STRAIN LOCALIZATION IN AN ORTHOTROPIC MATERIAL WITH PLASTIC SPIN

HAENGSOO LEE,* SEYOUNG IM,* and S.N. ATLURI†

*KAIST and †Georgia Institute of Technology

Abstract—The formation and development of strain localization via shear banding is examined for an orthotropic material for the purpose of studying the effect of the plastic spin and the anisotropy upon the strain localization. First, the material instability analysis is carried out for an orthotropic elastoplastic material with plastic spin. By way of finite element analysis, full numerical solution is next obtained for plane strain block of an elastoviscoplastic material under tension or compression, and the shear band development is simulated. The results show that the plastic spin may hasten or delay the initiation of shear band development depending upon the deformation or loading type—compression or tension, the initial orientation of the orthotropic axes, and the algebraic sign of the plastic spin.

1. INTRODUCTION

The formation of narrow bands of intense straining is observed on various deformation processes of metals and is a characteristic feature of large inelastic deformations. Often the large localized strain in a shear band precipitates a shear fracture. In other circumstances shear bands do not lead to fracture, but they persist, resulting in a highly inhomogeneous deformation in microscale, altering material properties and the overall macroscopic behavior of the continuum.

Such a localization is described by various physical mechanisms, and extensive theoretical, experimental, and numerical studies have been performed in order to identify the basic mechanisms of shear banding. Within a material instability framework, THOMAS [1961], HILL [1962], RICE [1977], and others elucidated the condition for the onset of localization. Within this framework, the localization of deformation is very sensitive to constitutive features, and the material instability analysis is useful for identifying such constitutive features. Indeed, consideration of some deviations from the von Mises type classical idealization may lead to prediction of realistic strains at the onset of localization: for example, yield surface vertex (RUDNICKI & RICE [1975]; NEEDLEMAN & RICE [1978]; NEMAT-NASSER [1992]), yield surface with a large curvature (MEAR & HUTCHINSON [1985]; TVERGAARD [1987]), deviation from plastic normality, i.e. deviation from an associated flow rule (RUDNICKI & RICE [1975]; RICE & RUDNICKI [1980]; BIGONI & ZACCARIA [1993]) or dilatancy effect induced by the nucleation and growth of microvoids (RUDNICKI & RICE [1975]; NEEDLEMAN & RICE [1978]; LEE [1989]).

In general cases where localization occurs in nonhomogeneous deformation fields, a full boundary value problem solution is needed for determining the shear band development. Such solutions can be found for the rate independent materials or rate dependent materials, for example: TVERGAARD et al. [1981] have carried out a finite element analysis of the plane strain tensile test with specified small initial thickness inhom-
They used the crossed triangular quadrilaterals for finite element meshes because of their ability to reproduce localized deformation modes. In a coupled thermoviscoplastic problem, the numerical study of adiabatic shear banding at high strain rates has been examined, among others, by Lemonds and Needleman [1986], Batra [1987], Wright and Walter [1987], Anand et al. [1988], Batra and Liu [1989], Shawki and Clifton [1989], and Batra and Kim [1990]. The dynamics of shear band development from internal material inhomogeneities under plane strain compressive loading is numerically analyzed by Needleman [1989] for investigating shear band propagation in a von Mises viscoplastic solid. Prevost and Lorent [1990] examined the dynamic strain localization in von Mises solids with strain softening. In their work, viscoplasticity has been introduced as a procedure to regularize the elastic-plastic solids. Zbib and Jibran [1992] recently examined the three dimensional aspects of shear banding under dynamic loading conditions using the explicit large-scale finite element code DYNA 3D.

Among a number of investigations dealing with the phenomenon of shear localization found in the literature, there has been only a little effort to examine the effect of anisotropy upon the onset of instability and the development of shear band for the anisotropic materials. In the large deformation analysis of materials with a substructure, the plastic spin concept was introduced for modeling the reorientation of this substructure, and the effect of plastic spin upon the simple shear as well as upon nonhomogeneous deformations has been examined by extensive studies (Lorent [1983]; Dafalias [1983, 1984, 1985]; Im & Atluri [1987]). The plastic spin was useful in determining the evolution of purely orientational variables for the orthotropic materials (Dafalias [1984, 1990a, 1990b]; Lorent & Dafalias [1992]). Moreover, in the recent work of Tvergaard and Van Der Giessen [1991] and Zhu et al. [1992], it has been shown that the back stress and the plastic spin have a significant influence upon the onset of instability. For a porous ductile material, Tvergaard and Van Der Giessen [1991] found that the localization behavior predicted by a kinematic hardening theory is sensitive to the corotational stress rates used for the finite strain generalization of the material model and a significant delay of final void-sheet fracture is predicted when plastic spin is neglected. Zhu et al. [1992] examined the stability of homogeneous deformation of bi-axial stretching using a linear stability analysis, and they found the softening effect of the plastic spin causing instability of positive strain hardening for the plane stress deformation with von Mises flow rule.

In this article, we are concerned with material instability analysis of an orthotropic elastic-plastic materials and finite element simulation for shear strain localization in an orthotropic elastoviscoplastic materials under dynamic plane strain uniaxial loading. We employ Hill's orthotropic yield criterion in material instability analysis of a rate-independent orthotropic material, and rely upon the plastic spin concept for representing evolution of orientation for the material orthotropic axes (Lorent & Dafalias [1992]). For the rate-dependent material modeling, the overstress type function is introduced based upon Hill's orthotropic potential.

The material instability analysis under a given plane strain uniaxial loading results in the critical hardening modulus and the corresponding shear band orientation for the orthotropic rate-independent materials, which emerges as the limit of the rate-dependent solid for which the dynamic shear band development is subsequently examined via finite element analysis. Moreover, the effect of plastic spin upon the onset of shear band in the orthotropic elastoplastic material is found via this instability analysis. The result of
this instability analysis indicates that the initial orientation of orthotropic axes has a significant effect upon the shear band orientation.

For finite element simulation of shear band development in the orthotropic rate-dependent material, we employ the four noded quadrilateral elements. The numerical results show that plastic spin may hasten or delay the initiation of shear band development depending upon the loading type—compression or tension, the initial orientation of the orthotropic axes, and the algebraic sign of the plastic spin constant.

II. ELASTOVISCOPLASTIC CONSTITUTIVE MODEL

Decompositions of rate of deformation tensor $D$ and material spin tensor $W$ into elastic and plastic (inelastic) parts may be written as

$$
D = D^e + D^p, \quad W = \omega + W^p,
$$

where $D^e$ and $D^p$ are elastic and plastic parts of $D$, respectively, $\omega$ is rigid body spin of substructure, and $W^p$ is the plastic spin. $W^p$ expresses the rate of rotation of the continuum with respect to its substructure in the process of inelastic deformations. The corotational rate $\dot{\alpha}^0$ of a material state variable tensor $\alpha$ with respect to $\omega$ is defined by

$$
\dot{\alpha}^0 = \dot{\alpha} - \omega \alpha + \alpha \omega = \dot{\alpha}' - \alpha W^p + W^p \alpha,
$$

where $\dot{\alpha}$ is the time rate of $\alpha$, and $\dot{\alpha}'$ is the Jaumann rate.

