The general Meshless Local Petrov-Galerkin (MLPG) type weak-forms of the displacement & traction boundary integral equations are presented, for solids undergoing small deformations. Using the directly derived non-hyper singular integral equations for displacement gradients, simple and straight-forward derivations of weakly singular traction BIE’s for solids undergoing small deformations are also presented. As a framework for meshless approaches, the MLPG weak forms provide the most general basis for the numerical solution of the non-hyper-singular displacement and traction BIEs. By employing the various types of test functions, several types of MLPG/BIEs are formulated. Numerical examples show that the present methods are very promising, especially for solving the elastic problems in which the singularities in displacements, strains, and stresses, are of primary concern.

1. Introduction

The boundary integral equations (BIEs) have been developed for solving PDEs during the passed several decades. The BIE methods become even more powerful, when several fast algorithms are combined, such as the panel-clustering method, multi-pole expansions, fast Fourier-transforms, wavelet methods, and so on. However, it is well known that hyper-singularities are encountered, when the displacement BIEs are directly differentiated to obtain the traction BIEs, usually for solving crack problems. The hyper-singularities also make it very difficult for the fast algorithms to be applied to traction BIEs, even after the some de-singularization. In contrast, as far back as 1989, Okada, Rajiyah, and Atluri [9] have proposed a simple way to directly derive the integral equations for gradients of displacements. It simplified the derivation processes by taking the gradients of the displacements of the foundation solutions as the test functions, while writing the balance laws in their weak- form. It resulted in “non-hyper-singular” boundary integrals of the gradients of displacements. Recently, this concept has been followed and extended for a directly-derived traction BIE[7], which is also “non-hyper-singular” [1/r^2], as opposed to being “hyper-singular” [1/r^3], as in the most common literature for 3D tBIEs.

The meshless methods have been investigated in recent years, besides the traditional element-based methods for solving BIEs. As a systematic framework for developing various meshless methods, the Meshless Local...
Petrov-Galerkin (MLPG) approach has been proposed as a fundamentally new concept\textsuperscript{[1-4]}. The many research demonstrates that the MLPG method is one of the most promising alternative methods for computational mechanics. In this paper, we write the local weak-forms of non-hyper-singular displacement & traction BIEs, in the local sub-boundary surfaces, through the MLPG approach. From this general MLPG approach, various boundary solution methods can be easily derived by choosing, a) a variety of the meshless interpolation schemes for the trial functions, b) a variety of test functions over the local sub-boundary surfaces, and c) a variety of numerical integration schemes. Such variants have been excellently summarized by Atluri and Shen\textsuperscript{[2,3]} in the complementary formulation of the MLPG approach for domain- solutions. In this paper, we focus on developing the general MLPG/BIE approach, and demonstrate its variants as the collocation method (MLPG/BIE1); MLS interpolation with its weight function as the test function (MLPG/BIE2); and general interpolation with the nodal trial function as the nodal test function (MLPG/BIE6), similar to those studied in Atluri and Shen\textsuperscript{[2,3]}. It should be pointed out that MLPG/BIE methods here are not limited to those variants that are presented here; and many other special suitable combinations are feasible, according to the problems to be solved. We implement the MLPG/BIE6 and solve some elastic problems, including fracture mechanics problems of non-planar crack-growth.

2. Non-Hyper-singular MLPG Displacement and Traction BIEs

The non-hypersingular displacement and traction BIEs for a linear elastic, homogeneous, isotropic solid, are summarized in this section. Consider a linear elastic, homogeneous, isotropic body in a domain $\Omega$, with a boundary $\partial \Omega$. The Lame' constants of the linear elastic isotropic body are $\lambda$ and $\mu$; and the corresponding Young's modulus and Poisson's ratio are $E$ and $\nu$, respectively. We use Cartesian coordinates $\xi_i$, and the attendant base vectors $\mathbf{e}_i$, to describe the geometry in $\Omega$. The solid is assumed to undergo infinitesimal deformations. The equations of balance of linear and angular momentum can be written as:

$$\nabla \cdot \mathbf{\sigma} + \mathbf{f} = 0; \quad \mathbf{\sigma} = \mathbf{\sigma}^t \quad (1)$$

