discussed, a discussion of each component of the service time is appropriate. The lag time $t_{l}\left(t_{l}<t_{f}\right)$ is defined as time elapsed before the first gap starts in the scanning period, which starts from the beginning of the move-up time and ends at the instant the pedestrian arrives during the move-up time. Because of the memoryless property of the Poisson process, the cumulative distribution function of the lag time can be specified as
$F\left(t_{l}\right)=\frac{F_{p}\left(t_{l}\right)}{F_{p}\left(t_{f}\right)}$
which is a scaled cumulative distribution function of the headway distribution of pedestrian flow $F_{p}$ given the condition that the lag time is less than the move-up time. The expectation of $t_{l}$ is then
$E\left(t_{l}\right)=\frac{\int_{0}^{t_{l}} t_{l} f_{p}\left(t_{l}\right) d t_{l}}{F_{p}\left(t_{f}\right)}$
The scanning time $t_{w}$ is the period during which the queuer is waiting for a gap in the pedestrian flow that is larger than the crossing time. The scanning period starts from the first gap after the lag time $t_{l}$. Its distribution has been extensively studied, and its main results, such as the expectation and variance, are presented in Mayne (17). Under the Poisson assumption of the pedestrian flow, the expectation of $t_{w}$ is given as
$E\left(t_{w}\right)=\frac{e^{\lambda_{p} \delta}-1}{\lambda_{p}}-\delta$
With Equations 4 and 5 prepared, the distribution of service time and its expectation can be derived. Let $P_{\text {qnd }}$ denote the probability of having a service time that equals $t_{f}$. According to Cases $1 a$ and $1 b$, $P_{\text {qnd }}$ can be derived as

$$
\begin{align*}
P_{\mathrm{qnd}} \equiv & P\left(t_{\mathrm{sq}}=t_{f}\right)=(1-L) P\left(N_{p}\left(t_{f}\right)=0\right) \\
& +(1-M)\left[L+(1-L) P\left(N_{p}\left(t_{f}\right)>0\right)\right] \tag{6}
\end{align*}
$$

where

$$
\begin{aligned}
M & =\text { yielding rate }, \\
N_{p}\left(t_{f}\right) & = \\
& \text { number of arrivals of pedestrians during move-up time } \\
& t_{f}, \text { and } \\
L= & \text { probability of having pedestrians waiting at crosswalk } \\
& \text { when preceding vehicle departs. }
\end{aligned}
$$

The first term in Equation 6, $(1-L) P\left(N_{p}\left(t_{f}\right)=0\right)$, corresponds to Case $1 a$, in which the driver does not need to make a yielding decision because no pedestrians are waiting or none arrives during the moveup time. The second term, $(1-M)\left[L+(1-L) P\left(N_{p}\left(t_{f}\right)>0\right)\right]>0$, corresponds to Case $1 b$, in which the driver decides not to yield even though pedestrians are left from previous gaps or are arriving during the move-up time.

Similarly, let $P_{q d}$ be the probability of a queuer yielding to the pedestrians and having a service time that is larger than $t_{f}$. Based on Cases $2 a$ and $2 b, P_{q d}$ can be obtained as
$P_{\mathrm{qd}} \equiv P\left(t_{\mathrm{sq}}>t_{f}\right)=M L+M(1-L) P\left(N_{p}\left(t_{f}\right)>0\right)$
Here, the LM term corresponds to Case $2 a$, in which pedestrians left from previous gaps and the driver decides to yield. The second term, $M(1-L) P\left(N_{p}\left(t_{f}\right)>0\right)$, corresponds to Case $2 b$, in which the driver yields to pedestrians who arrive during the move-up time. One can easily see that $P_{\mathrm{qnd}}+P_{\mathrm{qd}}=1$.

