ON SHEAR BAND FORMATION:  
I. CONSTITUTIVE RELATIONSHIP  
FOR A DUAL YIELD MODEL

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Abstract — A phenomenological constitutive relation, for capturing the shear band formation in a rate-independent elastic-plastic material, is established. The model takes into account both the J2-isotropic flow and a threshold shear stress-based flow. The elastic-plastic constitutive tensor is expressed explicitly in terms of elastic constants, the deviatoric stress tensor, the direction of the principal shear velocity-strain, and other material constants. This model particularly facilitates the resolution of the formation of the shear band even under material hardening conditions and does not demand an a priori knowledge of the orientation of the shear band. This is incorporated in an FEM, and the plane strain tensile test of ANAND and SPITZIG [1980] is numerically simulated. The computed results compare favorably with the experimental data. The shear band emerges more naturally as a solution to the boundary value problem, unlike the situations in solutions based on classical bifurcation methods. Nevertheless, the usefulness of the local instability condition (ORTIZ et al. [1987]) is also demonstrated.

I. INTRODUCTION

Fracture of materials due to localization of strain has attracted considerable attention, and a number of experimental and theoretical investigations have been carried out. Many of these studies were based on the localization occurring in an elastic-plastic specimen under uniaxial loading. Necking and shear band formation are the two commonly observed localization modes in such cases, and both these modes are understood to be outcomes of instability.

Based on the framework proposed by HILL [1958, 1961], a number of studies (NEEDLEMAN [1972]; HUTCHINSON & MILES [1974]; MILES [1975]; BURKE & NIX [1979]; MURAKAWA & ATLURI [1979]; REED & ATLURI [1983]) have been carried out to analyze the phenomenon of necking with an emphasis on the instability criterion. Even though the necking phenomenon can be reasonably described using an isotropic J2 flow theory with a smooth yield locus, the J2-criterion is found to be too stiff in describing the formation of the shear band. Consequently, more general constitutive theories that accommodate nonnormality, vertex formation, and so forth were introduced (RUDNICKI & RICE [1975]; NEEDLEMAN [1979], TVERGAARD et al. [1981a]; IWAKUMA & NEMAT-NASSER [1982]). Also, the consideration of macroscopic strain softening that is justifiable in porous materials, is shown to facilitate the shear band formation (TVERGAARD [1981b, 1987]; BAZANT [1988a, 1988b]). In this article, we introduce a new type of constitutive model, a dual-yield model, based more on certain phenomenological considerations.
ANAND and SPITZIG [1980] and KORBEL and MARTIN [1988] experimentally analyzed the shear band formation in maraging steel and in prestrained mild steel, respectively. These investigations, as well as the results of HATHERLY and MALIN [1984] and HUANG and GRAY [1989] conclusively demonstrate a noncrystalline nature of the shear band formation with microscopic evidence; i.e. the shear band runs across several grains without any deviation, irrespective of the orientation of the primary slip planes in each individual grain. Guided by this experimental observation, we hypothesize that the process of shear localization is a directionally preferred deformation, even in a continuum equivalent of a material. Consequently, a type of “directionally preferred strain” becomes a characteristic of the shear band. We believe that the sluggishness of the $J_2$-flow condition in capturing the shear band formation in a tensile specimen is due to its inadequacy in accounting for this “directionally preferred strain.” $J_2$-flow, being an energy-based criterion, implicitly averages the material response in all spatial directions, and, therefore, perhaps becomes too stiff in describing the directionally preferred phenomenon, i.e. intense plastic flow on the plane of shear band.

The appropriateness of the introduction of the “directionally preferred strain” can be understood in a different perspective as well. The earlier finite element simulations, which assumed only a $J_2$-flow theory for the constitutive description of the material, relied on bifurcation procedures to locate or orient the shear band and subsequently employed special elements to capture the shear localization. Even though the necking is also such an instability phenomenon, it is well known that the simulation of necking does not call for such a two-step procedure. Why? Perhaps, the averaging nature of the $J_2$-flow does not affect the necking being captured more naturally as a solution to the boundary value problem. In contrast to the earlier simulations, the finite element simulations of single crystal deformation of PEIRCE et al. [1982, 1983], DEVÉ et al. [1988] and HARREN et al. [1988], performed based on a Schmid's-law directed frame-work of HILL and RICE [1972], ASARO and RICE [1977], and ASARO [1979], capture shear localization as a natural solution. In our opinion, the reason for this lies in the nature of Schmid's constitutive law, which supports a slip-based “directionally preferred deformation” in single crystals and as a result of their accounting for the appropriate rotational features associated with the lattice structure.