With the state variables defined as the Cauchy stress $\sigma$ and a set of structure variables consisting of a second order tensor $\alpha$ and a scalar $k$, the rate forms of the constitutive relations for large deformation elastoviscoplasticity can be expressed as (see Dafalias [1990a])

$$
D^e = \mathbf{L}^{-1} : \dot{\sigma}^0, \quad D^p = \langle z \rangle^m \mathbf{N}^p, \quad W^p = \langle z \rangle^m \mathbf{\Omega}^p,
$$

$$
\dot{\alpha}^0 = \langle y_i \rangle^m \dot{\alpha}_i, \quad \dot{k}_i = \langle y_i \rangle^m \dot{k}_i,
$$

where summation is implied over $i$, $\mathbf{L}$ is elasticity tensor, $\langle \rangle$ is the Macauley bracket, $z$ and $y_i$ are overstress functions, and $\mathbf{N}^p$, $\mathbf{\Omega}^p$, $\dot{\alpha}_i$, and $\dot{k}_i$ define the direction of $D^p$, $W^p$, $\dot{\alpha}^0$, and $\dot{k}^0$. Invariance requirements under any superposed rigid-body rotation render the $z$, $y_i$, $\mathbf{N}^p$, $\mathbf{\Omega}^p$, $\dot{\alpha}_i$, and $\dot{k}_i$ isotropic functions of $\sigma$, $\alpha$, and $k$. If $\alpha$ is a purely orientational variable, then $\dot{\alpha}^0 = 0$ for any value of $\langle y_i \rangle^m$, which implies $\dot{\alpha}_i = 0$.

Consider an orthotropic material with orthonormal basis $n_i$ $(i = 1, 2, 3)$ along the axes of orthotropy. Then the purely orientational structure variables for orthotropic materials are written as (Liu [1982])

$$
\alpha_i = n_i \otimes n_i \quad \text{(no sum over } i)\)
$$

and its corotational rate associated with $\omega$ becomes always zero since $\alpha_i$ specify only the orientation of the orthotropic axes.

For Hill's orthotropic material, a yield criterion for an orthotropic plastically incompressible solid is given by

$$
f = J - k = 0
$$
with

\[ J(\sigma, \alpha_1, \alpha_2) = [A(\dot{\sigma}_{11} - \dot{\sigma}_{22})^2 + B(\dot{\sigma}_{22} - \dot{\sigma}_{33})^2 + C(\dot{\sigma}_{33} - \dot{\sigma}_{11})^2 \]
\[ + 2D\dot{\sigma}_{23}^2 + 2E\dot{\sigma}_{31}^2 + 2F\dot{\sigma}_{12}^2 \]^{1/2},

\[ k = \sqrt{2}(k_0 + R) \]

where \( A, B, C, D, E, \) and \( F \) are material constants and for the isotropic case \( A = B = C = 1, D = E = F = 3, \) and \( J = \sqrt{2}(\frac{1}{2} \sigma_{ij} \dot{\sigma}_{ij})^{1/2}; \) moreover \( k \) is the reference yield stress such that \( k = \sqrt{A + C} \dot{\sigma}_{11}^Y = \sqrt{A + B} \dot{\sigma}_{22}^Y = \sqrt{B + C} \dot{\sigma}_{33}^Y \) with \( \dot{\sigma}_{11}^Y, \dot{\sigma}_{22}^Y, \dot{\sigma}_{33}^Y \) being the uniaxial yield stress along each of the orthotropic axes. Note that \( k_0 \) is the initial yield strength and \( R \) represents the isotropic hardening. A superposed \(^*\) denotes tensor components in reference to \( n_i \) and \( \dot{\sigma}_{ij} \) is represented in terms of \( \sigma_{ij} \) and purely orientational variable \( n_i \) (or \( \alpha_i \)) as follows:

\[ \dot{\sigma}_{11} = n_1 \cdot \sigma \cdot n_1, \quad \dot{\sigma}_{22} = n_2 \cdot \sigma \cdot n_2, \quad \dot{\sigma}_{12} = n_1 \cdot \sigma \cdot n_2, \text{ etc.} \] (6)

Based upon the work of Dafalias [1990a], we employ an elastoviscoplastic constitutive model with plastic spin as follows:

\[ \dot{\sigma} = \mathbf{L} : \mathbf{D} - (z)\mathbf{N}^p - \mathbf{W}^p \mathbf{\sigma} + \mathbf{\sigma} \mathbf{W}, \]

\[ \dot{\alpha}_i = \alpha_i \mathbf{W}^p - \mathbf{W}^p \alpha_i, \]

\[ n_i^f = -\mathbf{W}^p n_i \]

with

\[ z = \left( \frac{J - \sqrt{2}(k_0 + R)}{V} \right). \]

Here a superposed "\( J \)" implies the classical Jaumann rate, and \( R \) depends upon the equivalent plastic strain rate \( \dot{\varepsilon}^p = (\frac{1}{2} \dot{D}^p : \dot{D}^p)^{1/2} \) and the equivalent plastic strain \( \varepsilon^p = \int \dot{\varepsilon}^p dt. \) We do not employ the work hardening model wherein the hardening is a function of plastic work, but we rely upon a strain hardening model wherein the hardening just depends upon the accumulated equivalent plastic strain \( \varepsilon^p \) (and the equivalent plastic strain rate \( \dot{\varepsilon}^p \)); hence \( \dot{\varepsilon}^p \) does not have to be equivalent to the plastic work rate, but it is simply a measure of the rate of accumulated plastic deformation in material at hand. The remaining variables, \( V \) and \( n, \) are material parameters. The \( \mathbf{N}^p \) is defined by \( \partial J/\partial \mathbf{\sigma} \) (associative flow rule).

