DEVELOPMENTS IN THE ANALYSIS OF INTERACTING CRACKS

R. JONES,* S. N. ATLURI, † S. PITT‡ and J. F. WILLIAMS‡

*Department of Mechanical Engineering, Monash University, Wellington Rd, Clayton, Victoria, Australia, 3168; †Center for Computational Modelling, Georgia Institute of Technology, Atlanta, GA, U.S.A.; ‡Department of Mechanical and Manufacturing Engineering, Melbourne University, Parkville, Victoria, Australia

(Received 19 June 1995)

Abstract—This paper presents an overview of the finite element alternating technique for the analysis of interacting cracks. To illustrate the ease and accuracy of this method the technique is used to analyse several problems associated with both widespread fatigue and multi-site damage, a problem which is attracting worldwide attention. Whilst this paper presents an overview of the technique for both two- and three-dimensional problems attention is focused on three-dimensional problems. In particular, the interaction effects between two fully embedded elliptical flaws and between two semi-elliptical surface flaws, and the effects of crack proximity and crack aspect ratio on the stress intensity factors are presented. For semi-elliptical surface flaws these results indicate that as the cracks approach each other the position of the point on the crack front with the highest stress intensity factor shifts. This subsequently suggests that surface cracks will tend to grow preferentially towards each other. The same trend is evidenced for fully embedded cracks. However, in this case there is no shift in the position of the maximum stress intensity factor. A discussion of the results in terms of stress intensity magnification factors is also presented.

1. INTRODUCTION

The downturn in the global economy and the end of the “cold war” coupled with the high acquisition costs associated with the purchase of modern military and civilian aircraft has resulted in greater utilisation of existing aircraft fleets. This trend, in operating existing aircraft approaching or beyond their intended design life, has been reflected in an increasing number of structurally significant defects. The long service life of “ageing” aircraft increases the possibility of a reduction, or loss, of structural integrity due to fatigue. The importance of understanding and managing ageing structures was highlighted by the failure of Aloha 737 on 28 April 1988. This failure was essentially due to the linking, into one large crack, of numerous small cracks at a number of fastener holes. This phenomenon has subsequently been termed “multi-site damage” (MSD). MSD presents unique problems to both the aircraft maintenance operator and the structural analyst. In essence, it occurs when groups of small cracks appear at about the same time and are located in a common area [1]. Even though each crack, considered individually, may be safe the level of interaction can degrade the damage tolerance of the structure beneath acceptable levels.

1.1. An Australian perspective

1.1.1. F/A-18 laboratory tests. Although the phenomenon was first observed in civilian aircraft, recent Australian work [2] has found that MSD plays a major role in determining the fatigue life of the F/A-18 aft bulk head, i.e. bulk head FS-488 [2]. In this case the various fatigue test(s) undertaken by McDonnell Douglas in support of the F/A-18 had indicated a large number of potential hot spots, including the FS488 aft bulkhead flange, mold line and wing attachment lug. To further assess the fatigue performance of the FS488 aft bulkhead a full-scale fatigue test was performed in Australia on a stand alone FS488 bulkhead. The test was performed to primarily
address the region of the wing attachment lug. In this test failure resulted from a fatigue crack approximately 6 mm deep. However, post-failure inspection of the test article revealed the presence of several hundred cracks within the critical region [2]. A fractographic evaluation of the specimen subsequently revealed that this population of cracks exhibited similar crack growth rates and that the cracks remained very small throughout the life of the component. Furthermore, the markings on the critical crack could essentially be traced back to the beginning of the test (see [2] for more details). The question thus arose as to the level of interaction of all of these cracks, particularly their effect on the so-called “dominant flaw”, and their combined effect on both the “local” load path and the damage tolerance of the bulkhead.

As a result of this test it was concluded [2] that existing NDI techniques could not be relied upon to find the critical crack. This conclusion highlighted the need to develop both advanced NDI techniques as well as new analysis tools for the assessment of structural integrity, particularly when the critical component contains large numbers of interacting dimensional flaws.