The constitutive relations of an isotropic elastic homogeneous solid are:

$$\mathbf{\sigma} = \lambda \mathbf{I} (\nabla \cdot \mathbf{u}) + 2\mu \mathbf{\varepsilon} \quad (2)$$
It is well known that the displacement vector, which is a continuous function of $\xi$, can be derived, in general, from the Galerkin-vector-potential $\varphi$ such that:

$$\mathbf{u} = \nabla^2 \varphi - \frac{1}{2(1 - \nu)} \nabla(\nabla \cdot \varphi)$$

(3)

Consider a point unit load applied in an arbitrary direction $e^p$ at a generic location $\mathbf{x}$ in a linear elastic isotropic homogeneous infinite medium. It is well-known that the displacement solution is given by the Galerkin-vector-displacement-potential:

$$\varphi^* = (1 - \nu)F^*e^p$$

(4)

in which $F^*$ is a scalar function$^{[7]}$. The corresponding displacements are derived, using (3), as:

$$u^*_{i}^p(x,\xi) = (1 - \nu)\delta_{pi}F^* - 0.5 F^*_{pi}$$

(5)

and the gradients of the displacements in (5) are:

$$u^*_{i,j}(x,\xi) = (1 - \nu)\delta_{pi}F^*_{kj} - 0.5 F^*_{pij}$$

(6)

By taking the fundamental solution $u^*_{i}^p(x,\xi)$ in Eq. (5) as the test functions, one may write the weak-form of the equilibrium Eq. (1). The traditional displacement BIE can be written as,

$$M^p(x) = L^K u^* / dS - l^W u^a / dS$$

(7)

Instead of the scalar weak form of Eq. (1), as used for the displacement BIE, we may also write a vector weak form of Eq. (1), by using the tensor test functions $u^*_{i,j}^p(x,\xi)$ in Eq. (6) [as originally proposed in Okada, Rajiyah, and Atluri$^{[9]}$], and derive a non-hypersingular integral equation for tractions in a linear elastic solid,

$$-t_b(x) = \int_{\Omega} t_q(\xi)n_a(x)\sigma^*_{ab}(x,\xi) dS + \int_{\Omega} D_p u_q(\xi)n_a(x)\Sigma^*_{abpq}(x,\xi) dS$$

(8)

where $u^*_{i,j}^p$, $\sigma^*_{ij}$ and $\Sigma^*_{abpq}$ are kernel functions, which were first given in Han and Atluri$^{[7]}$, the surface tangential operator $D_t$ is defined as,
\[ D_t = n_r e_{rst} \frac{\partial}{\partial \xi} \]  

(9)

It should be noted that the integral equations for \( u_p(x) \) and \( u_{p,k}(x) \) as in Eqs. (7) and (8) are derived independently of each other. On the other hand, if we derive the integral equation for the displacement-gradients, by directly differentiating \( u_p(x) \) in Eq. (7), a hyper-singularity is clearly introduced.

3. MLPG Approaches

Following the general idea as presented in [4], one may consider a local sub-boundary surface \( \partial \Omega_L \), with its boundary contour \( \Gamma_L \), as a part of the whole boundary-surface, as shown in Figure 1, for a 3-D solid. Eqs. (7) and (8) may be satisfied in weak-forms over the sub-boundary surface \( \partial \Omega_L \), by using a Local Petrov-Galerkin scheme, as:

\[ \int_{\partial \Omega_L} w_p(x) u_p(x) dS = \int_{\partial \Omega_L} w_p(x) dS \int_{\partial \Omega} t_j(\xi) u_j^p(x, \xi) dS 
- \int_{\partial \Omega_L} w_p(x) dS \int_{\partial \Omega} n_l(\xi) u_j^p(x, \xi) \sigma_{ij}^p(x, \xi) dS 
- \int_{\partial \Omega_L} w_b(x) t_b(x) dS \int_{\partial \Omega} t_q(\xi) n_a(x) \sigma_{ab}^p(x, \xi) dS 
+ \int_{\partial \Omega_L} w_b(x) dS \int_{\partial \Omega} D_q u_q(x) \Sigma_{abpq}(x, \xi) dS 
\]