Based on the representation theorem, Dafalias [1984, 1985, 1990a] and Lorent and Dafalias [1992] proposed the expression for the evolution of plastic spin

\[ \mathbf{D}^p = \eta_1 (\alpha_1, \mathbf{z} - \sigma \alpha_1) + \eta_2 (\alpha_2, \mathbf{z} - \sigma \alpha_2) + \eta_3 (\alpha_1, \sigma \alpha_2 - \alpha_2, \sigma \alpha_1), \] (8)

where \( \eta_i \) are scalar-valued functions of state variables and are considered as material constants measuring the plastic spin. Notice that \( \alpha_3 \) is not included in (8) since \( \alpha_1 + \)
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\[ \alpha_2 + \alpha_3 \] becomes the identity tensor. From eqns (3), (8), and \( \mathbf{N}^\rho = \partial J/\partial \sigma \), the plastic spin components in reference to \( \mathbf{n}_i \) can be expressed as follows:

\[
\dot{W}^P_{12} = \hat{\eta}_3 \dot{D}^P_{12}, \quad \dot{W}^P_{23} = \hat{\eta}_1 \dot{D}^P_{23}, \quad \dot{W}^P_{13} = \hat{\eta}_2 \dot{D}^P_{13}
\]

where

\[
\hat{\eta}_3 = \frac{J}{2F} (\eta_1 - \eta_2 + \eta_3), \quad \hat{\eta}_2 = \frac{J}{2F} \eta_1, \quad \hat{\eta}_1 = \frac{J}{2D} \eta_2.
\]

Throughout the analysis, we assume that the axis \( \hat{x}_3 \) is identical to the axis \( x_3 \) of a Cartesian coordinate system \( x_i \) fixed in space, while the axes \( \hat{x}_1 \) and \( \hat{x}_2 \) form an angle \( \theta \) with \( x_1 \) and \( x_2 \) (Fig. 1).

III. BIFURCATION ANALYSIS FOR THE RATE INDEPENDENT SOLIDS

For quasistatically and isothermally deforming rate-independent solids, shear band instabilities can be characterized in terms of bifurcation of the equilibrium path, and the bifurcation analysis is useful for determining the critical strain or stress and the corresponding direction of bands at the onset of instability. For the rate-dependent solids, however, the only solution of bifurcation analysis is the trivial one and localization bifurcation does not occur because there is no loss of ellipticity in quasistatic problems, and wave speeds remain real as long as stress levels remain small compared to elastic stiffness.

In this section, the bifurcation analysis for the orthotropic rate-independent solids which emerge as the limit of rate-dependent solids is used to examine the effect of plastic spin and initial orthotropic orientation upon the onset of shear localization.

Constitutive relations for plastic solids are generally expressed as a relation between some objective rate of Cauchy or Kirchhoff stress and the rate of deformation tensor. The plastic flow rule for orthotropic elastoplastic solids can be written as

\[
\mathbf{D}^p_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}},
\]

Fig. 1. Schematic illustration of coordinate system in the orthotropic material.
where $\lambda$ is a loading index and "$f$" is a orthotropic yield criterion in (5). A loading index $\lambda$ is obtained by employing the following consistency condition and the definition of equivalent plastic strain rate

$$\frac{df}{dt} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \alpha_{1ij}} \dot{\alpha}_{1ij} + \frac{\partial f}{\partial \alpha_{2ij}} \dot{\alpha}_{2ij} - \frac{\partial k}{\partial \dot{\varepsilon}^p} \dot{\varepsilon}^p$$

(11)

$$= \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \frac{\partial k}{\partial \dot{\varepsilon}^p} \dot{\varepsilon}^p = 0$$

$$\dot{\varepsilon}^p = \left( \frac{2}{3} D_{ij}^p D_{ij}^p \right)^{1/2} = \lambda \left( \frac{2}{3} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} \right)^{1/2}.$$  (12)

Use is made of the following relation

$$\frac{\partial f}{\partial \alpha} : \dot{\alpha} + \frac{\partial f}{\partial \alpha_1} : \dot{\alpha}_1 + \frac{\partial f}{\partial \alpha_2} : \dot{\alpha}_2 = \frac{\partial f}{\partial \varepsilon} : \dot{\varepsilon}^0.$$  

This is apparent from the physics that the yield function is affected only by the change in stress tensor observed by an observer attached to the orthotropic axes $\hat{x}_i$, and it is a straightforward matter to confirm this via algebra. From eqns (10-12) the plastic flow rule can be rewritten as

$$D^p_{ij} = \frac{N_{rs} \dot{\alpha}^0_{rs}}{h^{1/2} N_{mn} N_{mn}} N_{ij},$$

(13)

where

$$N_{ij} = \frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial J}{\partial \sigma_{ij}} = \frac{\partial J}{\partial \dot{\sigma}_{rs}} \frac{\partial \dot{\alpha}_{rs}}{\partial \sigma_{ij}}, \quad h = \frac{\partial k}{\partial \dot{\varepsilon}^p}.$$  

In eqn (13), "$h$" denotes the strain hardening modulus. Then the rate of deformation tensor $D$ is written as

$$D_{ij} = D_0^p + D_0^p = L_{ijkl}^0 \dot{\phi}^0_{kl} + \frac{N_{rs} \dot{\alpha}^0_{rs}}{h N_{mn} N_{mn}} N_{ij}.$$  (14)

The eqn (14) may be inverted easily, then the constitutive relations for orthotropic elasto-plastic solids are as follows:

$$\dot{\sigma}^0 = L^{tan} : D$$

$$\dot{\sigma}^f = L^{tan} : D - W^p \sigma + \sigma W^p,$$

with

$$L_{ijkl}^{tan} = L_{ijkl} - \frac{L_{ijrs} N_{rs} N_{pq} L_{pqkl}}{h N_{pq} N_{pq} + N_{pq} L_{pqrs} N_{rs}}.$$
The component of plastic spin tensor can be expressed in terms of the rate of deformation tensor and the direction of initial orthotropic axes (eqn 9). For the deformation in the $x_1$-$x_2$ plane (Fig. 1), the plastic spin components are as follows:

$$W_{12}^p = W_{12}^p = \hat{\eta}_3 \delta_{12} = \hat{\eta}_3 \left[ \frac{D_{12}^p - D_{11}^p}{2} \sin 2\theta + D_{12}^p \cos 2\theta \right],$$

(16)

$$W_{23}^p = W_{31}^p = 0,$$

where $W_{12}^p = W_{23}^p$ because the $\hat{x}_3$ and $x_3$ are identical, and $\cos \theta$, $\sin \theta$ are the components of $\mathbf{n}_i$ in reference to $x_{\alpha}$ ($\alpha = 1, 2$).