1.1.2. **Mirage III O cracking.** The importance of understanding and managing widespread fatigue damage has also gained international visibility. In Australia the importance of understanding and managing widespread fatigue damage was highlighted by fatigue cracking in Mirage III O aircraft in service with the RAAF. In this case full-scale fatigue testing of Mirage III O fighter aircraft wings at the Swiss Federal Aircraft Factory (F + W) resulted in fatigue cracks at the innermost bolt holes along the rear flanges of the main spars, with failure associated with cracking at bolt hole number 9. The test was then continued with a starboard RAAF Mirage III O wing (2190 h service) and a port Swiss Mirage III S wing (510 h service). After a relatively short test life cracks were then found at the innermost bolt holes. Cracking was also found in a number of other locations including (fuselage) frame 26. Further cracking was experienced in the lower (tension) wing skin both at the fairing hole (nearest the main spar) and at the fuel decant hole in the lower wing skin (see [2, 4] for more details). In the RAAF wing a failure occurred from cracks which developed at the single leg anchor nut (SLAN) rivet holes associated with bolt hole number.

Following this test crack indications were confirmed at identical locations, in the main spar, in wings of the RAAF Mirage III O fleet. A detailed laboratory test program was then undertaken and it subsequently found that the existence of the two SLAN rivet holes meant that cracking, which was both complex and three-dimensional in nature, developed at the (two) SLAN holes as well as at the main bolt holes. We were thus faced with the problem of distributed and interacting (three-dimensional) flaws.

The F + W fatigue test undertaken in Switzerland also resulted in a major failure, together with a number of nearby cracks, in the fuselage frame 26. The major crack occurred at hole 18 on the left side of frame 26A and extended across the entire flange and well into the web. Other cracks occurred in the region between holes 1 and 23 and included a 9 mm crack in the inner strap plate and small (less than 3 mm long) cracks in holes 4, 7, 8, 18, 20 and 22. Prior to the major crack being detected significant cracking had also occurred in the outer strap plate in the bottom of the frame. At the time of the failure this region contained cracks with lengths of 40, 31, 20, 18 and 12 mm.

1.1.3. **The RAAF Macchi recovery program.** The importance of maintaining continued airworthiness was further highlighted by the November 1990 failure of an RAAF Macchi aircraft (viz. A7-076) which suffered a port wing failure whilst in an estimated 6 g manœuvre. In this case it was found that failure was caused by fatigue cracking originating from the “D17” rivet hole in the lower spar cap [5]. As a result of this event a Macchi Recovery Program was initiated to determine the structural condition of the fleet and to reassess the fatigue lives and management philosophies of the main structural components. A tear down inspection program, which involved
Developments in the analysis of interacting cracks

10 post LOTEX wings, two fuselages, two fins and five horizontal tail planes being destructively inspected (see [5] for details), was then undertaken.  

Six of the wings showed significant cracking indications and of the, approximately, 1000 holes which were examined 100 revealed fatigue cracks, including major cracking in the D series rivet holes. This program revealed the fatigue critical locations in the centre section lower spar boom to be bolt holes 3–6 and 17–20. The flaws were highly three-dimensional in nature and, from the failure investigation of aircraft A7-076, the failure process had involved a number of interacting cracks. In this case cracking had progressed from a web attachment fastener hole through the flange as well as from the nearby wing attachment fastener (rivet) hole (Fig. 1). Fractographic evidence also indicated multiple crack origins at the root of the rivet hole (Fig. 1). To assist in establishing the critical crack size a detailed three-dimensional finite element analysis of this cracking was performed. This established that, once the main crack had grown past the flange, the stress intensity factor was essentially equal to its fracture toughness. A more detailed description of this program, its underlying philosophy and the Macchi Aircraft Structural Integrity Management Plan is given in [5].  

The mutual influence of the adjacent cracks increases the complexity of predicting both the fatigue crack growth rate and the failure mechanisms. Unfortunately, few solutions exist in the literature for interacting cracks in finite geometry bodies. For interacting three-dimensional cracks the sparsity of solutions is even greater. The challenge is therefore to develop analytical tools, which are simple to use, and which will allow accurate and rapid assessment of structural integrity. 

