Expected Value – Variance Analysis
(E-V Frontier Analysis)

Chapter 6, Robison & Barry's Book

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CHAPTER

SIX

EQUILIBRIUM ANALYSIS UNDER RISK

As established in Chap. 1, a distinction between decision theory and economic theory is the former's emphasis on the selection of either a preferred choice or a set of preferred choices. In contrast, economic theory addresses changes in a preferred choice as changes occur in one or more of the underlying parameters. Under risk, these parameters may include changes in the variability of choice outcomes or changes in the expected level of choice outcomes. Thus economic theory under risk extends the firm's equilibrium analysis from the certainty case to the risk case. Using the expected value–variance framework, this extension yields a stochastic counterpart of the certainty case in which income and substitution effects resulting from shifts in EV efficient sets are distinguished by an investor's risk aversion characteristics.

In this chapter we introduce the fundamental properties of equilibrium analysis that serve as the analytic framework for model evaluations in later chapters. We begin by establishing the concept of expected value–variance analysis and show how maximizing the expected utility of a choice from an EV efficient set under certain conditions is equivalent to maximizing the certainty equivalent of the choice. Following this, we define the income and substitution effects under risk and illustrate them with an application to a savings problem.

ORIGINS OF EXPECTED VALUE–VARIANCE ANALYSIS

This chapter introduces an analytic framework that characterizes expected utility solutions in terms of their expected values E and variances V. To justify this approach, we will review the debate over the use of EV models and the more general EU models. Then we will show that the EV approach is more strongly justified as an analytic or deductive tool than as an empirical tool, although most studies using the EV approach have emphasized empirical applications.

EV Models and Decision Theory

Since its development by Markowitz as a portfolio selection tool, the EV model has been a popular method for ordering choices into efficient and inefficient sets. The EV efficient set is defined as the choices or sets of choices that provide minimum variance for alternative levels of expected returns. The efficient set is considered to contain the preferred choice for a well-defined set of investors. In contrast, the inefficient set does not contain the preferred choice.

Earlier, the conditions justifying the use of the EV model included assumption of a quadratic utility function or normally distributed outcomes. Tobin showed that, if the investor's utility function were quadratic, so that preferences were expressed only about the expected value and variance of choices, the preferred choice would be a member of the EV set (see also Chap. 4). Samuelson showed that a risk-averse expected utility maximizer's preferred choice would be from an EV set as long as the choices' outcomes were normally distributed and thus fully characterized by their expected values and variances.

Another justification for the EV approach was shown by Tobin but has generally been overlooked. He showed that expected utility–maximizing decisions are always members of an EV set when choices are represented by various combinations of a risky asset and a safe asset. The resulting choice set has no choices that are excluded from the EV set. Meyer has since shown that Tobin's condition is a special case of a more general condition requiring linear combinations of a random variable (Meyer).[1]

Thus the EV approach is justified on the basis of four conditions: (1) quadratic utility, (2) normality, (3) choices involving a single random variable, and (4) choices involving linear combinations of the random variable. None of these conditions except 4 is very satisfactory relative to the characteristics of most empirical situations. Quadratic utility implies that marginal utility becomes negative beyond some level of monetary outcome and that the investor being modeled is characterized by increasing absolute risk aversion. Few variables take on values that range from negative to positive infinity as normality implies or are symetrically distributed. And most decision situations concern choices involving more than one risky asset. These are, of course, all only sufficient conditions.

These shortcomings of conditions underlying the EV approach have made its justification in empirical analysis dependent on the ability to approximate results obtained with the more general EU model. Porter, for example, showed that EV sets of randomly constructed stock portfolios were consistent with EU models, except for portfolios having small expected values and variances. Tsiang demonstrated that various restrictions on skewness could yield a close correspondence between the EV and EU results. Levy and Markowitz showed similar effects of EV analysis as an effective approximating approach to portfolio selection. Moreover, the appropriateness of quadratic utility has been defended as a second-order Taylor series approximation to

all risk-averse utility functions. Thus the debate involving EU and EV models as decision tools has largely focused on the approximating capacity of the EV model.