Next shall come derivation of the probability $L$, which describes the chance that a queuer would encounter a pedestrian waiting at the beginning of the move-up time, as discussed in Case $2 a$ and shown in Figure $2 a$. By using the one-step-forward iteration developed in Wie et al. $(14,15)$, the probability $L$ can be specified as
$L=L(1-M)+(1-M)(1-L) P\left(N_{p}\left(t_{f}\right)>0\right)$
where the first term, $L(1-M)$, is the probability that pedestrians are waiting when the preceding vehicle starts its move-up time but its driver refuses to yield. The second (last) term corresponds to the scenario that no pedestrians are left from previous gaps when the preceding vehicle initiates the move-up time. However, at least one pedestrian arrives during the move-up time, and the driver refuses to yield. Both scenarios result in the case that pedestrians are waiting when the move-up time starts. Reorganizing Equation 8, one obtains
$L=\frac{(1-M) P\left(N_{p}\left(t_{f}\right)>0\right)}{(1-M) P\left(N_{p}\left(t_{f}\right)>0\right)+M}$
Given the Poisson arrival of the pedestrians flow in Equation 1, $P_{\mathrm{qd}}, P_{\mathrm{qnd}}$, and $L$ can be derived as
$P_{\mathrm{qd}}=M\left[1-e^{-\lambda_{p} t}(1-L)\right]$
$P_{\mathrm{qnd}}=1-M\left[1-e^{-\lambda_{p f} f}(1-L)\right]$
$L=\frac{(1-M)\left(1-e^{-\lambda_{p} t_{f}}\right)}{(1-M)\left(1-e^{-\lambda_{p} t_{f}}\right)+M}$
Now, the cumulative distribution function of the service time for queuers is given by
$F_{\mathrm{sq}}\left(t_{\mathrm{sq}}\right)=\left\{\begin{array}{lll}0 & t_{\mathrm{sq}}<t_{f} \\ P_{\mathrm{qnd}} & t_{\mathrm{sq}}=t_{f} \\ P_{\mathrm{qnd}}+M\left[\begin{array}{c}(1-L)\left(f_{p} * f_{w}\right)\left(t_{\mathrm{sq}}\right) \\ +L \int_{0}^{t_{\mathrm{sq}}} f_{w}(\tau-\delta) d \tau\end{array}\right] & t_{\mathrm{sq}}>t_{f}\end{array}\right.$
where $f_{p}$ is the probability density function of pedestrian headways and $f_{w}$ is the probability density function of the scanning time $t_{w}$. Here, $f_{p} * f_{w}\left(t_{\text {sq }}\right)$ is the convolution of $f_{p}$ and $f_{w}$ as
$\left(f_{p} * f_{w}\right)\left(t_{\mathrm{sq}}\right)=\int_{\delta}^{t_{\mathrm{g}}} \int_{0}^{t} f_{p}\left(t_{l}\right) f_{w}\left(\tau-\delta-t_{l}\right) d t_{l} d \tau$
From the cumulative distribution, one obtains the expectation of the service time as
$E\left(t_{\mathrm{sq}}\right)=M(1-L) F_{p}\left(t_{f}\right) E\left(t_{l}\right)+P_{\mathrm{qd}}\left(E\left(t_{w}\right)+\delta\right)+P_{\mathrm{qnd}} t_{f}$
By entering Equations 4 and 5 and 10 through 12 into Equation 15, the mean service time for queuers is finally obtained as
$E\left(t_{\mathrm{sq}}\right)=\frac{M\left(1-e^{-\lambda_{p} t_{f}}\right)}{\lambda_{p} e^{\lambda_{p} \delta}}+(1-M) t_{f}-M L \frac{1-e^{\lambda_{p}\left(\delta-t_{f}\right)}}{\lambda_{p}}$
Equation 16 is one of the most important results in this study, and it directly leads to the traffic capacity formula. When the pedestrian flow rate is zero ( $\lambda_{p}=0$ ), the probability $L$ is also zero (from Equation 12). The mean service time tends toward
$\lim _{\lambda_{p} \rightarrow 0} E\left(t_{\mathrm{sq}}\right)=t_{f}$
which simply means queuers are able to pass the crosswalk successively with a move-up time $t_{f}$ without any impedance.