This paper discusses one such technique which is based on the finite element alternating method, which has the advantage that the cracks need not be modelled explicitly. Initially attention is focused on the technique as developed for three-dimensional problems [6, 7]. For three-dimensional problems the alternating finite element technique makes extensive use of the analytical solution for a three-dimensional elliptical flaw subject to arbitrary crack face loading. In this context one of the first relevant solutions was obtained by Green and Sneddin [8], who solved the problem of a penny-shaped crack, subject to uniform tension at infinity. Kassir and Sih [9] solved the problem of uniform shear loading along the crack face and obtained an exact solution in terms of two harmonic potential functions. The generalisation of these

Fig. 1. Cross section view of failure surfaces in the Macchi spar.
solutions for the cases when the crack surface was subjected to various degrees of polynomial pressure distribution was the focus of many other important studies. The work of Segedin [10] proposed the use of a certain type of ellipsoidal harmonics and their partial derivatives which satisfy Laplace’s equation. This approach was subsequently used by both Kassir and Sih [11] and Shah and Kobayashi [12].

In the work of Shah and Kobayashi [12], the contributions of the potentials to each stress component on the crack surface were not linearly independent polynomial functions, and so it was necessary to make a judicious choice for the potential functions for each degree of loading. Their analysis considered only normal loading, and, in addition, they were limited to just cubic polynomial distributions on the crack. This work was subsequently generalised by Vijayakumar and Atluri [7] who considered arbitrary normal as well as shear loading. These authors derived expressions for stress intensity factors near the flaw border, as well as for stresses in the far-field, for these generalised loadings. The key to implementing this solution in the finite element alternating technique was the development by Nishioka and Atluri [6] of a general procedure for evaluating the necessary elliptic integrals.

The Schwartz Neumann alternating technique was originally applied to three-dimensional fracture mechanics by Shah and Kobayashi [10]. However, this solution suffered from a number of drawbacks, viz:

(i) The analytical solution was limited to just cubic loading on the crack face.
(ii) The solution was limited to media bounded by straight surfaces.

Whilst Smith and Kullgren [13] introduced the finite element method into the alternating technique, again using only a cubic polynomial, Nishioka and Atluri [6] were the first authors to use a full analytical solution for the complete polynomial. This technique has subsequently been successfully applied to solve a range of three-dimensional problems, viz thick plates [6], pressure vessels [14] and aircraft attachment lugs [15], and was recently extended by Jones et al. [16], to include arbitrary interacting cracks.

One major advantage of this technique is that by combining the finite element method with the analytical solution we enable accurate results to be obtained using only a relatively coarse mesh. Furthermore, since cracks are not modelled explicitly this means that the crack configuration can be changed without complex remeshing, and, as the crack geometry changes, it also removes the need for tedious remeshing. Consequently, for problems associated with MSD the finite element alternating method provides a very efficient and cost-effective method of analysis.

2. THE THREE-DIMENSIONAL FINITE ELEMENT ALTERNATING METHOD

The basic steps in the finite element alternating technique can be explained as follows (see [6, 7] for more details):

(1) The stresses in the uncracked body are first obtained using the finite element method.
(2) The tractions, at the locations of the cracks, in the uncracked body are determined.
(3) The tractions at the crack faces are then reversed and the problem is converted into that of solving for the same cracked structure, where the crack faces are subject to the (reversed) tractions determined in step (2), and where the outer boundaries (surfaces) of the body are stress-free.
(4) The analytical solution for a crack, in an infinite body, subjected to these (reversed) tractions, is then used to calculate both the stress intensity factors and the stresses at the boundary of the body.

Initially the resultant stresses (tractions), obtained from the infinite body solution, at
Developments in the analysis of interacting cracks

the external surfaces (boundaries) will not satisfy the requirement that the external boundary be stress-free.

(5) If the resultant stresses, as calculated in step (4), at the external boundary are below a user-defined tolerance then the required solution has been obtained.

(6) If not, the residual stresses (tractions) at the boundaries of the body are reversed, and steps (1)–(5) are repeated, using these reversed stresses as a new load case.

(7) This iterative loop is continued until convergence occurs [see step (5)]. At this stage the final solution is the sum of the individual solutions obtained in each of the iterative loops.