The EV Model as a Deductive Tool

Decision theory models and the resulting efficient sets are valuable for their empirical content. They identify efficient choices and the resulting measures of financial performance (e.g., expected values and variances). They contribute importantly to improving the quality of decision making when the objective function is already known, and to predicting the magnitude of outcomes from various decision alternatives. But the empirical measures of expected values and variances of efficient choice distributions are not necessary in an analysis that serves to deduce economic theory under risk. The deductive or analytic results are valuable not for their measurability but for their characterization of relationships between variables and for showing the direction of change in relevant variables as changes occur in other factors that comprise the decision environment. These relationships and changes are basic to an analytic model.

Analytic models must of necessity be developed in as simple a fashion as possible, while still reflecting as rich a content as possible. Only a limited number of relationships can be analyzed simultaneously through deduction. Thus most analytic models formulated under risk contain only a single random variable. Adding more variables quickly complicates the derivations and tends to obscure the underlying relationships. In fact, stochastic dominance conditions are indeterminate for the evaluation of multiple risky assets under most circumstances (Hader and Russell).

Restricting the analytic model to be linear combinations of a random variable suggests that Tobin's rationale for EV analysis is applicable. That is, with all choices formed by combinations of a risky and a safe asset, only EV efficient choices are available. With this approach, we can seek to build an EV analytic model that yields results consistent with those obtained from the more general EU models. The point of departure for doing this graphically is to characterize any solution to the EV set in terms of maximizing a linear tangent line to the EV set where the line has slope $\lambda/2$. Since the choice of the slope is arbitrary, any solution from the EV set can be described using this approach.

In Fig. 6.1, the tangency point for the isoexpected utility line $E\left[U(\bar{y})\right]$ and the EV set AB occurs at choice C, yielding expected wealth $E(y_c)$ and variance $\sigma^2(y_c)$. To further develop this approach, let the linear tangent line in Fig. 6.1 be extended until it reaches the vertical expected return axis at y_{CE}. Since y_{CE} has zero variance, it is considered a completely certain return that is equivalent in terms of expected utility to the risky choice with expected return $E(y_c)$. Thus we can call y_{CE} the certainty equivalent to risky expected return $E(y_c)$ and refer to the line between y_{CE} and $E(y_c)$ as the certainty equivalent line. Moreover, the optimal risky solution can be obtained by maximizing the certainty equivalent subject to the restriction that the choice occurs from the EV