Having recognized the need for the introduction of such a “directionally preferred strain” in the present analysis, our next concern is to account for it with a suitable yield model. It is possible to modify the $J_2$-flow condition for this purpose, as in many other earlier investigations, but we choose to retain the Mises condition as it is and add another threshold shear stress-based yield condition to account for the “directionally preferred strain” so that the model is built on a phenomenological foundation. This is somewhat analogous to the model of HILL and HUTCHINSON [1975] and also of ZBIB and AIFANTIS [1988], who employed two different elastic shear moduli; one for a direction parallel to the plane of the shear band and another for a direction perpendicular to it in their analysis of an elastic rubber-like material. In our model, we assume that the position as well as the movement of each yield function in stress space is independent of the other yield condition. This assumption implies that the results of the commonly used uniaxial tensile test and the pure shear test can be complementary and not necessarily mutually redundant for describing a material’s constitutive response. Also, we show that this model essentially corresponds to a particular type of nonproportional loading, leading to a vertex forming yield surface, and does not violate the normality and plastic incompressibility conditions.
II. DUAL YIELD MODEL

II.1. Directionally preferred velocity strain

Here we implicitly restrict our attention to planar deformations, without much loss of generality. The total velocity strain is additively decomposed into reversible elastic and irreversible plastic components, and the latter is further decomposed into two parts: (i) the velocity strain dictated by the Levy–Mises theory and (ii) the directionally preferred velocity strain described in the introduction. Thus we postulate that,

\[ D_{ij} = (D_{ij})_e + (D_{ij})_p + (D_{ij})_s, \]

where \( D_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}) \); the subscript e refers to elastic; p refers to plastic as determined by the conventional \( J_2 \)-flow theory; s refers to the directionally preferred plastic component; and \( v \) refers to velocity of the material particle in the global cartesian system \( x_i \).

Let the inclination of the plane of the maximum shear velocity strain in a differential material element, to the \( x_1 \) axis of the global coordinate system, be \( \theta \), as shown in Fig. 1. The orientations of the planes of maximum shear velocity strains in the neighbouring differential elements in a continuum may, in general, be different. However, when such maximum-shear velocity strain planes align in the same orientation on a global plane, we show that the shear band is nucleated on that global plane.

Consider another coordinate system with its \( x'_1 \) axis placed along the plane of maximum shear velocity strain and the \( x'_3 \) axis along \( x_3 \) axis. In the coordinate system \( (x'_i) \), let \( (D_{12})_s \) and \( (D_{21})_s \) be the only non-zero components of the “directionally preferred velocity strain.” We postulate the evolution equations for these strains as,

\[ (D'_{12})_s = \lambda_s \tau'_{12} \]

\[ = (D'_{21})_s = \lambda_s \tau'_{21}. \]
(In this model we have assumed $\tau'_{12} = \tau'_{21}$), where $\lambda_s$ is a material dependent scalar parameter and $\tau'$ is Kirchhoff stress tensor expressed in the ($x''$) coordinate system. Therefore,

$$ (D_{ij})_s = \lambda_s U_{ij} \tau'_{12}, \tag{4} $$

where $U_{ij} = 0$ except $U_{12} = U_{21} = 1$. The stress and the shear velocity strain components in the different coordinate systems are related by the following coordinate transformations:

$$ (D_{ij})_s = A_{mi} A_{nj} (D'_{mn})_s \tag{5} $$

$$ \tau'_{12} = A_{1k} A_{2l} \tau_{kl} \tag{6} $$

where $A_{ij}$ is the coordinate transformation tensor. $A_{11} = A_{22} = \cos \theta$; $A_{22} = -A_{21} = \sin \theta$; and the rest of the terms of $A_{ij}$ are zero. Combining eqns (4), (5), and (6) we obtain,

$$ (D_{ij})_s = \lambda_s A_{mi} A_{nj} U_{mn} A_{1k} A_{2l} \tau_{kl}, \tag{7} $$

which simplifies to,

$$ (D_{ij})_s = \lambda_s M_{ijkl} \tau_{kl}, \tag{8} $$

where,

$$ M_{ijkl} = A_{1k} A_{2l} (A_{1i} A_{2j} + A_{2i} A_{1j}). \tag{9} $$

It must be noted here that these directionally preferred velocity strain does not contribute to any volume change. An examination of both $(D_{ij})_p$ and $(D_{ij})_s$ would reveal that this type of decomposition of irreversible plastic velocity strain is simply one of the ways of introducing nonproportional loading. Nevertheless, we believe that the merit, for this type of decomposition of the plastic velocity strain, lies in its phenomenological reasoning and the convenience of mathematical operations.