2.1. Basic formulation: infinite geometry

As can be seen in the previous section the alternating finite element method makes extensive use of the analytical solution for a three-dimensional crack, in an infinite domain, subjected to arbitrary crack face loading. Consequently, for the sake of completeness, we will now briefly outline this solution. Let us begin by defining $u_\alpha (\alpha = 1, 2, 3)$ and $\sigma_{\alpha\beta} (\alpha, \beta = 1, 2, 3)$ as the displacements and stresses, respectively, in a homogeneous, isotropic linear elastic solid. Hooke’s law then tells us that

$$\sigma_{\alpha\beta} = G(u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{2\nu}{1-2\nu}\delta_{\alpha\beta}u_{\gamma\gamma}) \quad (1)$$

In the absence of body forces the Navier displacement equations of equilibrium, in rectangular Cartesian coordinates $x_\alpha (\alpha = 1, 2, 3)$, are

$$u_{\alpha,\alpha} + (1 - 2\nu)u_{\alpha,\beta} = 0 \quad (2)$$

Let $R$ denote the surface of the crack, which for simplicity we will define as lying in the plane $(z =) x_3 = 0$. As is often the case in fracture problems it is convenient to consider the complementary problem in which the surface $R$ of crack is subjected to arbitrary tractions $\sigma_{3\alpha}$. Following the work of Trefftz the solution of this problem can be expressed in terms of four harmonic functions $\psi$ and $\phi_\alpha (\alpha = 1, 2, 3)$ in the form

$$u_\alpha = \phi_\alpha + x_3\psi_\alpha \quad (3)$$

so that the Navier displacement equilibrium equations are satisfied if

$$\phi_{\alpha,\alpha} + (3 - 4\nu)\psi_3 = 0 \quad (4)$$

The problem can be further simplified by expressing $\psi$ and $\phi_\alpha$ in the form

$$\psi = \nabla \cdot \vec{f} = f_{\alpha,\alpha} \quad (5)$$

and

$$\phi_1 = (1 - 2\nu)(f_{1,3} + f_{3,1}) - (3 - 4\nu)f_{1,3},$$

$$\phi_2 = (1 - 2\nu)(f_{2,3} + f_{3,2}) - (3 - 4\nu)f_{2,3},$$

$$\phi_3 = -(1 - 2\nu)(f_{1,1} + f_{2,2}) - 2(1 - \nu)f_{3,3}. \quad (6)$$

Then the governing equations, viz:

$$\psi_{,\alpha\alpha} = 0 \quad (7)$$

$$\phi_{\alpha,\beta\beta} = 0 \quad (8)$$

$$\phi_{\alpha,\alpha} + (3 - 4\nu)\psi_3 = 0 \quad (9)$$

are satisfied if the three functions $f_{\alpha} (\alpha = 1, 2, 3)$ are harmonic.

For near elliptic flaws the cracks will be taken to be an ellipse in the $(z =) x_3 = 0$ plane. Let us now assume that the geometry of the flaw can be described by the equation

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1, \quad (10)$$
where \( a_1 \) and \( a_2 \) are the major and minor axes of the flaw respectively, i.e. \( a_1^2 > a_2^2 \). For simplicity the geometry of the crack surface is more conveniently described in ellipsoidal coordinates \( \xi_\alpha \) \((\alpha = 1, 2, 3)\), where these are the roots of the cubic equation \( \omega(\xi) = 0 \), where

\[
\omega(\xi) = 1 - \frac{x_1^2}{a_1^2 + \xi} - \frac{x_2^2}{a_2^2 + \xi} - \frac{x_3^2}{\xi}. \tag{11}
\]

With this formulation it has been found that suitable forms for the potential functions in the Trefftz's formulation are

\[
f_\alpha = \sum_k \sum_l C_{\alpha,k,l} F_{kl} \tag{12}\]

where

\[
F_{kl} = \frac{\partial^{k+l}}{\partial x_1^k \partial x_2^l} \int_{\xi_3}^{\infty} [\omega(s)]^{k+l+1} \frac{ds}{\sqrt{Q(s)}}. \tag{13}
\]

In this notation \( \omega(s) = P(s)/Q(s) \) and

\[
P(s) = (s - \xi_1)(s - \xi_2)(s - \xi_3), \tag{14}\]
\[
Q(s) = s(s + a_1^2)(s + a_2^2). \tag{15}\]

If we denote the partial derivative of \( f_\alpha \) with respect to \( x_\beta \) \((\beta = 1, 2, 3)\) as \( f_{\alpha,\beta} \) we can then write

\[
f_{\alpha,\beta} = \sum_k \sum_l C_{\alpha,k,l} F_{kl,\beta}, \tag{16}\]
\[
f_{\alpha,\beta y} = \sum_k \sum_l C_{\alpha,k,l} F_{kl,\beta y} \tag{17}\]

and

\[
f_{\alpha,\beta y \delta} = \sum_k \sum_l C_{\alpha,k,l} F_{kl,\beta y \delta}. \tag{18}\]