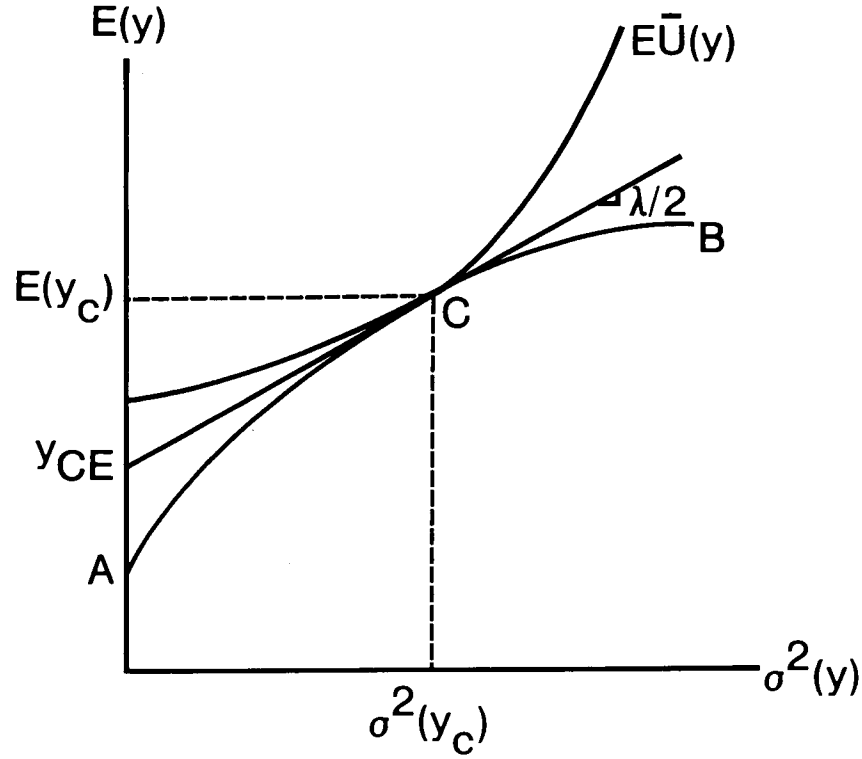

Figure 6.1 Equilibrium point C established by the tangency between an isoexpected utility line $E\bar{U}(y)$ and an EV set ACB identified by the y_{CE} linear tangent.

set and that its slope equals the equilibrium slope at the tangency between the isoexpected utility line $E\bar{U}(y)$ and the EV set, a slope designated as $\lambda/2$. This objective can be expressed as:

$$\max y_{CE} = E(y) - \frac{\lambda}{2}\sigma^2(y)$$

In this framework, the maximization of y_{CE} is equivalent to finding an optimal solution based on the equality of slope $\lambda/2$ between the EV set and the isoexpected utility lines, as illustrated by the tangency at C in Fig. 6.1. Maximizing the expected utility of a choice, where the expected value and variance are the objects of utility, is equivalent to maximizing its certainty equivalent. Moreover, it follows that if a change occurs in the shape or location of the EV set, then the optimal slope $\lambda/2$ must change to reflect a new point of tangency between an investor's isoexpected utility lines and the revised EV set. The optimal slope, then, can be described as a functional relationship:

$$\lambda = \lambda(r, \sigma_\epsilon^2, W_0, \ldots)$$

where r and σ_ϵ^2 are the mean and variance, respectively, of the return on the risky asset, W_0 is the level of allocable resources, and "..." represents other variables affecting the location of the EV set.

The general problem is to formulate an analytic framework that shows how the optimal value of the control variable responds to changes in other variables

that shift and/or rotate the EV set. This in turn involves a description of how $\lambda(\ldots)$ responds to changes in the EV set. This problem is approached by dividing the adjustment into two parts: an income effect and a substitution effect. The income effect is identical to the results obtained by Cass and Stiglitz for the generalized EU model. The substitution effect, to be derived later, requires a limiting assumption that may not always yield identical results for the EV and the EU models. The net results of the income and substitution effects, however, nearly always yield results for the EV model that conform to those for the EU model.

This use of the EV model is based on its strength as an analytic tool rather than as a decision theory tool. In this light, the primary analytic strengths of the EV model include: its relative ease in deriving optimal solutions and conducting equilibrium analysis; the natural relationship between the concepts of risk and variability and the statistical concept of variance; the ease with which it can be graphed using expected values and variances as the two dimensions; and finally, although not considered here, the extension of micro results to aggregative analysis. Although not justified by Tobin's single risky asset argument, the EV model permits a natural framework for analyzing multiple risky assets which is not analytically possible within the generalized EU framework. These strengths make the EV approach well-suited for modeling and analyzing many types of decision situations.

ASSET CLASS AND THE INCOME EFFECT UNDER RISK

Under conditions of certainty, an increase in an investor's wealth will increase the purchase of normal assets, leave constant the purchase of neutral assets, and reduce the purchase of inferior assets. Since all investors are considered to maximize profits in a certain world, and hence do exactly the same thing, these differences in asset purchases arising from a change in wealth are attributed to the characteristics of the assets, not to the investor. Under risk, however, this is not true. As wealth increases, investors facing the same investment choices may respond differently because of differences in their risk attitudes. Thus equilibrium analysis under risk utilizes information about these risk attitudes.

We begin the analysis by specifying a set of investors who can choose between a risky and a safe asset in formulating their portfolios. If these investors experience an increase in risk-free wealth, we must identify the conditions under which they will increase, leave constant, or decrease their holdings of the risky asset. The classification of risky assets as normal, neutral, or inferior is based on the generalization of a proof (contained in App. 6A) from Cass and Stiglitz that has the following result:

Theorem 6.1 If there are two assets, one risky and one safe, the total purchases of the risky asset increase, remain unchanged, or decrease with increases in initial wealth as there is decreasing, constant, or increasing absolute risk aversion.