Within a thin band with the unit normal vector $\mathbf{m}$ in the current configuration (Fig. 2), the compatibility and equilibrium must be satisfied across the band interfaces (Rudnicki & Rice [1975]).

$$\Delta \left( \frac{\partial v_j}{\partial x_i} \right) = g_i m_j$$

(17)

$$\mathbf{m} \cdot \Delta \sigma = 0,$$

(18)

where $\Delta$ denotes the difference between the field inside the band and the field outside the band and $\mathbf{g}$ is the vector representing discontinuity across the band. The differences of the spin tensor and the rate of deformation tensor then can be written

$$\Delta D_{ij} = \frac{1}{2} (g_i m_j + m_i g_j), \quad \Delta W_{ij} = \frac{1}{2} (g_i m_j - m_i g_j).$$

(19)

The difference of plastic spin tensor are given as

$$\Delta W_{ij}^p = \hat{\eta}_3 (\delta_{i1} \delta_{j2} - \delta_{i2} \delta_{j1}) \left( \frac{\Delta D_{12}^p - \Delta D_{11}^p}{2} \sin 2\theta + \Delta D_{12}^p \cos 2\theta \right),$$

(20)

Fig. 2. A material element with a shear band.
where

$$\Delta D_{ij}^p = \frac{N_{kl} L_{kij}^{tan} D_{pq}}{h \sqrt{\frac{2}{3} N_{mn} N_{mn}}} N_{ij}$$

under the assumption of continuous bifurcation (see Rice & Rudnicki [1980]). Rice and Rudnicki [1980] showed that localization first occurs as a continuous bifurcation, in which the material inside and outside the localized zone is assumed to continue to be under elastic-plastic loading at the inception of localization. Then the material at hand is conceived of being replaced by the so called “linear comparison solid,” (Hill [1958]; Raniecki & Bruhns [1981]) so that the tangent modulus $L_{ijkl}^{tan}$ is not dependent upon the strain rate, that is, loading or unloading branch. Hence, the values $L_{ijkl}^{tan}$ remain the same inside and outside the band at the point of bifurcation, and the following difference can be found.

$$\Delta \delta_{ij}^I = L_{ijkl}^{tan} \Delta D_{kl} - \Delta W_{ijk}^p \sigma_{kj} + \sigma_{ik} \Delta W_{kj}^p$$

(21)

$$\Delta \delta_{ij} = L_{ijkl}^{tan} \Delta D_{kl} - \Delta W_{ijk}^p \sigma_{kj} + \sigma_{ik} \Delta W_{kj}^p + \Delta W_{ik} \sigma_{kj} - \sigma_{ik} \Delta W_{kj}.$$

(22)

From the eqns (18) through (22), we can obtain the following characteristic equation.

$$[m_i L_{ijkl}^{tan} m_j + S_{jk} + R_{jk}]g_k = 0,$$

(23)

where

$$S_{jk} = \hat{\eta}_3 m_i (\sigma_{i1} \delta_{j1} - \sigma_{i2} \delta_{j1} + \delta_{i2} \sigma_{1j} - \delta_{i1} \sigma_{2j})$$

$$\times \left\{ \frac{N_{22} - N_{11}}{2} \sin 2\theta + N_{12} \cos 2\theta \right\} \frac{N_{pq} L_{pqkl}^{tan} m_i}{h \sqrt{\frac{2}{3} N_{mn} N_{mn}}}$$

$$R_{jk} = \frac{1}{2} (m_i \sigma_{ij} m_k - m_j \sigma_{ik} m_i - \sigma_{kj} + m_p \sigma_{pq} m_q \delta_{jk}),$$

where $S_{jk}$ and $R_{jk}$ are terms which arise due to the plastic spin and the difference between $\hat{\sigma}$ and $\hat{\sigma}'$, respectively. For the existence of a nontrivial solution, the determinant of coefficient matrix in eqn (23) must vanish, and the condition for localization is written as

$$\det [m_i L_{ijkl}^{tan} m_j + S_{jk} + R_{jk}] = 0.$$  

(24)

For the initially specified orthotropic axes and the plastic spin coefficient $\eta$, the solution to eqn (24) gives the critical strain hardening modulus $h$ and the corresponding shear band orientation $m$. As the deformation progresses, however, the orthotropic axes are rotated according to the evolution equation of $\eta_i$ (7) for the nonzero value of $\eta_i$. Determination of the orthotropic axis orientation needs the complete solution for an elastoplastic boundary value problem under an actual hardening behavior. However, we here limit ourselves to seeking the shear band orientation simply by carrying out material instability analysis for a fixed orientation of the orthotropic axes without taking into account the rotation of orthotropic axes associated with the plastic spin.
IV. SHEAR BAND DEVELOPMENT IN ELASTOVISCOPLASTIC MATERIALS

In the foregoing section, the bifurcation analysis has been discussed in relation to the material instability of a rate-independent orthotropic material. From the viewpoint of the acceleration waves, wherein the velocity is continuous across the singular surface of wave front, but velocity gradient and acceleration are discontinuous there, the bifurcation eqn (23) is nothing but the condition for the stationary acceleration wave or the acceleration wave of vanishing wave speed (HILL [1962]; RICE [1977]). Indeed the dynamical equation for the singular surface may be written as (RICE [1977])

\[ m_i \Delta \dot{\sigma}_{ij} = [m_i L^\tan_{ijkl} m_l + S_{jk} + R_{jk}] g_k = \rho c^2 g_j, \]  

(25)

where \( c \) is the wave speed in a given material and \( \rho \) is the density. Note that the vanishing wave speed \( c = 0 \) will yield the bifurcation eqn (23). For vanishing wave speed, disturbance cannot propagate, and the hyperbolicity of the dynamical equation is lost, so that the initial value problem is not well-posed any longer. For the rate-dependent material discussed in Sec. II, it follows from eqn (7) that the discontinuity of stress rate \( \Delta \dot{\sigma} \) is the same as eqn (22) except that the elastoplastic modulus \( L^\tan_{ijkl} \) is replaced by the elastic modulus \( L_{ijkl} \) because the viscoplastic terms are continuous inside and outside the band. Accordingly, the dynamical equation for a singular surface in an elastoviscoplastic material may be written as

\[ [m_i L_{ijkl} m_l + S_{jk} + R_{jk}] g_k = \rho c^2 g_j. \]  

(26)

Because the magnitude of \( S_{jk} \) and \( R_{jk} \) are proportional to stress, these two terms will be negligible compared with the elastic acoustic tensor term \( m_i L_{ijkl} m_l \) as long as the magnitude of stress is much smaller than the elastic modulus. For the acoustic tensor \( m_i L_{ijkl} m_l \), there are no eigenvalues corresponding to vanishing elastic wave speed (\( c > 0 \) always), and thus there exists no stationary acceleration wave, i.e. no eigenvectors representing discontinuity (NEEDLEMAN & TVERGAARD [1992]) for an elastoviscoplastic material.

For viscoplastic materials representing softening behavior, however, the shear band of finite thickness will develop from the growth of a small disturbance induced by inhomogeneities in some material properties like flow strength or porosity, which exist in the form of initial imperfection, or induced by wave reflection (NEEDLEMAN [1989]; PREVOST & LORET [1990]). In the limit of such rate-dependent material, the rate-independent material appears and the corresponding shear band will ultimately reduce to a stationary discontinuity of zero width. As opposed to the ill-posedness for an initial value problem for the rate-independent material in the presence of vanishing wave speed or softening behavior, the dynamical equation for the rate-dependent material retains hyperbolicity and the well-posedness even under the softening material behavior. We can therefore integrate discrete finite element equation for an initial/boundary value problem concerning the shear band development without any pathological mesh dependency resulting from spurious length scale effects (PREVOST & LORET [1990]).