II.2. Shear yield function and potential

To distinguish the special yield function of the present study from the Mises ($J_2$-flow) yield function, we designate it as "Shear Yield function $F_s". It is expressed as,

$$ F_s = \left( \frac{1}{2} \right) \{(\tau'_{12})^2 + (\tau'_{21})^2 \} - \tau_m^2 = 0, \tag{10} $$

where $\tau_m$ is a material parameter representing the critical true shear stress at which the directionally preferred shear-flow takes place on the preferred plane of orientation, i.e. the plane of maximum shear velocity strain in each different material element. If we define a special shear potential $Q_s$, also with the simplification of eqn (10) using $\tau'_{12} = \tau'_{21}$, as

$$ Q_s = F_s = A_{1i} A_{2j} A_{1k} A_{2l} \tau_{ij} \tau_{kl} - \tau_m^2 \tag{11} $$
where \( A_{mn} \) are the direction cosines of the axis of maximum shear velocity strain with respect to the chosen global axis \( x_i \). It can be easily shown that,

\[
(D_{ij})_s = \lambda_s \left\{ \frac{\partial Q_s}{\partial \tau_{ij}} \right\}.
\]  

(12)

Eqn (12) implies that the directionally preferred velocity strain is normal to the special shear yield surface. It should be noted that although the special yield function and the potential, \( F_s \) and \( Q_s \), respectively, depend on the orientation of the maximum shear velocity strain with respect to the chosen global axes, they are, nevertheless independent of the orientation of the global coordinate system.

The physical significance of the shear potential \( Q_s \) can be understood by examining the first term of the RHS of eqn (10). This type of plastic strain does not violate the incompressibility requirement, and, consequently, the distorsional energy criterion associated with the plastic deformation is also preserved.

II.3. Constitutive relationship

In the last subsection, we established a special shear yield function and the potential, which are identical to each other. Also, the directionally preferred velocity strain is established as a function of the shear potential in eqn (12). The plastic velocity strain component of the conventional J2-flow theory is given as,

\[
(D_{ij})_p = \lambda_p \left\{ \frac{\partial Q_p}{\partial \tau_{ij}} \right\},
\]  

(13)

where the yield function and the potential are,

\[
F_p = Q_p = \frac{1}{2} \tau_{ij} s_{ij} - \left( \frac{1}{2} \sigma_m^2 \right) = 0
\]

\[= \frac{1}{2} s_{ij} s_{ij} - \left( \frac{1}{3} \right) \sigma_m^2 = 0,
\]  

(14)

where \( \lambda_p \) is a scalar function, \( s_{ij} \) is the deviatoric component of Kirchhoff stress, and \( \sigma_m \) is uniaxial equivalent stress determined experimentally.

Using the potentials, yield functions, and the velocity strain components expressed in eqns (10) to (14), it is possible to establish a constitutive equation in rate form relating an objective stress rate and the total velocity strain as (the derivation is provided in the Appendix),

\[
(\dot{\tau}_{ij})_0 = E_{ijkl} D_{kl},
\]  

(15)

where the constitutive tensor turns out to be symmetric, and it is expressed in an explicit form as,

\[
E_{ijkl} = C_{ijkl} + (k_1/k_0) s_{ij} s_{kl} + (k_2/k_0) (s_{ij} R_{kl} + R_{ij} s_{kl}) + (k_3/k_0) R_{ij} R_{kl}.
\]  

(16)

\( C_{ijkl} \) is isotropic elastic constitutive tensor; \( s_{ij} \) is the deviatoric stress tensor; \( R_{ij} \) is expressed in terms of the coordinate transformation tensor \( A_{ij} \) (described in eqn 6),

\[
R_{ij} = A_{1i} A_{2j} + A_{2i} A_{1j}.
\]  