(10)

where \( w_b(x) \) is a test function. If \( w_b(x) \) is chosen as a Dirac delta function, i.e. \( w_b(x) = \delta(x, x_m) \) at \( \partial \Omega_L \), we obtain the standard "collocation" method for displacement and traction BIEs, at the collocation point \( x_m \). With some basic identities of the fundamental solution, one may
obtain the fully desingularized dBIE and tBIE for the standard "collocation" methods as,

\[ 0 = \int_{\partial \Omega} t_j(\xi)u_j^*(x, \xi) \, dS - \int_{\partial \Omega} n_i(\xi)[u_j(\xi) - u_j(x)] \sigma_{ij}^p(x, \xi) \, dS \]  
\((11) a\)

\[ 0 = \int_{\partial \Omega} [t_q(\xi) - n_p(\xi)\sigma_{pq}(x)]n_a(x) \sigma_{ab}^*(x, \xi) \, dS \]

\[ + \int_{\partial \Omega} [D_p u_q(\xi) - (D_p u_q)(x)]n_a(x) \Sigma_{abpq}^*(x, \xi) \, dS \]  
\((11) b\)

If \(w_b(x)\) is chosen such that it is continuous over the local sub boundary-surface \(\partial \Omega_L\) and zero at the contour \(\Gamma_L\), one may apply Stokes' theorem to Eq. (10), and re-write it as:

\[ \frac{1}{2} \int_{\partial \Omega_L} w_p(x)u_p(x) \, dS_x = \int_{\partial \Omega_L} w_p(x) \, dS_x \int_{\partial \Omega} t_j(\xi)u_j^*(x, \xi) \, dS_x \]

\[ + \int_{\partial \Omega_L} w_p(x) \, dS_x \int_{\partial \Omega} D_{ij}(\xi)u_{ij}(\xi)G_{ij}^*(x, \xi) \, dS_x \]  
\((12) a\)

\[ + \int_{\partial \Omega_L} w_p(x) \, dS_x \int_{\partial \Omega} n_i(\xi)u_j(\xi)\phi_{ij}^*(x, \xi) \, dS_x \]

\[ - \frac{1}{2} \int_{\partial \Omega_L} t_b(x)w_b(x) \, dS_x = \int_{\partial \Omega_L} D_a w_b(x) \, dS_x \int_{\partial \Omega} t_q(\xi)G_{ab}^q(x, \xi) \, dS_x \]

\[ - \int_{\partial \Omega_L} t_q(\xi) \, dS_x \int_{\partial \Omega} n_a(x)w_b(x)\phi_{ab}^q(x, \xi) \, dS_x \]

\[ + \int_{\partial \Omega_L} D_a w_b(x) \, dS_x \int_{\partial \Omega} D_p u_q(\xi)H_{abpq}^*(x, \xi) \, dS_x \]  
\((12) b\)

where \(G_{ab}^q\), \(\phi_{ab}^q\) and \(H_{abpq}^*\) are kernel functions[^7].

If the test function \(w_b(x)\) is chosen to be identical to a function that is energy-conjugate to \(u_p\) (for dBIE) and \(t_b\) (for tBIE), namely, the nodal trial function \(\hat{u}_p(x)\) and \(\hat{u}_b(x)\), respectively, we obtain the local weakforms of the symmetric Galerkin dBIE and tBIE.