When the yielding rate $M$ is zero, indicating that all the drivers refuse to yield to pedestrians, the mean service time also reduces to $t_{f}$.
$E\left(t_{\mathrm{sq}} \mid M=0, \lambda_{p}>0\right)=t_{f}$
Finally, when the yielding rate equals one, indicating that all the drivers fully respect the priority of the pedestrian flow, $L=0$ (from Equation 12). The mean of the service time becomes
$E\left(t_{\mathrm{sq}} \mid M=1, \lambda_{p}>0\right)=\frac{1-e^{-\lambda_{p} t_{f}}}{\lambda_{p} e^{-\lambda_{p} \delta}}$

In this case, the mean service time drops to the classic formula for absolute priority-based unsignalized intersections that results from the stepwise gap acceptance function as derived in Gartner et al. (5), Poeschl (8), and Heidemann and Wegmann (10).

## TRAFFIC CAPACITY

With expectation of the mean service time in place, the capacity accounting for the impact of the probabilistic yielding behavior can be determined. The "capacity" is defined as the maximum possible
number of departures per unit of time. More precisely, by following the definition in Heidemann and Wegmann, let $N_{v}$ denote the number of departures from an infinite queue on the road (10). Then, the capacity $\lambda_{c}$ is
$\lambda_{c} \equiv \lim _{t \rightarrow \infty} \frac{N_{v}(t)}{t}$
provided that the limit exists. Denote $\rho$ as the traffic intensity (or equivalently the saturation rate), which is given as
$\rho=\lambda_{v} E\left(t_{\mathrm{sq}}\right)$
where $\lambda_{\nu}$ is the arrival rate of the vehicular traffic and $E\left(t_{\mathrm{sq}}\right)$ is given in Equation 16. The capacity can then be determined by letting the saturation rate $\rho$ equal one, which gives rise to the capacity formula as
$\lambda_{c}=\frac{1}{E\left(t_{\mathrm{sq}}\right)}=\frac{\lambda_{p} e^{-\lambda_{p} \delta}}{M\left(1-e^{-\lambda_{p} t_{f}}\right)+(1-M) t_{f} \lambda_{p} e^{-\lambda_{p} \delta}-M L\left(e^{-\lambda_{p} \delta}-e^{-\lambda_{p} t_{f}}\right)}$

Equation 22 is the central result from this study that quantitatively determines how a probabilistic yielding behavior affects traffic flow capacity. Now, examination of the capacity formula in some special cases is in order. The first special case is that pedestrian flow is zero, which implies no impedance on the vehicular traffic. From Equation 22, one obtains
$\lim _{\lambda_{p} \rightarrow 0} \lambda_{c}=\frac{1}{t_{f}}$
This simply means that the capacity tends toward the maximum value, which is the reciprocal of the move-up time.

If $M=0$, indicating that all drivers do not yield to pedestrians, then $L=1$. The capacity also reduces to $1 / t_{f}$ :
$\lambda_{c}\left(M=0, \lambda_{p}>0\right)=\frac{1}{t_{f}}$
Conversely, if all drivers yield to pedestrians ( $M=1$ ), then $L=0$ (from Equation 12). The capacity equation becomes
$\lambda_{c}\left(M=1, \lambda_{p}>0\right)=\frac{\lambda_{p} e^{-\lambda_{p} \delta}}{1-e^{-\lambda_{p} t_{f}}}$
which is the classic capacity formula obtained by using the stepwise gap acceptance function when pedestrian flow (or the major stream in a two-way stop-controlled intersection) has absolute priority (5, 16, 17).

## SIMULATION VALIDATION

To validate the capacity model, empirical data should be collected at unsignalized crosswalks that operate at capacity for extended periods. However, this requirement can hardly be met in reality because at-capacity operation for extended periods of time usually warrants the installation of traffic signals. As such, validation of the capacity model against empirical data is very difficult, if not infeasible.

In consequence, stochastic simulations are performed to validate the proposed capacity formula. The impacts of the yielding rate, the pedestrian flow rate, the move-up time, and the crossing time are also discussed. Under the assumptions mentioned in the section on probabilistic yielding behavior, the capacity formula given in Equation 22 is derived without any approximations, and therefore, is the exact solution. Stochastic simulations were developed to assure the correctness of the derivation. The program was coded by using C\# language.