(17)
The material constants are,

\[ k_0 = \frac{\sigma_m^2}{9\mu^2}(3\mu + H_p)(\mu + H_t - \tau^\mu_{12}) - (\alpha_p \cdot \alpha_s)\tau_m \]

\[ k_1 = - (\alpha_p)(\mu + H_t - \tau^\mu_{12}) \]

\[ k_2 = (\alpha_p \cdot \alpha_s)\mu \tau_m \]

\[ k_3 = - (\alpha_s)(\sigma_m^2/9)(3\mu + H_p), \]

\[ \mu \] is elastic shear modulus; \( H_p \) is the tangent modulus of the stress–plastic strain variation of the uniaxial tension test, and \( H_t \) is the tangent modulus of the variation of the shear stress—directionally preferred shear strain along the shear plane, obtained in a pure-shear experiment. \( \sigma_m \) and \( \tau_m \) are flow stresses corresponding to the two types of material yielding. \( \tau^\mu_{12} \) is the shear stress in a coordinate system with its \( x^\mu_i \) inclined at an angle of \( (\pi/4 - \theta) \) with the global \( x \) and \( x^\mu_3 \) placed along \( x_3 \), and this term arises due to the rotation of the shear band.

Here \( \alpha_p \) and \( \alpha_s \) are binary switches, which depend on the yield conditions as given below (for monotonic loading):

\[ \alpha_p = 0 \text{ when } \epsilon_m \leq [\sigma_m/(2\mu(1+\nu))] \text{ or } (d\epsilon_m/dt)_p < 0 \]

\[ = 1 \text{ when } \epsilon_m > [\sigma_m/(2\mu(1+\nu))] \text{ and } (d\epsilon_m/dt)_p \geq 0 \]

and

\[ \alpha_s = 0 \text{ when } \gamma_m \leq (\tau_m/\mu) \text{ or } (d\gamma_m/dt)_s < 0 \]

\[ = 1 \text{ when } \gamma_m > (\tau_m/\mu) \text{ and } (d\gamma_m/dt)_s \geq 0. \]

For \( \alpha_p = 1 \) and \( \alpha_s = 0 \), \( E_{ijkl} \) reduces to the conventional constitutive matrix of the Mises plasticity (J2-flow); for \( \alpha_p = 0 \) and \( \alpha_s = 1 \), \( E_{ijkl} \) reduces to a case corresponding to an exclusively directionally preferred deformation and \( \alpha_p = \alpha_s = 1 \) represents certain special cases where both the types of yield conditions operate concomitantly.

II.4. Objectivity

Objectivity of a stress-rate in its simplest physical interpretation implies that the components of the stress-rate should be measured in a coordinate system that is participating in the rotational motion of a material element and not in a spatially fixed coordinate system. Since there is an infinite variety of coordinates that participate in the motion of the material element, there is an infinite variety of objective stress-rates. In the following text, we try to pick an objective stress-rate that is meaningful in the present context.

First consider the spin corresponding to the velocity strain \( D \) of eqn (1), which can be defined as,

\[ \omega_{ij} = \left( \frac{1}{2} \right) (v_{i,j} - v_{j,i}) \]
such that,

\[ u_{i,j} = D_{ij} + \omega_{ij}. \]  \hspace{1cm} (19)

It is well known from the kinematics of deformation that \( \omega_{ij} \) can be interpreted as: (i) the angular velocity of the principal directions of \( D_{ij} \) or (ii) half the relative rotation in the counterclockwise direction of a material element lying along the \( x_1 \) and \( x_2 \) directions (see Fig. 1). Since \( \omega_{ij} \) is a skew symmetric tensor, it can be written as:

\[ \omega_{ij} = e_{ijk} \omega_k. \]

In the present two-dimensional context,

\[ \omega_{ij} = e_{ij3} \omega \quad (i,j = 1,2); \quad e_{ij3} = \text{cyclic tensor}, \]

where \( \omega \) can be interpreted as half the relative rotation of a material element lying along \( x_1 \) and \( x_2 \) directions, due to the total velocity gradient \( (v_{i,j}). \) On the other hand, if we use the decomposition,

\[ (v'_{i,j})_s = (D'_{ij})_s + \omega'^*_{ij}, \]  \hspace{1cm} (20)

where

\[ \omega'^*_{ij} = (\frac{1}{2}) [(v'_{i,j})_s - (v'_{j,i})_s]. \]  \hspace{1cm} (21)