4. Numerical Implementation

4.1. Moving Least Squares for 3D Surface

It is very common to adopt the moving least squares (MLS) interpolation scheme for interpolating the trial functions over a 3-D surface, as it has been
done successfully in meshless domain methods\textsuperscript{[4]}. Unfortunately, the moment matrix in the MLS interpolation sometimes becomes singular, when it is applied to three-dimensional surface approximation, if Cartesian coordinates are used. An alternative choice is to use the curvilinear coordinates\textsuperscript{[5]}, or choose a varying polynomial basis instead of the complete basis\textsuperscript{[8]}. However, these algorithms require the local geometry information, which hinder the truly meshless implementation. In the present study, the enhanced MLS approximation has been implemented, after reconditioning the singular moment matrix, while still using the global Cartesian coordinates to approximate the trial function over a surface.

Consider a local sub-part of the boundary $\partial \Omega$, of a 3-D solid, denoted as $\partial \Omega_x$, the neighborhood of a point $x$, which is a local region in the global boundary $\partial \Omega$. To approximate the function $u$ in $\partial \Omega_x$, over a number of scattered points $\{x_I\}, (I = 1, 2, \ldots, n)$ (where $x$ is given, in the global Cartesian coordinates by $x_1, x_2$ and $x_3$), the moving least squares approximation $u(x)$ of $u$, $\forall x \in \partial \Omega_x$, can be defined by

\begin{equation}
    u(x) = p^T(x)a(x) \quad \forall x \in \partial \Omega_x
\end{equation}

where $p^T(x) = [p_1(x), p_2(x), \ldots, p_m(x)]$ is a monomial basis of order $m$; and $a(x)$ is a vector containing coefficients, which are functions of the global Cartesian coordinates $[x_1, x_2, x_3]$, depending on the monomial basis. They are determined by minimizing a weighted discrete $L_2$ norm, defined, as:

\begin{equation}
    J(x) = \sum_{i=1}^{m} w_i(x)[p^T(x_i)a(x) - \hat{u}_i]^2 \equiv [Pa(x) - \hat{u}]^T W[Pa(x) - \hat{u}]
\end{equation}

where $w_i(x)$ are the weight functions and $\hat{u}_i$ are the fictitious nodal values. The stationarity of $J$ in Eq. (14), with respect to $a(x)$ leads to following linear relation between $a(x)$ and $\hat{u}$,

\begin{equation}
    A(x)a(x) = B(x)\hat{u} \quad , \quad A(x) = P^T W P \quad B(x) = P^T W \quad \forall x \in \partial \Omega_x
\end{equation}

The MLS approximation is well defined only when the matrix $A(x)$ in Eq. (15) is non-singular. It needs to be reconditioned, if the monomial basis defined in the global Cartesian coordinate system for an approximation of $u$ as in Eq. (13), becomes nearly linearly dependent on a 3-D surface. One may define a local set of orthogonal coordinates, $x'_i$ as in Figure 1, on $\partial \Omega_x$. One may rewrite Eq. (13) as:
Suppose $\partial \Omega_x$ becomes nearly planar, which may be defined in the local-set of orthogonal coordinates $x'_i$, for instance, as $x'_3 = \text{constant}$. It is then clear that the monomial basis in Eq. (16), in terms of $x'_i$ becomes linearly dependent. In fact, one may make the basis to be *linearly independent* again in Eq. (16), for instance, for $x'_3 = \text{constant}$, by setting the corresponding coefficients $a'(x)$ to be zero. When this is done, the order of the vector $p'(x)$ is correspondingly reduced; and thus, correspondingly, the order of $A(x)$ in Eq. (15) is reduced. Thus, it can be seen that if one proceeds with a full monomial basis, with $m$ basis functions in $x$ coordinates in Eq. (16), and if the points on $\partial \Omega_x$ are not all in the same plane, the matrix $A(x)$ in Eq. (15) will have the full rank of $m$. On the other hand, if $\partial \Omega_x$ becomes almost planar, say normal to $x'_3$, then the rank of $A(x)$ is clearly only $(m - n)$, where $n$ is the reduction in the number of basis due to the fact that $x'_3 = \text{constant}$. Thus, by simply monitoring the eigen-values of $A(x)$, and if a set of eigen-values becomes nearly or precisely zero, we automatically detect that $\partial \Omega_x$ is becoming nearly planar. In addition, it implies that the normal to the surface can be determined from the lowest eigenvalue of matrix $A(x)$ when it is singular or nearly-singular, without the local geometry information. It makes the present method to be truly meshless, which does need any background cells to define the geometry as well as the normal direction, if the boundary integrals are handled based on the nodal influence domain.