The components of the stresses can then be written in the form

\[
\sigma_{11} = 2\mu(f_{3,11} + 2v f_{3,22} - 2f_{1,31} - 2v f_{2,32} + x_3 (\nabla \cdot \hat{f}),_{11}), \tag{19}\]
\[
\sigma_{22} = 2\mu(f_{3,22} + 2v f_{3,11} - 2f_{2,32} - 2v f_{1,31} + x_3 (\nabla \cdot \hat{f}),_{22}), \tag{20}\]
\[
\sigma_{12} = 2\mu((1 - 2v)f_{3,12} - (1 - v)(f_{1,32} + f_{2,13}) + x_3 (\nabla \cdot \hat{f}),_{12}), \tag{21}\]
\[
\sigma_{33} = 2\mu(-f_{3,33} + x_3 (\nabla \cdot \hat{f}),_{33}), \tag{22}\]
\[
\sigma_{31} = 2\mu(-(1 - v)f_{3,13} + v(f_{1,11} + f_{2,21}) + x_3 (\nabla \cdot \hat{f}),_{13}), \tag{23}\]
\[
\sigma_{32} = 2\mu(-(1 - v)f_{2,33} + v(f_{1,12} + f_{2,22}) + x_3 (\nabla \cdot \hat{f}),_{23}). \tag{24}\]

Let us now express the tractions along the crack surface in the form

\[
\sigma^{(0)}_\alpha = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} A_{\alpha,m-n,n} x_1^{2m-2n+i} x_2^{2n+j}, \tag{25}\]

where \( \alpha = 1, 2, 3 \) and the values of \((i, j)\) specify the symmetries of the load with respect to the axes of the ellipse (see [6, 7] for more details). The solution in terms of the potential function is then assumed to be of the form

\[
f_\alpha = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{K} \sum_{l=0}^{l} C_{\alpha,k-1,l} F_{2k-2l+i,2l+j}. \tag{26}\]

Substituting for \( f_\alpha \) and its various partial derivatives in Eqns (19)-(24) thereby
enables the coefficients of \( C \) to be obtained directly from the coefficients of \( A \) via a simple matrix equation of the form (see [6] for details)

\[
\{ A \} = [B]\{ C \}.
\]

Once the coefficients \( C \) have been determined, for the given crack surface loadings, the stress intensity factors corresponding to these loads can be directly computed. For the mode I problem the relevant equation takes the form

\[
K_I = 8\mu \left( \frac{\pi}{a_1a_2} \right)^{1/2} A^{1/4} \sum_{i=0}^{1} \sum_{k=0}^{M} \sum_{l=0}^{k} (-2)^{2k+i+j}(2k+i+j+1)! \\
\times \left( \frac{\cos \theta}{a_1} \right)^{2k-2l+i} \left( \frac{\sin \theta}{a_2} \right)^{2l+j} C_{2a-1,i}^{(i,j)}, 
\]

where \( \theta \) is the elliptic angle and

\[
A = a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta. 
\]

### 3. SEMI-ELLIPTICAL SURFACE FLAWS IN A SEMI-INFINITE BODY

#### 3.1. A single flaw

To illustrate this approach let us first consider a single semi-elliptical surface flaw in an infinite body. For numerical purposes, in accordance with common practice, the body was regarded as being infinite if its edges were further than 5 times the semi-major crack length away from the crack centre. To model this situation with the finite element alternating technique only one quadrant needs to be considered, because of the two planes of symmetry inherent in this problem, and the resultant (uncracked) mesh used consisted of a total of 216 20-noded brick elements, i.e. a \( 6 \times 6 \times 6 \) mesh. For the purposes of this analysis, the material properties were chosen as \( E = 70 \text{ GPa} \) and \( v = 0.33 \), and a remote tensile stress of 1000 MPa was assumed. In the first set of cases considered the surface (half) crack length \( a_1 = 18 \text{ mm} \), the width \( 2b = 400 \text{ mm} \) and the crack aspect ratio \( a_2/a_1 \), where \( a_2 \) is the (variable) crack depth (Fig. 2). In this instance the maximum stress intensity factor occurs at point \( Q \) on the crack face which is deepest into the solid. To illustrate the accuracy of this technique the values obtained for the maximum stress intensity factor, for different crack aspect ratios, were compared with two different sets of published values [17, 18]. The result of this analysis can be seen in Fig. 3. The mean difference between the finite element alternating technique values and those of Rooke and
Cartwright [17] was 3.8%, whilst that between the finite element alternating technique and those published by the FAA [18] was 4.6%. Given the uncertainties quoted (approximately 5%) in the published values these differences were deemed to be well within acceptable limits. Furthermore the values obtained also tend to lie between those given in [17] and those given in [18].