Thus \( \omega'^*_{ij} \) characterizes the relative rotation of the material element due to the “directionally preferred deformation” alone. \( \omega'^*_{ij} \) do not contribute to the rigid body rotation of the material.* Therefore, to obtain a meaningful objective stress-rate in the present context, we measure the components of the stress-rate in a coordinate system spinning at the rate \( (\omega_{ij} - \omega'^*_{ij}). \) Thus, the objective stress-rate is defined as,

\[ (\dot{\mathbf{t}}_{ij}) _0 = \dot{\mathbf{t}}_{ij} - (\omega_{ik} - \omega'^*_{ik}) \tau_{kj} + (\omega_{kj} - \omega'^*_{kj}) \tau_{ik}. \]  \hspace{1cm} (22)

II.5. Yield surface

In this section we present the graphical representation of the dual yield model for plane stress condition. Consider a coordinate system inclined at \( \phi = \gamma \) in which the shear stress components vanish and \( \tau^o_{22} \) and \( \tau^o_{11} \) represent principal stresses. The shear-yield function can be written as,

\[ \tau^o_{22} - \tau^o_{11} = \pm 2\tau_m/\sin 2(\theta - \gamma). \]  \hspace{1cm} (23)

The locus of the shear-yield function expressed in eqn (23) superimposed on the \( J_2 \)-based yield loci for plane stress condition is depicted in Fig. 2. As per our original assumption, both the yield functions can be independent of each other, and the move-

*This is because the “directionally preferred shear deformation” does not rotate the material axis \( (x_1). \) \( \omega'^*_{ij} \) show the spin \( (-\dot{\theta}). \) \( \dot{\theta} \) is given in eqn. (A15).
Fig. 2. Representation of the yield surfaces corresponding to \( J_2 \)-flow and the special shear stress based yield condition (plane stress). I: \( (r_{22}^2 - r_{11}^2)^2 + (r_{11}^1)^2 = (\sigma_m)^2 \) (\( J_2 \)-flow). II: \( r_{22}^2 - r_{11}^1 = \pm 2\tau_m/\sin (2\theta - \eta) \) (Shear-yield).

Since the rotations of both the surfaces in the stress space can be mutually independent. The \( J_2 \)-ellipse expands radially when there is material hardening. In the case of shear-yield, planar surface moves away from the origin for material hardening.

Let us describe a characteristic angle associated with the formation of the shear band to be,

\[ \beta = |\theta - \eta| - \pi/4 \]

for \( 0 \leq \theta < \pi/2 \) and \( 0 \leq \eta < \pi/2 \). This characteristic angle is nothing but the angle between the principal shear stress and the principal shear strain. When \( \beta \) equals zero, eqn (23) reduces to a tresca-type condition in two dimensions. Increase in \( \beta \) is associated with a type of geometric hardening.

The constitutive tensor as expressed in eqn 16 has four additive terms in the RHS. For a purely elastic deformation (\( \alpha_p = 0 \) and \( \alpha_s = 0 \)) only the first term becomes operational. When the plastic deformation is entirely dictated by the Mises plasticity (\( \alpha_p = 1 \) and \( \alpha_s = 0 \)) the first and the second terms become operational and the other terms vanish. This implies that the Mises ellipse falls well between the parallel lines of the special shear based yield function. If the special yield function totally controls the plastic deformation (\( \alpha_p = 0 \) and \( \alpha_s = 1 \)), only the first and the fourth terms become functional. This means that the parallel lines of the special yield function intersect the Mises ellipse, forming vertices, and the deformation moves on any of the linear segments of the bounded surface. The case where \( \alpha_p = 1 \) and \( \alpha_s = 1 \) is a special one corresponding to any of the vertices of the yield surface, and this case activates all the four terms in...
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the constitutive tensor. Whenever there is a transition from one yield function to the other, at the time of the transition this special case remains functional.

III. DISCUSSION

III.1. Validation

In this section, we validate our phenomenological constitutive model using the experimental results of ANand and SPritzig [1980]. They performed a particular type of plane-strain tension test using maraging steel specimens of aspect ratio 4 and recorded the overall fracture strain and the angle of orientation of the shear band at the time of fracture. They presented the material behaviour as a plot between the true stress and the true strain.