This classification of risky assets is in turn based on the absolute risk aversion characteristics of the investor. A corollary to Theorem 6.1 is that, if the purchase of the risky asset increases, remains constant, or decreases with increases in risk-free wealth, depending on whether absolute risk aversion $R'(y)$ decreases, remains constant, or increases, then so does the variance of the choice. Together, the theorem and corollary imply that a risky asset is a normal, neutral, or inferior good depending on whether $R'(y) \lesseqgtr 0$.

Under conditions of certainty, the income effect is defined as a change in demand for an asset resulting from an increase in real income with prices held constant. Under risk, however, prices of assets are often random variables and, to identify the income effect, their probability distributions are assumed to remain constant as risk-free wealth increases. Thus the income effect under risk is the change in demand for an asset resulting from an increase in risk-free wealth while holding probability distributions constant.

Within the expected value–variance framework the income effect is expressed by a parallel shift in the EV set. To show the effect graphically, let the EV set be curve ACB in Fig. 6.2, with preferred choice C occurring at the point of tangency between an isoexpected utility curve $E\left[\bar{U}(\bar{y})\right]$ and the EV set ACB. Preferred choice C has expected value $E(y_C)$ and variance $\sigma^2(y_C)$. Now assume that the investor receives an increase in risk-free wealth that causes an upward parallel shift in the EV set. The upward parallel shift reflects the addition of risk-free wealth to all previous portfolios in the amount $(1+r)\Delta W_0$, where ΔW_0 is the change in wealth and r is the safe return. The increase in risk-free wealth also extends the EV set from B to B' because more risky assets can be held than before. Graphically the new EV set is $A'C'B'$.[2]

Now consider how investors within the three classes of absolute risk aversion will adjust to the parallel shift in the EV set. For the decision maker with constant absolute risk aversion, $R'(y) = 0$, the risky asset is a neutral good and the investor's holdings remain unchanged. The preferred choice is C' with the same variance as before. Because the shift is parallel, the slope at C', given by a linear tangent, is the same as the slope at C.

For the investor with decreasing absolute risk aversion, $R'(y) < 0$, the new risky asset is a normal good; its purchase increases with an increase in risk-free wealth. This investor will move from the original choice B to a location on the new EV set $A'C'B'$ that lies to the right of C'. This new choice is represented by C''. Since the slope of the EV set declines as one moves from A' to B', the equilibrium slope at C'' is less than the slope at C'.

A similar analysis indicates that risky assets are inferior goods for investors with increasing absolute risk aversion. In this case, the equilibrium choice lies