3.2. Two identical surface flaws

Having established the validity of the model for one crack, the problem was subsequently extended to investigate two interacting semi-elliptical surface cracks, with their centres being a distance \( d \) apart, in a semi-infinite body (Fig. 4). In the following cases the mesh density was essentially the same as described above, the crack aspect ratio \( (a_2/a_1) \) was fixed at 0.8 and the “crack separation distance” \( d \) was held constant at 100 mm. The crack separation ratio \( (\lambda = 2a_1/d) \) was then allowed to vary, and values of the mode I stress intensity factor were calculated at 5° increments along the crack front.

The results of this analysis can be seen in Fig. 5. This graph indicates that, for two interacting cracks, the point of maximum stress intensity factor was no longer the point which lies deepest in the solid. The latter behaviour only occurred when the cracks were relatively far apart \( (\lambda < 0.3) \) and were (essentially) acting independently of each other. As the crack spacing decreased, the position of the point with the largest \( K_1 \) gradually shifted away from the point furthest into the solid \( (\theta = 90°) \).
Developments in the analysis of interacting cracks

Fig. 5. Variation in maximum stress intensity factor with elliptic angle, $0 < \theta < 90$, for one of a pair of semi-elliptical surface flaws in a semi-finite body. In each case $a_2/a_1 = 0.8$.

towards the other crack ($\theta = 0^\circ$). Physically this means that, as crack spacing decreases, interaction effects become more pronounced and the cracks begin to grow preferentially towards each other rather than progressing further into the bulk of the solid.

A more common way of expressing these results is to use a normalised stress intensity factor, called the stress intensity magnification factor, $F_m$, defined by

$$F_m = \frac{K_1(\theta)}{\sigma_0 \sqrt{E(k)}} \sqrt{\frac{\pi a_2}{a_1}} \left(\frac{a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta}{a_1^2}\right)^{1/4}$$

where $E(k)$ is the complete elliptic integral of the second kind, $k^2 = (a_1^2 - a_2^2)/a_1^2$, and $\theta$ is the elliptic angle. The expression in the denominator is in fact the stress distribution for a single crack in an infinite elastic solid subject to uniform tension at infinity. The stress magnification factor is useful as it provides an indication of the percentage increase in the stress intensity factor due to the interaction effect of the other crack.

The magnification factors for the previous set of cases ($a_2/a_1 = 0.8$ and $d = 100$ mm) were calculated by first employing the finite element alternating technique to evaluate the stress intensity factors on a single semi-elliptical surface flaw. The results of this analysis can be seen in Fig. 6. This graph clearly illustrates that, in all cases, the magnification factor was greatest at the free surface of the solid. As expected, the magnification factors increase as the cracks get closer together. However, the rate of increase was more pronounced at the free surface than at the point on the crack front deepest into the solid.

To investigate the effect of crack aspect ratio on the stress intensity factor, and on the magnification factors, attention was then focused on the point of the crack which was deepest into the solid. In all cases the distance between the crack centres was fixed at 100 mm. Two different crack separation ratios were considered: viz $\lambda = 0.5$ and $\lambda = 0.8$. As $d$ was fixed at 100 mm the first case implies that $a_1 = 25$ mm whilst the second implies $a_1 = 40$ mm. The values of $a_2$ were then allowed to vary. The relationship between stress intensity factor (at $\theta = 90^\circ$) and crack aspect ratio is illustrated in Fig. 7. It must be remembered that the position of the maximum stress intensity factor was different for these two curves. Figure 7 shows that as the cracks get larger (i.e. as $a_2$ increases) the stress intensity factor at $\theta = 90^\circ$ also increases.