In passing, we remark that one could perform a linear stability analysis for such rate-dependent problems, instead of the bifurcation analysis, to obtain information about the onset of instability and orientation of shear band (ZHU et al. [1992]). However, we here rely upon numerical analysis via finite element method.
In this section, we consider numerical simulation of plane strain shear band development in an orthotropic elastoviscoplastic material via finite element method. We are concerned with the effect of orthotropy and the plastic spin upon the shear band development. We take a rectangular block under plane strain uniaxial tension or compression prescribed in terms of uniform end velocity as shown in Fig. 3. The figure states additional conditions relevant to the initial/boundary value problem at hand: among others, the translation in the $x_1$ direction is constrained to be zero at the center, $x_1 = x_2 = 0$ to eliminate the rigid body motion. The orthotropic axes are assumed to take the same orientation as stated in Sec. II, i.e. as shown in Fig. 1.

Following Needlemann [1989], we prescribe an initial inhomogeneity in the form of flow stress.

$$k_0(x_1,x_2) = \bar{k}_0[1 - \zeta \exp\{-[(x_1 - x_{10})^2 + (x_2 - x_{20})^2]/r_0^2\}].$$

(27)

To introduce thermal softening induced by adiabatic heating, the material is assumed to exhibit strain softening in terms of the static flow strength $k$ or $R$ such that

$$R = \left(\frac{H_\alpha}{C_r + C_s/\dot{\varepsilon}_p}\right)\left[1 - \exp\left\{-\left(\frac{C_r + C_s}{\dot{\varepsilon}_p}\right)(\ddot{\varepsilon}_p - (\ddot{\varepsilon}_p - \ddot{\varepsilon}_p^0))\right\}\right].$$

(28)

Material properties and the values of parameters in stress–strain relation (7), (28), and inhomogeneity (27) are given as

$$V = 100 \text{ Mpa}, \quad H_\alpha = -1500 \text{ Mpa}, \quad C_r = 0.5, \quad C_s = -0.5 \quad \ddot{\varepsilon}_p^0 = 0.05,$$

$$n = 10, \quad \bar{k}_0 = 460 \text{ Mpa}, \quad \zeta = 0.02, \quad r_0 = 0.1 \text{ mm}, \quad x_{10} = x_{20} = 0.$$

These values of parameters do not correspond to any specific material, but we have prescribed these so that the corresponding stress–strain relation simulates softening, which may originate from adiabatic heating or other sources. The reference flow stress-equivalent plastic strain curve for these values of parameters is shown in Fig. 4 for various values of $\dot{\varepsilon}_p$. 

![Fig. 3. The plane strain specimen under the dynamic loading.](image-url)
Based on the results of Stickels and Mould's [1970] experiments, we used the following material properties for carbon steel.

\[ E_3 = 210.85 \text{ GPa}, \quad \nu_{12} = 0.3, \quad E_2 = 213.95 \text{ GPa}, \quad \nu_{23} = 0.25, \]
\[ E_3 = 220.64 \text{ GPa}, \quad \nu_{31} = 0.3333, \quad G_{12} = 84.75 \text{ GPa}, \quad G_{23} = 83.70 \text{ GPa}, \]
\[ G_{31} = 81.15 \text{ GPa}, \quad \rho = 7800 \text{ kg/m}^3, \]
\[ A = 0.551, \quad B = 0.7247, \quad C = 0.333, \quad F = 0.633, \]

where \( A, B, C, \) and \( F \) are calculated from the plastic strain ratio \( R \) in their experimental results. The resulting elastic wave speeds, which are obtained from eqn (26), are as follows:

\[ c_1 = 6021 \text{ m/s}, \quad c_2 = 3296 \text{ m/s}, \quad c_3 = 3276 \text{ m/s}. \]

The finite element analysis is based on an updated Lagrangian formulation in which the material state variables are evaluated in the most recently updated configuration. Introduce a rectangular Cartesian coordinate \( \mathbf{x} \) and \( \mathbf{X} \) which are the positions of a material point in the current and reference configuration, respectively. Then the displacement vector \( \mathbf{u} \) is defined by

\[ \mathbf{u} = \mathbf{x} - \mathbf{X}. \] (29)

The equation of momentum balance is expressed in terms of the Cauchy stress \( \sigma_{ij} \) as

\[ \sigma_{ij,j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \] (30)
where $\rho$ is the mass density of the body and $\mathbf{b}$ is the body force vector. In the absence of body forces, the dynamic principle of virtual work can be written as

$$\int_{V'} \sigma_{ij} \delta u_{i,j} \, dV = \int_{S'^a} \hat{T}_i \delta u_i \, dS - \int_{V'} \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i \, dV,$$

where $V'$ and $S'^a$ are the volume and traction prescribed surface, respectively, in the current configuration, and $\hat{T}$ is the prescribed traction vector. After the finite element discretization of the field variables, we can obtain the equations of motion as follows:

$$\mathbf{M} \frac{\partial^2 \mathbf{d}}{\partial t^2} = \mathbf{P} - \mathbf{F},$$

(32)

where $\mathbf{M}$, $\mathbf{P}$, and $\mathbf{F}$ are the mass matrix, the external load vector, and the stress divergence vector, respectively, and $\mathbf{d}$ is a nodal displacement vector. We adopt the central difference scheme for integrating the equations of motion (32) as follows (HUGHES [1987]):

$$\mathbf{d}_{n+1} = \mathbf{d}_n + \Delta t \frac{\partial \mathbf{d}_n}{\partial t} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 \mathbf{d}_n}{\partial t^2},$$

$$\frac{\partial^2 \mathbf{d}_{n+1}}{\partial t^2} = \mathbf{M}^{-1}(\mathbf{P}_{n+1} - \mathbf{F}_{n+1})$$

(33)

where the subscripts $n$ and $n+1$ refer to the configurations at time $t_n$ and $t_{n+1}$, respectively. For the stable numerical solution, we restrict the time step size $\Delta t$ by the following condition.

$$\Delta t \leq \Delta t_{\text{crit}} = \frac{1}{c} \min[h_1, h_2], \quad c = \max[c_i(i = 1,2,3)],$$

where $h_1$ and $h_2$ are the side lengths of rectangular element and $c_i (i = 1,2,3)$ are the elastic wave speeds of the orthotropic materials. A lumped mass matrix is used in eqn (32), because this has been found preferable for explicit time integration procedures. At each time increment, the stress state is updated using the Euler scheme (see Appendix).

The “crossed” triangular quadrilateral has the ability to easily reproduce the localization modes (TVERGAARD et al. [1981]) but it has the preferred localization mode which is produced in the direction of element diagonal and is dependent upon the aspect ratio of an element. In this work, the localization orientation in orthotropic materials depends upon the orientation of orthotropic axes, and this orientation may be obtained from the instability analysis for the underlying rate-independent material which emerges as the limit of elastoviscoplastic material under consideration. Then the crossed triangular quadrilateral mesh must be designed such that the diagonal may be in the direction of the shear band orientation predicted from the material instability analysis. This may be much involved in the presence of plastic spin, which is closely related to the rotation of
the orthotropic axes. To avoid such extra endeavor, however, we employ the four noded quadrilateral elements for the present finite element analysis. We rely upon the selective reduced integration scheme such that the value at the center point of an element is used for computing the dilatational contribution while the values of the two by two Gaussian points is used for computing the deviatoric parts via the so-called strain projection method (HUGHES [1987]).