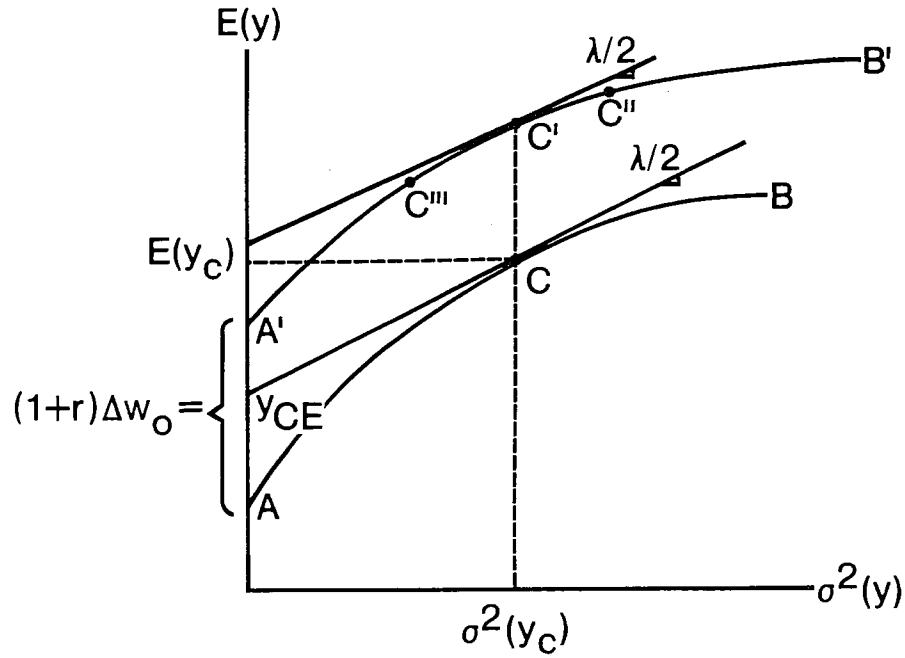

Figure 6.2 Income effects under risk measured on expected value–variance set ACB.

to the left of C', say at C''', with an increased slope.

To summarize the income effect, for an increase in risk-free wealth that extends the EV set and shifts it upward in a parallel fashion, the equilibrium slope decreases, remains constant, or increases for investors with decreasing, constant, or increasing absolute risk aversion, respectively. Moreover, since changes in the slope of an EV set correspond to changes in the holdings of risky assets, the purchases of risky assets decrease, remain constant, or increase for investors with increasing, constant, or decreasing absolute risk aversion, respectively.

THE SUBSTITUTION EFFECT UNDER RISK

Under conditions of certainty, the substitution effect is the change in demand for an asset resulting from a change in relative prices after compensating the decision maker for a change in real income; i.e, keeping utility fixed. Under risk, the substitution effect is the change in quantity demanded resulting from a change in the probability distribution of price after compensating the investor for a change in risk-free income.

To illustrate, consider an investor who is in equilibrium at point C on EV set ACB in Fig. 6.3. Let the probability distribution change so that the expected return on the risky asset increases. This change causes the EV set to rotate counterclockwise to a new location $AC'B'$. The substitution effect resulting