The corresponding magnification factors were calculated in the manner described
Fig. 6. Variation in magnification factors versus elliptic angle, \( \theta \), for a pair of semi-elliptical surface flaws in a semi-infinite body. In each case \( a_2/a_1 = 0.8 \).

Fig. 7. Variation in maximum stress intensity factor with crack aspect ratio for one of a pair of identical semi-elliptical surface flaws in a semi-infinite body. In all cases \( d = 100 \) mm.

above, and the results can be seen in Tables 1 and 2. For both values of the crack separation ratio (\( \lambda = 0.5, 0.8 \)) there was little change in the magnification factor at \( \theta = 90^\circ \). In the situation of two cracks it would therefore appear that, at the point where the cracks protrude deepest into the solid, the crack separation is a more crucial factor than the crack shape.

Table 1. Magnification factors for a pair of semi-elliptical surface flaws in a semi-infinite solid (\( \theta = 90^\circ, \lambda = 0.8 \)).

<table>
<thead>
<tr>
<th>( a_2/a_1 )</th>
<th>( k )</th>
<th>( E(k) )</th>
<th>( K_I ) (N mm(^{-3/2}))</th>
<th>Magnification factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.995</td>
<td>1.01599</td>
<td>3.49E + 03</td>
<td>1.0025</td>
</tr>
<tr>
<td>0.2</td>
<td>0.98</td>
<td>1.02859</td>
<td>5.02E + 03</td>
<td>1.0299</td>
</tr>
<tr>
<td>0.3</td>
<td>0.95</td>
<td>1.06047</td>
<td>6.13E + 03</td>
<td>1.0585</td>
</tr>
<tr>
<td>0.4</td>
<td>0.92</td>
<td>1.08793</td>
<td>6.89E + 03</td>
<td>1.0586</td>
</tr>
<tr>
<td>0.5</td>
<td>0.87</td>
<td>1.12845</td>
<td>7.41E + 03</td>
<td>1.0548</td>
</tr>
<tr>
<td>0.6</td>
<td>0.8</td>
<td>1.17848</td>
<td>7.73E + 03</td>
<td>1.0491</td>
</tr>
<tr>
<td>0.7</td>
<td>0.714</td>
<td>1.23568</td>
<td>7.89E + 03</td>
<td>1.0395</td>
</tr>
<tr>
<td>0.8</td>
<td>0.6</td>
<td>1.29842</td>
<td>7.93E + 03</td>
<td>1.0269</td>
</tr>
</tbody>
</table>
Table 2. Magnification factors for a pair of semi-elliptical surface flaws in a semi-infinite solid ($\theta = 90^\circ; \lambda = 0.5$)

<table>
<thead>
<tr>
<th>$a_2/a_1$</th>
<th>$k$</th>
<th>$E(k)$</th>
<th>$K_1$ (N mm$^{-\frac{1}{2}}$)</th>
<th>Magnification factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.99</td>
<td>1.01599</td>
<td>2.75E + 03</td>
<td>0.9984</td>
</tr>
<tr>
<td>0.2</td>
<td>0.96</td>
<td>1.02859</td>
<td>3.52E + 03</td>
<td>0.99035</td>
</tr>
<tr>
<td>0.3</td>
<td>0.91</td>
<td>1.06047</td>
<td>4.59E + 03</td>
<td>1.00277</td>
</tr>
<tr>
<td>0.4</td>
<td>0.84</td>
<td>1.08793</td>
<td>5.15E + 03</td>
<td>0.9996</td>
</tr>
<tr>
<td>0.5</td>
<td>0.75</td>
<td>1.12845</td>
<td>5.55E + 03</td>
<td>0.9994</td>
</tr>
<tr>
<td>0.6</td>
<td>0.64</td>
<td>1.17848</td>
<td>5.82E + 03</td>
<td>0.9998</td>
</tr>
<tr>
<td>0.7</td>
<td>0.51</td>
<td>1.23568</td>
<td>6.00E + 03</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.36</td>
<td>1.29842</td>
<td>6.11E + 03</td>
<td>1.0001</td>
</tr>
</tbody>
</table>

4. THE FINITE ELEMENT ALTERNATING METHOD IN TWO DIMENSIONS

In the previous section we saw that the three-dimensional finite element alternating method provided an efficient and cost-effective analysis method. To illustrate the simplicity and accuracy of this approach for two-dimensional problems, which differs only in that a different analytical solution is used, let us consider a flat plate, containing multiple cracks, under uniform uniaxial tension. For the purposes of this investigation the plate was assumed to be 10 mm thick and have dimensions 100 x 100 mm. The plate was also assumed to be an aluminium alloy with $E = 71$ GPa and $\nu = 0.32$, and the remote (uniform) stress was taken to be 68.9 MPa.