Once coefficients $a(x)$ in Eq. (15) are determined, one may obtain the approximation from the nodal values at the local scattered points, by substituting them into Eq. (13), as

$$u(x) = \Phi^T(x)\hat{u} \quad \forall x \in \partial \Omega_x \tag{17}$$

where $\Phi(x)$ is the so-called shape function of the MLS approximation, as,

$$\Phi(x) = p^T(x)A^{-1}(x)B(x) \tag{18}$$
The weight function in Eq. (14) defines the range of influence of node $I$. Normally it has a compact support. The possible choices are the Gaussian and spline weight functions with compact supports.

It should be pointed out that the shape functions given in Eq. (18) are based on the fictitious nodal values. This introduces an additional complication, since all the nodal values in BIEs are the direct boundary values, a situation which is totally different from the domain meshless methods. As a practical way, a conversion matrix is used to map the fictitious values to true values and applied to the system equations.

### 4.2. Variants of the MLPG/BIE: Several Types of Interpolation (Trial) and Test functions, and Integration schemes

The general MLPG/BIE approaches are given through Eqs. (10) and (12). For a so-called meshless implementation, we need to introduce a meshless interpolation scheme to approximate the trial functions over the surface of a three-dimensional body. Considering a meshless approximation, one may define the non-overlapping cells for the whole boundary as,

\[ \partial \Omega = \bigcup_{i=1}^{NC} \text{Cell}_i \quad \text{and} \quad \text{Cell}_i \cap \text{Cell}_j = 0 \quad \text{for} \quad \forall i \neq j \]  

where $NC$ is the number of cells. Thus, one may obtain the boundary integrals, in the meshless form, as,

\[ \int_{\partial \Omega} f(x, \xi) \, dS = \sum_{i=1}^{NC} \int_{\text{Cell}_i} f(x, s) \, ds \]  

in which the curvilinear coordinates $s$ are used.

In addition, if the variables are interpolated from the nodal ones as in Eq. (17) and the range of influence of each node is compact, one set of overlapped nodal domains may be defined as,

\[ \partial \Omega = \bigcup_{i=1}^{NN} \text{DOI}_i \quad \text{and} \quad \text{DOI}_i \cap \text{DOI}_j \neq 0 \quad \text{for} \quad \exists i \neq j \]  

where $NN$ is the number of nodes and $\text{DOI}_i$ is the influence domain of node $i$. The boundary integrals can be re-written in the truly meshless form, as,
\[
\int_{\partial \Omega} f(x, \xi) \, dS = \sum_{i=1}^{NN} \int_{\partial \Omega_i} f(x, s) \, ds \tag{22}
\]

This integration scheme has been applied to BNM for solving potential problems\(^5\).

Besides the traditional collocation methods, the weakly-singular BIEs in Eq. (11) can also be easily used for developing the meshless BIEs in their numerically tractable weak-forms. All piece-wise continuous functions can be used here as the test functions. For example, all the weight functions in MLS approximation discussed above, are suitable for such purposes. In addition, one may also directly use the nodal shape function as the nodal test functions as in Eq. (18), which leads to the symmetric BIEs in Eq. (12). In summary, the several variants, of the general MLPG/BIE approach, are defined through a suitable combination of the trial-function interpolation scheme, the choice for the test functions, and the choices for the integration scheme.

5. Numerical Experiments

Several problems in three-dimensional linear elasticity are solved to illustrate the effectiveness of the present method. The numerical results of the MLPG/BIE method as applied to problems in 3D elasto-statics, specifically (i) a cube, (ii) a hollow sphere, (iii) a concentrated load on a semi-infinite space, and (iv) non-planar fatigue growth of an elliptical crack, are discussed.