In the simulation, vehicles are queued up at a virtual crosswalk. The arrival instants of the pedestrians are generated by following the Poisson arrival. A constant move-up time is applied for vehicles that move from the second to the first position of the queue. During the move-up time, if pedestrians are waiting or arriving, the driver needs to make a yielding decision on the basis of a random number, which is generated in relation to the yielding rate. In the experiments, the authors assumed that the move-up time $t_{f}$ is 2 s and the crossing time $\delta$ is 6 s . Simulations were run for $3,600 \mathrm{~s}$, with a resolution of 0.01 s . The simulation results with different combinations of pedestrian flow rates and yielding rates were averaged over 10 runs.

The comparisons are shown in Figure 3 for different yielding rates (such as $0.2,0.4,0.6$, and 0.8 ). The figure shows that the simulation results precisely match the analytical solutions for different pedestrian flow rates and yielding rates, and this matching confirms
the correctness of the derivation under the assumptions made in the section on probabilistic yielding behavior. In addition, one can find that the capacity drops from the maximum value of $1 / t_{f}$ with the increase in the pedestrian flow rate.

The surface plots of the capacity with respect to the yielding rate and pedestrian flow rate are shown in Figure 4. Again, the surface plots from the analytical solution (Equation 22) and the stochastic simulation precisely match each other without any noticeable differences. At low yielding rates, the capacity tends to be more sensitive given higher pedestrian flow rates. A sharp decrease in the capacity can be observed in Figure 4 when the yielding rate increases from 0 to 0.1 at a high pedestrian flow rate. This difference can be explained as follows. At a high pedestrian flow rate, the scanning period $t_{w}$ dominates the service time for those yielding drivers and is a lot higher than the move-up time. Once the probability of having a service time larger than the move-up time $\left(P_{\mathrm{qd}}\right)$ becomes positive, for example, when the yielding rate increases from 0 to 0.1 , the average service time increases dramatically and results in a sharp capacity drop.

The impacts of different parameters, including the move-up time and the crossing time, are demonstrated in Figure 5. As the crossing time $\delta$ increases, the capacity drops because the yielding driver's scanning time becomes longer and so does the mean service time. A longer move-up time also leads to a reduction in


FIGURE 3 Comparisons of analytical and simulated capacities for yielding rate of (a) 0.2 and (b) 0.4.


FIGURE 3 (continued) Comparisons of analytical and simulated capacities for yielding rate of (c) 0.6 and (d) 0.8

(a)

FIGURE 4 Surface plots of analytical and simulated capacities: [a) analytical solution.
(continued)

(b)

FIGURE 4 (continued) Surface plots of analytical and simulated capacities: (b) stochastic simulation.


FIGURE 5 Capacities with respect to different move-up and crossing times for yielding rate of (a) 0.3 (and $t_{f}=2 \mathrm{~s}$ ) and (b) 0.9 (and $t_{f}=2 \mathrm{~s}$ ).


FIGURE 5 (continued) Capacities with respect to different move-up and crossing times for yielding rate of $(c) 0.3$ and (d) 0.9.
capacity. This change can be explained from two respects. On the one hand, the service time for nonyielding drivers, which is equal to the move-up time, becomes longer with an increase in the move-up time. On the other hand, given a longer move-up time, the probability of having pedestrians arrive during the move-up time also rises and results in an increase in the frequency that a driver encounters a pedestrian during the move-up time. Therefore, the capacity is negatively related to all parameters, including the crossing time, the move-up time, the yielding rate, and the pedestrian flow rate.

## PRACTICAL APPROXIMATION

Though the capacity formula given in Equation 22 precisely matches the simulation result, the complexity may prevent practical applications. In this section, a much simpler and more straightforward approximation, which still gives quite close estimations to the exact solutions, is proposed.