For the uncracked problem a relatively coarse mesh consisting of nine elements was used (see Fig. 8) with the (various) cracking states under consideration located in element 5. For these test cases two crack configurations were considered, viz: (a) a row of four horizontal cracks, and (b) a column of four cracks, and the resultant stress intensity factors were compared with standard handbook values [17].

4.1. Case (a)

In this test case a row of four equal length cracks were "placed" along the x-axis. The length of each crack was designated as $2a$ and the spacing between the crack centres as $2b$ (Fig. 9). The results reveal that the maximum stress intensity factors occur at locations $A$ and $B$ as shown in Fig. 9. The resultant stress intensity factors, normalised by dividing by $K_o = \sigma_0(\pi a)$, are shown in Fig. 10 together with the

![Fig. 8. Schematic diagram showing the details of the mesh used.](image-url)
values obtained from [17]. In general the values of $K_i/K_0$ agree with the published values to within 2.6%.

4.2. Case (b)

The second series of test cases involved a stack of four horizontal cracks. The geometry for this and the nomenclature adopted are as shown in Fig. 11. In reference [17] two sets of stress intensity factors are published for this crack configuration. The first of these two sets are for the tips of the outermost cracks. These cracks have the highest stress intensity factors. The other series of results refer to the tips of the inner cracks. These cracks are partly shielded by the outer ones and hence the stress intensity factors at these locations are lower than for the outermost crack.

As in the previous test case the stress intensity factors, thus calculated, were normalised with respect to $K_0$ and the resultant values, plotted as a function of $a/c$, are shown in Figs 12 and 13. From Figs 12 and 13 it can be seen that the results agree with the published values to within 0.5%.

5. CONCLUDING REMARKS

This paper has outlined the alternating finite element technique as applied to both three- and two-dimensional fracture problems. For interacting cracks the mutual
Developments in the analysis of interacting cracks

influence of the adjacent cracks significantly increases the complexity of predicting both the fatigue crack growth rates and the failure mechanisms. The challenge was to develop analytical tools, which are simple to use, and which will allow accurate and rapid assessment of structural integrity. This paper discusses one such technique based of the finite element alternating method [6, 7, 19]. One advantage of this method is that the cracks need not be modelled explicitly. To illustrate its simplicity particular attention has been focused on the effects of interacting surface and embedded cracks in a three-dimensional solid. In this case it has been shown that the stress intensity factors depend upon a number of variables including:

(i) crack configuration, viz whether the cracks are surface flaws or are fully embedded,

(ii) individual crack geometry, viz the shape of the crack itself, as measured by the crack aspect ratio \((a_2/a_1)\),

---

Fig. 11. Crack configuration for the second series of test cases. "o" designates an outer crack and "i" an inner crack.

Fig. 12. Stress intensity factors for the outermost crack in a vertical stack of four horizontal cracks.

Fig. 13. Stress intensity factors for the middle crack in a stack of four horizontal cracks.
(iii) the separation between the cracks, as determined by \( \lambda (= 2a_J/d) \). In most instances the crack interaction effects can be neglected until the cracks are closely spaced.

From the problems detailed in this paper the following trends are evident:

(a) For a single semi-elliptical surface flaw in a semi-infinite body the point with the highest stress intensity factor is always the point on the crack face lying deepest in the solid. However, for two interacting surface flaws this situation changes and the cracks interact in such a way as to promote crack growth towards each other, i.e. link-up.

(b) For two interacting semi-elliptical surface flaws the crack separation ratio has a greater influence on the stress intensity factor at the point deepest in the solid than does the crack aspect ratio.

(c) For a single fully embedded elliptical flaw the point with the highest stress intensity factor lies on the semi-minor axis. This is still the case for two interacting fully embedded elliptical flaws and also promotes crack growth towards each other.

REFERENCES