V. NUMERICAL RESULTS AND DISCUSSION

V.1. Results of bifurcation analysis for the rate-independent material

In Section III, we obtained the characteristic eqn (24) for the condition of a localization bifurcation. In this section, the solution of eqn (24) is numerically examined in order to obtain the critical hardening modulus and the plane of localization at the onset of localization. Because the hardening modulus “h” is a decreasing function of strain in general, we seek the orientation m for which the value of h is maximum so that the condition for the onset of the shear band is first met. Rather than searching for critical stress level or critical hardening modulus of a given material, i.e. for a given hardening or softening behavior, we leave the detailed hardening behavior \( k = k(\varepsilon_p) \) not specified and we seek for the critical hardening modulus \( h_{cr}/E_2 \) for a given stress state. The comparison of \( h_{cr}/E_2 \) corresponding to the various values of plastic spin parameters enables us to investigate the influence of the plastic spin upon bifurcation instability, i.e. hastening or delay of shear band formation. In the calculations, the loading type is specified as plane strain uniaxial state, and all material parameters are kept fixed except the angle of initial orthotropic axes and that we introduce \( \eta \) such that \( \eta = (\varepsilon_p/2F)(\eta_1 - \eta_2 + \eta_3) \) for convenience, which is employed for representing the plastic spin instead of \( \dot{\gamma}_3 \).

For the case of plane strain uniaxial tension with \( \sigma_{22}/E_2 = 0.00215 \) and \( \sigma_{12}, \sigma_{11} \) zero, Fig. 5(a) and 5(b) show the critical hardening modulus \( h_{cr}/E_2 \) and the corresponding angle of shear band \( \varphi_0 \) for the various angles of orthotropic axes \( \theta \).

As shown in these figures, the plastic spin and the initial orientation of orthotropic axes have a significant influence upon the onset of shear band and the corresponding band orientation for the present constitutive model at hand. Particularly the figures show that the plastic spin hastens or delays the onset of shear band depending upon the initial orientation of orthotropic axes; in the range of \( 0^\circ < \theta < 45^\circ \), the hastening occurs for the negative value of \( \eta = -1 \) while delay occurs for the positive value of \( \eta = 1 \) in the same range of \( \theta \). As might be expected, however, there is no influence of plastic spin parameter \( \eta \) near \( \theta = 0^\circ \) and \( 90^\circ \) because \( \delta_{12} \) and accordingly \( \dot{\gamma}_3 \) disappear from eqn (8) for these values of \( \theta \) under plane strain uniaxial loading. Moreover, it is noticed that for \( \theta = 45^\circ \), which is corresponding to a biaxially symmetric state with shear stress, there is no hastening or delay due to the plastic spin.

Except for the cases of \( \theta = 0^\circ, 45^\circ, 90^\circ \), the critical hardening modulus \( h \) is positive in the absence of plastic spin, which implies the onset of shear band may occur on the positive hardening range. The most favorable orientation of \( \dot{x}_i \) axes for the onset of shear band, i.e. the orientation that yields the maximum \( h_{cr} \) is near \( \theta = 65^\circ \) in the absence of plastic spin. However, the presence of plastic spin (\( \eta = 1 \) or \( -1 \)) changes such a trend substantially.

Fig. 5(b) shows that the shear band orientation is strongly dependent upon the initial orientation of orthotropic axes due to anisotropy. It is noted that there are two band
orientations for $\theta = 0^\circ$ and $90^\circ$ due to the symmetry of geometry, loading, and material properties with respect to the $x_1-x_3$ and the $x_2-x_3$ plane. There appear two band orientations $\varphi_0 = 45^\circ$ and $\varphi_0 = 135^\circ$ also for $\theta = 45^\circ$, which is corresponding to a biaxially symmetric stress state with shear in reference to $\hat{x}_i$. For the other values of $\theta$, there exist no symmetry, and the shear band occurs in a nonsymmetric manner. There was found only a small influence of plastic spin upon the shear band orientation when the effect of orthotropic axis rotation, which is closely related to the plastic spin, is neglected as in this material instability analysis. As will be seen in the next section, however, the presence of plastic spin may substantially change the orientation of orthotropic axes, and accordingly have a significant effect upon the shear band orientation ultimately.

If the yield criterion (5) degenerates to the isotropic case ($A = B = C = 1$, $D = E = F = 3$), then $h_{cr}/E_2$ becomes zero and $\varphi_0 = 45^\circ$, $135^\circ$ regardless of $\theta$ values. However, $h_{cr}/E_2$, and $\varphi_0$ are in general dependent upon the combination of $A-F$ in the orthotropic yield criterion, except for the case of plane strain uniaxial stress state in reference to $\hat{x}_i$ ($\theta = 0^\circ$, $\theta = 90^\circ$) and biaxially symmetric stress state with shear in reference to $\hat{x}_i$ ($\theta = 45^\circ$) wherein $h_{cr}/E_2$ becomes zero.

Fig. 6(a) and (b) shows the results for the case of plane strain uniaxial compression. From Fig. 6(a), it is seen that the hastening or the delaying effect of the plastic spin upon
the onset of shear band is completely reversed in this case compared with the case of plane strain tension. Moreover, as shown in Fig. 6(b), the orientation of shear band shows quite a different orientation in comparison to the plane strain tension except for $\theta = 0^\circ$ and $90^\circ$. This is due to the fact that the stress state for the compression in reference in $\dot{x}_i$ is different from the stress state for the tension.

Fig. 7 shows the effect of stress magnitude on the critical hardening modulus $h_{cr}/E_2$ and the corresponding angle of shear band $\varphi_0$ for the special case of uniaxial tension with $\theta = 22.5^\circ$, $\eta = 0$. Their changes depending upon the magnitude of stress can be neglected for the realistic stress level in the onset of localization.

V.2. Finite element results for the elastoviscoplastic material

In order to determine the postlocalization response the numerical solution of full initial/boundary value problem for the specimen in Fig. 3 is obtained for the elastoviscoplastic solid, discussed in Sec. II and IV, by finite element analysis. A uniform mesh involving $60 \times 120$ four noded quadrilaterals are employed to model the specimen. The block size $h_0 = 2H_0$ is taken to be 2 mm.
We first consider plane strain uniaxial tension with the prescribed dynamic loading \( V_0 = 3 \text{ m/s} \) and rise time \( t_1 = 0.1 \times 10^{-6} \) (see Fig. 3). The reason for such low velocity condition is that our prime interest lies in examining the influence of constitutive features such as plastic spin and anisotropy upon the shear banding, so that it is necessary to separate the influence of the constitutive features from the inertia effect, and hence to suppress the inertia effect by taking a slowly increasing velocity profile with low velocity. However, a slowly increasing velocity profile requires too many time steps to advance the explicit solution scheme to the post localization range. The velocity condition shown in Fig. 3 enables us to obtain the solution up to the range of postlocalization without any significant inertia effect.