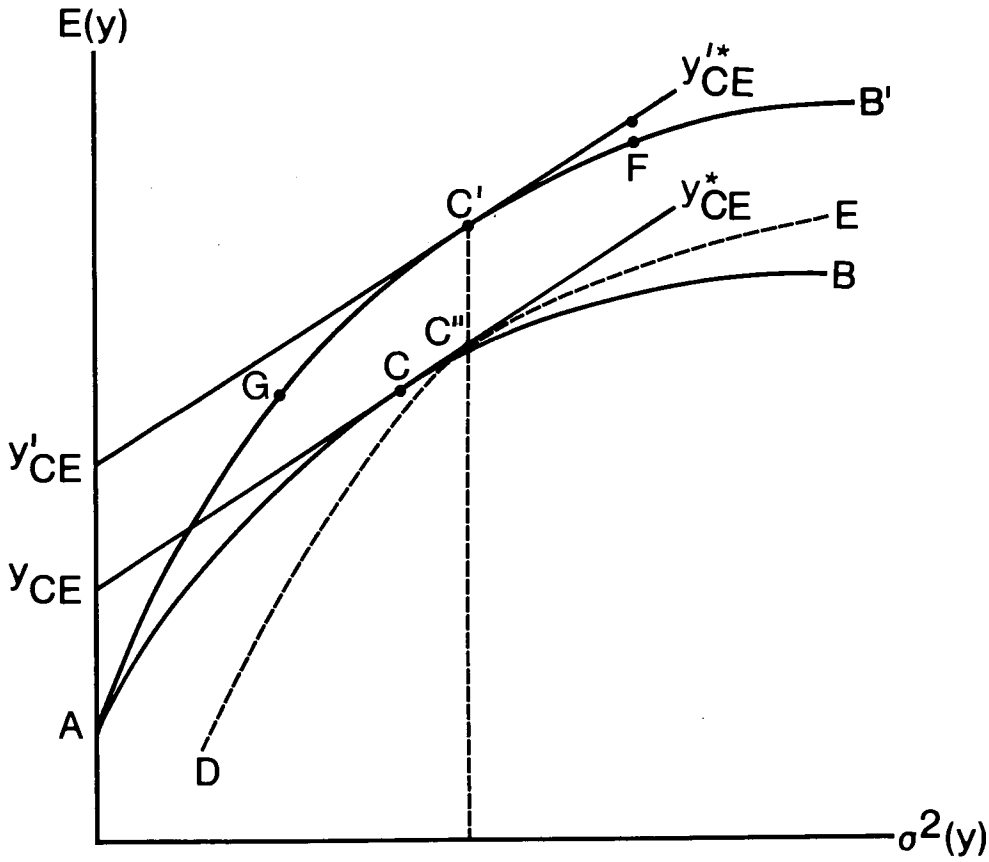

Figure 6.3 Substitution effect under risk obtained by measuring the adjustment of a constant absolute risk-averse investor.

from this change can be found in two ways. One way is to subtract the income effect from the total change in the EV set, with the remainder representing the substitution effect. The second way is to find the size of the adjustment in the equilibrium solution for those investors whose income effect is zero. As shown above, these investors display constant absolute risk aversion.

Following the second approach, which is based on an initial equilibrium at C, we first subtract real income until the investor is returned to his or her initial isoexpected utility line. An important condition is the shape of this isoexpected utility line. Freund showed that the line was linear when the risky asset was normally distributed. Moreover, Pratt's second-order approximation of the risk premium also yields a linear relationship:

$$y_{CE} = E(y) - \frac{\lambda}{2}\sigma^2(y) \tag{6.1}$$

But this relationship is the same as the linear tangent line used earlier in the chapter to obtain the initial characterization of the solution, where λ

was a function of the parameters defining the location of the EV set. Thus, to identify the substitution effect under risk, we assume that an investor with constant absolute risk aversion has an isoexpected utility line that is linear in expected value and variances. This specification is, of course, an approximation when the distributions are not normal. However, the complete generality of the income effect plus the more limiting assumptions imposed on the specification of the substitution effect yield highly reliable analytic results compared to those obtained from the more general expected utility model.[3]

To find the substitution effect, let the isoexpected utility line $y_{CE}y^*_{CE}$ in Fig. 6.3 be the straight line described in Eq. (6.1) with slope $\lambda/2$. Initially, the preferred choice occurs at tangency point C between the isoexpected utility line $y_{CE}y^*_{CE}$ and EV set ACB. Let an increase in expected return on the risky asset, or a decrease in its variance, rotate the EV set from ACB to $AC'B'$. The substitution effect is found by subtracting risk-free income from set $AC'B'$, causing it to shift downward in a parallel fashion until a new tangency at C'' occurs between the isoexpected utility line $y_{CE}y^*_{CE}$ and the shifted EV set DE. Any change in the demand for the risky asset between choices C and C'' is due to the substitution effect.

A more convenient measure of the substitution effect is based on the observed EV set $AC'B'$ rather than the unobserved set DE. Since the variances at C'' and C' on EV sets DE and $AC'B'$, respectively, are the same, and the slopes at choices C'' and C' are equal, the change in demand for risky assets between C and C' is the same as the change in demand between C and C''. Thus, since EV set $AC'B'$ is observed and DE is not, the substitution effect is more conveniently measured as the difference between C and C'.

Finally, we add the income effect to Fig. 6.3. For upward parallel shifts, the equilibrium slope decreases, remains constant, or increases for investors with decreasing, constant, or increasing absolute risk aversion, respectively. Adjusting from equilibrium point C' on EV set $AC'B'$, investors with decreasing absolute risk aversion choose a new equilibrium position to the right of C' with a lower slope (i.e., F). Decision makers with increasing absolute risk aversion will choose an equilibrium position to the left of C' (i.e., G) with a higher slope. Those with constant absolute risk aversion choose equilibrium position C' without a slope change, signifying no income effect.

MATHEMATICAL FORMULATION

To this point, the income and substitution effects associated with shifts in EV sets have been analyzed in terms of equilibrium slopes. Extending the adjustment theory to evaluate holdings of individual assets and specific changes in risk-free wealth and probability distributions requires a mathematical description.

The substitution effect is found by expressing the certainty equivalent model as in (6.1), holding λ constant, and observing an asset's change in response to a given change in the parameters of the EV set. Similarly, the income effect