The complexity of Equation 22 is mainly attributed to the determination of the probability $L$. Therefore, in this simplification, $L$ is assumed to equal zero, and therefore, is independent from the pedestrian flow rate and the yielding rate. Let $\left(\tilde{\lambda}_{c}\right)$ denote the approximated capacity. By letting $L=0$ in Equation 22, one obtains the simplified capacity formula as

$$
\begin{equation*}
\tilde{\lambda}_{c}=\frac{\lambda_{p} e^{-\lambda_{p} \delta}}{M\left(1-e^{-\lambda_{p} t_{f}}\right)+(1-M) t_{f} \lambda_{p} e^{-\lambda_{p} \delta}} \tag{26}
\end{equation*}
$$

Equation 26 greatly simplifies the exact solution in Equation 22 without the necessity of calculating the probability $L$. From Equation 26, one can also obtain the corresponding mean service time under the approximation as
$E\left(\tilde{t}_{\mathrm{sq}}\right)=M \frac{\left(1-e^{-\lambda_{p} t_{f}}\right)}{\lambda_{p} e^{-\lambda_{p} \delta}}+(1-M) t_{f}$

This simplified equation (Equation 27) of the mean service time also has an interesting implication, though it is not rigorous. The first term is the product of the yielding rate and the service time for those yielding drivers. The second term is the proportional service time for those nonyielding queuers who experience only the move-up time. Therefore, Equation 27 is simply a weighted sum of the service times for yielding and nonyielding drivers.

Despite its simplicity, solutions of the simplified formula at the boundaries are still the same as those from the exact formula. If the pedestrian flow is zero $\left(\lambda_{p}=0\right)$, Equation 26 tends toward $1 / t_{f}$ :
$\lim _{\lambda_{p} \rightarrow 0} \tilde{\lambda}_{c}=\frac{1}{t_{f}}$

When the yielding rate is zero $(M=0)$, the simplified capacity formula reduces to $1 / t_{f}$ :
$\tilde{\lambda}_{c}\left(M=0, \lambda_{p}>0\right)=\frac{1}{t_{f}}$
If all the drivers fully respect the priority of pedestrians $(M=1)$, the simplified capacity formula then reduces to
$\tilde{\lambda}_{c}\left(M=1, \lambda_{p}>0\right)=\frac{\lambda_{p} e^{-\lambda_{p} \delta}}{1-e^{-\lambda_{p} t_{f}}}$
which is again the classic capacity formula with absolute priority and is the same as Equation 25.

The comparisons between the exact solutions and the approximations are shown in Figure 6, which shows that the approximation generally gives very close estimations of the exact solutions. With


FIGURE 6 Comparisons between exact solution from Equation 22 and approximation from Equation 26 for yielding rate of $(a) 0.2,(b) 0.4$, and $(c) 0.6$.


FIGURE 6 (continued) Comparisons between exact solution from Equation 22 and approximation from Equation 26 for yielding rate of ( $d$ ) 0.8 .
higher yielding rates, the approximations are more accurate. This is because the probability $L$ tends toward zero given a high yielding rate, as those from Equation 12. The approximation is also more accurate if the pedestrian flow rate is high because the scanning time $t_{w}$ becomes dominant if the pedestrian flow is high. As a consequence, the contribution of the last term in the mean service time in Equation 16, which contains the probability $L$, is negligible. Owing to its great simplicity and satisfactory accuracies, the authors recommend the simplified capacity formula given in Equation 26 for practical applications.

## CONCLUSION

The uncertainty in yielding behavior at unsignalized crosswalks leads to a complicated interaction pattern between traffic and pedestrian flows in which neither of the flows holds absolute priority. This pattern is vastly different from information in the existing literature, which normally assumes that one flow has absolute priority over the other.

Quantifying the reduced capacity because of the probabilistic yielding behavior is essential for evaluating the level of service and performance measures such as traffic delay and queue length. In this paper, a capacity model that explicitly takes the yielding probability into consideration was developed. It can be viewed as a generalization of the capacity equation with absolute priority. If the yielding rate equals one, it reduces to the classic capacity formula with absolute priority. The experiments show that the solutions from the proposed capacity models precisely match the results from stochastic simulations.

An approximation that greatly simplifies the exact formula was also developed, but it still gives very close estimations. The simplified formula given in Equation 26 is recommended for practical applications because of its simplicity and satisfactory accuracies.

The proposed capacity model, as well as the distribution of the service time, can be further applied for evaluating performance measures (i.e., determining traffic delay and queue length by sophisticated queuing formulas, a topic that the authors are now studying). The model can also be extended to multilane traffic scenarios through careful consideration of independent yielding behavior across multiple lanes.

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