5.1. Cube under Uniform Tension

The first example is the standard patch test, shown in Figure 2. A cube under the uniform tension is considered. The material parameters are taken as \( E = 1.0 \), and \( \nu = 0.25 \). All six faces are modeled with the same configurations with 9 nodes. Two nodal configurations are used for the testing purpose: one is regular and another is irregular, as shown in Figure 2.
In the patch tests, the uniform tension stress is applied on the upper face and the proper displacement constraints are applied to the lower face.

The satisfaction of the patch test requires that the displacements are linear on the lateral faces, and are constant on the upper face; and the stresses are constant on all faces. It is found that the present method passes the patch tests. The maximum numerical errors are $1.7 \times 10^{-7}$ and $3.5 \times 10^{-7}$ for two nodal configurations, respectively, which may be limited by the computer.

5.2. Non-planar Crack Growth

An inclined elliptical crack with semi-axes $c$ and $a$, subjected to fatigue loading, is shown in Figure 3. Its orientation is characterized by an angle, $\alpha$. The present meshless method is applied to solve this problem. The nodal configuration is used to model the crack inclined at 45 degrees with 249 nodes. The exact solution for a tensile loading $\sigma$ is given in [10].

As a mixed-mode crack, the distribution of all three stress intensity factors, $K_I$, $K_{II}$ and $K_{III}$, along the crack front are shown in Figure 4, after being normalized by $K_0 = \sigma \sqrt{\pi a / 2}$. It can be seen that a good agreement of the present numerical results with the theoretical solution is obtained.

![Figure 3 Inclined elliptical crack under tension](image)

![Figure 4 Normalized stress intensity factors along the crack front of an inclined elliptical crack under tensile load](image)
The fatigue growth is also performed for this inclined crack. The Paris model is used to simulate fatigue crack growth, as:

\[ \frac{da}{dN} = C(\Delta K_{\text{eff}})^n \]  

in which the material parameters \( C \) and \( n \) are taken for 7075 Aluminum as \( C = 1.49 \times 10^{-8} \) and \( n = 3.21 \)\(^6\). The crack growth is simulated by adding nodes along the crack front. The newly added points are determined through the K solutions. 7 increments are performed to grow the crack from the initial size \( a = 1 \) to the final size \( a = 2.65 \). The normalized stress intensity factors during the crack growing are given in Figure 5, which are also normalized by \( K_0 \). The results show that \( K_I \) keeps increasing while \( K_{II} \) and \( K_{III} \) are decreasing during the crack growth. It confirms that this mixed-mode crack becomes a mode-I dominated one, while growing. The shape of the final crack is shown in Figure 6. It is clear that while the crack, in its initial configuration, starts out as a mixed-mode crack; and after a substantial growth, the crack configuration is such that it is in a pure mode-I state.

![Figure 5](image-url) Normalized stress intensity factors for the mixed-mode fatigue growth of an inclined elliptical crack
6. Closure

In this paper, we presented the “Meshless Local Petrov-Galerkin BIE Methods” (MLPG/BIE), by using the concept of the general meshless local Petrov-Galerkin (MLPG) approach developed in Atluri et al\textsuperscript{[1-4]}. The several variants of the MLPG/BIE solution methods are also formulated, in terms of the varieties of the interpolation schemes for trial functions, the test functions, and the integration schemes. With the use of a nodal influence domain, truly meshless BIEs are also presented. The MLS surface-interpolation, with the use of Cartesian coordinates, is enhanced for the three dimensional surface without the requirement of a mesh or cells, to define the local geometry. It leads to the truly meshless BIE methods with the use of the nodal influence domain for the boundary integrations. The accuracy and efficiency of the present MLPG approach are demonstrated with numerical results.

References

3. Atluri, S. N.; Shen, S.: The meshless local Petrov-Galerkin (MLPG) method: A simple & less-costly alternative to the finite element and