To confirm the convergence of the numerical solution versus mesh refinement, we obtain the equivalent plastic strain distribution depending upon the three mesh refinements—40 × 80, 60 × 120, and 80 × 160, as shown in Fig. 8. These results are calculated for uniaxial tension with \( \theta = 0^\circ \), \( \eta = 0 \) at \( U/h_0 = 0.012 \) where \( U = \int_0^t V(t) \, dt \) is the end displacement. Here we show the upper right-hand part of the specimen (Fig. 3) for modeling due to symmetry. Note that such symmetry holds only for \( \theta = 0^\circ \) and \( \theta = 90^\circ \), and that a full modeling is needed for other values of \( \theta \). Fig. 8(d) shows the spatial variation of \( \overline{\varepsilon_p} \) along \( x_2 = 0.0005 \text{ mm} \). Figure 8 shows a good convergence of solution according to mesh discretization for the most part of the block except that the
convergency in the strain distribution of high gradient inside the band is a little delayed for rough meshes. That is, as the number of elements increases, it leads to a slightly earlier onset of localization because the finer mesh can resolve larger gradients. However, when a sufficiently fine mesh, such as 60 × 120 elements or 80 × 160 elements, is used, the width of shear band remains approximately the same, and there is no difficulty in capturing the formation and development of shear band in the finite element analysis via 60 × 120 mesh discretization.
The solution convergence depending upon the time step size, which influences upon integration of the equation of motion (32) and upon integration of the rate type constitutive eqn (7), is confirmed via numerical experiment of comparing stress at a given point for decreasing time step size as shown in Fig. 9. For time step size smaller than the stability limit, sufficient convergence in relation to the integration of the equation of motion as well as the rate type constitutive equation is found to be attained. For actual computation, the step size \( \Delta t = 2 \times 10^{-9} \) sec is employed.

For various angles of initial orthotropic axes, the equivalent plastic strain contours are shown for the three spin parameters in Figs. 10 through 13. These results are obtained at \( U/h_0 = 0.012 \). Figs. 10 through 13 show the localization modes according to the initial orientation \( \theta \) of orthotropic axes and the plastic spin parameter \( \eta \). For each value of \( \theta \), it is noted that the spin parameter \( \eta \) has the hastening or delay effect upon the shear band development as well as localization mode. We note that the delay or hastening tendency due to the plastic spin observed in these figures is in agreement with the results predicted from the instability analysis in Fig. 5. For the case of \( \theta = 45^\circ \), the material instability analysis (Fig. 5) gives the same localization modes and band orientations \( \varphi_0 = 45^\circ \) and \( 135^\circ \) for various \( \eta \), for the material instability analysis has not accounted for the rotation of the orthotropic axes closely related to the plastic spin, i.e. it simply considers the effect of plastic spin for a fixed orientation of the orthotropic axes. On the other hand, the finite element analysis for the rate dependent materials in Fig. 12 shows the two shear band orientations \( \varphi_0 = 45^\circ \) and \( 135^\circ \) for \( \eta = 0 \), and these two bands are found to grow at the same rate. For \( \eta = 1 \) and \( \eta = -1 \), the two bands seem to appear initially, but only one of these two continues to grow and form a shear band ultimately, i.e. \( \varphi_0 \equiv 48^\circ \) for \( \eta = -1 \) and \( \varphi_0 \equiv 132^\circ \) for \( \eta = 1 \), respectively (see Fig. 14).

This means that as the deformation progresses the orthotropic axes are rotated, and the direction and amount of the rotation are dependent upon the plastic spin \( \dot{\varphi}_{12}^p = \dot{\eta}_3 \dot{\varphi}_{12} = (J/k_0) \eta \dot{\varphi}_{12} \). The magnitude of \( \dot{\varphi}_{12}^p \) takes the maximum at \( \theta = 45^\circ \) because \( \dot{\varphi}_{12} \) becomes maximum for this angle under the uniaxial stress state. Hence the effect of the orthotropy axis rotation related to the plastic spin is most prominent for this value of \( \theta \). The effect of plastic spin on the evolution of orientation of anisotropic axes has

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**Fig. 9.** The normal stress versus end displacement for various time step sizes at (a) \( x_1 = x_2 = H_0/2 \) (inside the band) (b) \( x_1 = 0, \ x_2 = 3h_0/4 \) (outside the band) under uniaxial tension with \( \theta = 0^\circ, \ \eta = 0. \)**
been examined by Dafalias [1990b] for biaxial stretching and simple shear. For the assumed values $D_{22} = -D_{11} > 0$ and $0^\circ < \theta < 90^\circ$, it is easy to show that the axis $\hat{x}_i$ tends to line up along the $x_2$ axis for $\eta > 0$, and along the $x_1$ axis for $\eta < 0$ as the axial strain increases, that is, the orthotropic axes $\theta$ are rotated clockwise for negative val-

Fig. 10. The development of strain localization at $U/h_0 = 0.012$ for the uniaxial tension, $\theta = 0^\circ$.

Fig. 11. The development of strain localization at $U/h_0 = 0.012$ for the uniaxial tension, $\theta = 22.5^\circ$. 
Fig. 12. The development of strain localization at $U/h_0 = 0.012$ for the uniaxial tension, $\theta = 45^\circ$.

Fig. 13. The development of strain localization at $U/h_0 = 0.012$ for the uniaxial tension, $\theta = 67.5^\circ$. 

values of $\eta$ or counterclockwise for the positive values of $\eta$. Hence the aforementioned results for the band orientation $\varphi_0 \approx 48^\circ$ for $\eta = -1$ and $\varphi_0 \approx 132^\circ$ for $\eta = 1$ at the initial value of $\theta = 45^\circ$ are consistent with the results from the material instability analysis in Fig. 5(b), which shows a jump in shear band orientations between the values
Fig. 14. The course of shear band development for the uniaxial tension, $\theta = 45^\circ$. (a) $\eta = -1$, (b) $\eta = 1$.

of $\theta < 45^\circ$ and the values of $\theta > 45^\circ$. As in the case of $\theta = 45^\circ$, more than one band may appear initially as shown for $\theta = 22.5^\circ$ and $\theta = 67.5^\circ$ in Figs. 11 and 13. As deformation proceeds, however, only the band with consistent with the material instability analysis in Fig. 5(b) continues to grow and develops into shear band as seen from Fig. 15.
For plane strain uniaxial compression with $V_0 = -3\text{m/s}$, $t_1 = 0.1 \times 10^{-6}$, the equivalent plastic strain contours at $U/h_0 = -0.012$ are shown in Figs. 16 through 19 in the same way as in the case of tension. These results show the consistency with the results of the material instability analysis (Fig. 6) just as in the aforementioned plane strain
uniaxial tension. For the case of $\theta = 45^\circ$, the finite element results (Fig. 18) show the critical angle of shear band $\phi_0 \approx 41^\circ$ and $\phi_0 \approx 139^\circ$ for $\eta = -1$ and $\eta = 1$, respectively. As discussed in uniaxial tension test for $\theta = 45^\circ$, this is due to the fact that the orthotropic axes are rotated counterclockwise for $\eta < 0$ and clockwise for $\eta > 0$ when $D_{22} = -D_{11} < 0$. 

Fig. 17. The development of strain localization at $U/h_0 = -0.012$ for the uniaxial compression, $\theta = 22.5^\circ$.

Fig. 18. The development of strain localization at $U/h_0 = -0.012$ for the uniaxial compression, $\theta = 45^\circ$. 
Comparing the results for the tension with those for the compression, for example, Fig. 11 and Fig. 17 for $\theta = 22.5^\circ$, or Fig. 13 and Fig. 19 for $\theta = 67.5^\circ$, we find that the delay or hastening effect is completely reversed for the same value of $\eta$ when the loading type is changed from tension to compression or vice versa. Moreover, the band orientation also changes totally depending on the loading type; for example the band orientation for $\eta = -1$ and $\theta = 22.5^\circ$ under tension is $\phi_0 \approx 57^\circ$ (Fig. 11) while the band orientation for $\eta = -1$ and $\theta = 22.5^\circ$ under compression is $\phi_0 \approx 147^\circ$ (Fig. 17). This is due to the fact that the stress state in reference to $\hat{x}_i$ is different between tension and compression, and the evolution of the orthotropic axes is different from each other depending upon the deformation type (or the loading type) as discussed in the case of $\theta = 45^\circ$.

For the case of $\theta = 22.5^\circ$ and $\eta = 0$, the curve of average stress versus end displacement is shown in Fig. 20. The average stress is given by

$$\sigma_{ave} = \frac{1}{2H} \int_{-h_0}^{H_0} [T_z(x_2)]_{x_2=h_0} dx_1,$$

where $H$ is the current width at $x_2 = h_0$. As the plastic deformation continues stress decreases along with shear band formation. As seen in Fig. 20, there are no significant inertia effects observed in the present dynamic loading.

VI. CONCLUSION

From the foregoing bifurcation analysis and the finite element results, we may draw the following conclusions for the shear band formation and development in the present orthotropic material under plane strain uniaxial loading:
1. Since the stress state in reference to $\hat{x}_i$ coordinate depends upon the orientation of the orthotropic axes, the formation and development of shear band in an orthotropic material is strongly dependent upon the orientation of the initial orthotropic axes.

2. The plastic spin, one of the constitutive features, has a substantial hastening or delay effect upon the shear band formation and development depending upon the initial orientation of the orthotropic axes and the deformation mode (or stress state). Moreover, the change in the orientation of orthotropic axes due to the influence of plastic spin, which is dependent upon the deformation or loading mode, may have a significant effect upon the orientation and development of the shear bands in the elastoviscoplastic material.

3. Initially more than one band may appear in the rate-dependent material, however, only the bands with orientation consistent with the material instability analysis continue to grow and develop into shear bands.

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REFERENCES


APPENDIX

Let the current configuration of the body be the configuration at time \( t_n \) and a subsequent configuration be the configuration at time \( t_{n+1} = t_n + \Delta t \). Next, let \( Q \) be the rotation tensor which is given as a solution of the initial value problem:

\[
\ddot{Q}(t) = \omega(t)Q(t), \quad Q(0) = I, \tag{A1}
\]

where the time interval of integration is \([t_n, t_{n+1}]\) and \( \omega \) is the substructural spin tensor. Then we have the following result

\[
\dot{\sigma} = Q^T \sigma^0 Q, \tag{A2}
\]

where

\[
\sigma = Q^T \sigma Q.
\]

This equation means that after transformation, the evolution equation for stress involves only the material time derivative of \( \sigma \) instead of the Jaumann derivative of \( \sigma \) (Rubinstein & Atluri [1983]; Nagtegaal & Veldpaus [1984]; Hughes [1984]).

Using (A2) and (7), the evolution equation for the stress \( \dot{\sigma} \) then may be written as

\[
\dot{\sigma} = \dot{L} : [\bar{D} - \langle z \rangle^n \bar{N}^p], \tag{A3}
\]

where the overbar "---" denotes a tensor after transformation.

Then, a procedure for integrating the evolution eqn (7) over a time increment \( \Delta t \) may be summarized as:

1. Calculation of strain increment tensor \( \Delta e_{ij} \) and substructural spin tensor \( \omega_{ij} \) as

\[
\Delta e_{ij} = \frac{1}{2} \left( \frac{\partial \Delta u_i}{\partial x_j^{(n+1/2)}} + \frac{\partial \Delta u_j}{\partial x_i^{(n+1/2)}} \right),
\]

\[
\omega_{ij} = \frac{1}{2 \Delta t} \left( \frac{\partial \Delta u_i}{\partial x_j^{(n+1/2)}} - \frac{\partial \Delta u_j}{\partial x_i^{(n+1/2)}} \right) - \langle z \rangle^n \left[ \eta_1 (\alpha_{1}^{(n)} \sigma^{(n)} - \sigma^{(n)} \alpha_{1}^{(n)}) 
+ \eta_2 (\sigma_{2}^{(n)} \sigma^{(n)} - \sigma^{(n)} \sigma_{2}^{(n)}) \right]_{ij}.
\]
where $\Delta u_i$ is the displacement increment and $x_i^{(n+1/2)} = x_i^{(n)} + \frac{1}{2} \Delta u_i = x_i^{(n+1)} - \frac{1}{2} \Delta u_i$ is the position vector component at the midpoint configuration.

2. Calculation of the rotation tensor $Q_n, Q_{n+1/2}, Q_{n+1}$ from the eqn (A1) (Hughes [1984]).

3. Transformation of the strain increment, current stress, direction of plastic flow, and elasticity tensor.

$$\Delta \bar{\varepsilon} = Q_{n+1/2}^T \Delta \varepsilon Q_{n+1/2}, \quad \bar{\sigma}_n = Q_n^T \sigma_n Q_n,$$

$$\bar{N}_n^p = Q_n^T N_n^p Q_n, \quad \bar{L}_n = Q_n^T (Q_n L_n Q_n) Q_n$$

4. Update the stress

$$\sigma_{n+1} = Q_{n+1}(\bar{\sigma}_n + \bar{L}_n: \Delta \bar{\varepsilon} - \Delta t (z)^n \bar{N}_n^p) Q_{n+1}^T.$$