We incorporated the dual yield constitutive model in an FEM algorithm and simulated the test carried out by ANAND and SPITZIG [1980]. The details of the finite element procedure employed are presented in the accompanying paper. Considering symmetrical deformation, only the first quadrant of the specimen geometry is analyzed. The nodes falling on the $x_1$-axis are constrained in $x_2$-direction, and those falling on the $x_2$-axis are constrained in $x_1$-direction. The top surface EF, shown in Fig. 1, is subjected to

![Fig. 3. Neck formation in a tensile specimen strained to (a) 0.04; (b) 0.08; (c) 0.12; (d) 0.16 (considering only $J_2$ flow condition).](image)
step-wise incremental displacements in the $x_2$-direction. Experimental material data are supplied using the following empirical equations:

$$\sigma_m = k_p \epsilon_m^{n_1}$$  \hspace{1cm} (25)

$$\tau_m = k_s \gamma_m^{n_2}$$  \hspace{1cm} (26)

where, $k_p = \sigma_y / \left( \sigma_y / (2 \mu(1 + \nu)) \right)^{n_1}$ and $k_s = \tau_y / \left( \tau_y / \mu \right)^{n_2}$. Here, $\sigma_m$ and $\tau_m$ are effective uniaxial stress and equivalent shear stress, respectively, and the subscript $y$ corresponds to yield conditions in the respective cases; $\epsilon_m$ and $\gamma_m$ are effective uniaxial strain and shear strain, respectively; $n_1$ and $n_2$ are material hardening exponents.

We considered three different cases of the simulation. In the first case, the material input is chosen in such a way that only the $J_2$-flow condition becomes functional, and, similarly, in the second case, only the special yield is functional. But the third case allows both the flows. In all these cases, a Young’s modulus of 207 Gpa and a Poisson’s ratio

\begin{verbatim}
1 0.4967E-01
2 0.1332E+00
3 0.2168E+00
4 0.3003E+00
5 0.3839E+00
6 0.4674E+00
7 0.5510E+00
8 0.6345E+00
9 0.7181E+00
10 0.8016E+00
\end{verbatim}

\begin{verbatim}
1 0.8298E-01
2 0.2258E+00
3 0.3687E+00
4 0.5115E+00
5 0.6544E+00
6 0.7973E+00
7 0.9401E+00
8 0.1083E+01
9 0.1226E+01
10 0.1369E+01
\end{verbatim}

Fig. 4. Strain contours in a tensile specimen strained to 0.16 (considering only $J_2$-flow condition). (a) Effective strain ($\epsilon_m$); (b) Effective shear strain ($\gamma_m$).
of 0.3 are used. In case 1, \( \sigma_y \) is assigned a value of 1539 Mpa with the exponent \( n_1 = 0.0335 \) and in case 2, \( \tau_y = 866.2 \) Mpa and \( n_2 = 0.0366 \). In case 3, \( \sigma_y \) and \( n_1 \) were same as that of case 1, but \( \tau_y \) and \( n_2 \) are set to 940 Mpa and zero, respectively. The material properties for case 1 and case 2 are chosen in such a way that they exactly satisfy the stress–strain variation published by Anand and Spitzig [1980] for plane strain condition and, in case 3, though not exact, nearly correspond to it. In case 1, a reducing taper from loading ends to the center is maintained by keeping the width GH (Fig. 1) to be 2% less than EF to trigger necking at GH. In cases 2 and 3, another imperfection in the form of a second phase is also incorporated at the center, with the shear strength of the second phase being 50% of the matrix phase. Case 1 has no second phase imperfection.

The results are shown in Figs. 3–8 corresponding to cases 1, 2, and 3 for total engineering tensile strain from 0.0 to 0.16. In Fig. 3a–d, the deformed mesh clearly shows the development of necking. Also, the effective plastic strain contour, which is morphologically identical to the effective shear strain (shear strain on the plane of principal shear strain) contour, is shown in Fig. 4. This is always the case for \( J_2 \)-plasticity. This indicates an absolute lack of tendency for shear band formation. On the other hand, the deformed configurations shown in Fig. 5a–d indicate the emergence of a shear band in a pronounced form with absolutely no necking. The effective strain contours in Fig. 6 show a strong bias toward the shear band formation analogous to the bias in Fig. 4 for the neck formation. Actually, Case 3 is a more realistic one and is used for the experimental verification of the constitutive relationship of this study. The deformed config-

![Fig. 5. Shear Band formation in a tensile specimen strained to (a) 0.04; (b) 0.08; (c) 0.12; (d) 0.16 (considering only the Shear stress based yield condition).](image-url)
urations in Fig. 7a–d display both diffused necking and shear band formation. The angle of orientation of the shear band with the tensile axis (~39°) and the reduction of neck width, reasonably agree with the experimental results for the overall tensile strain 0.16. The contour diagrams in Fig. 8a–d show a gradual transition from the necking mode to the shear band mode as the operating flow type switches from \( J_2 \)-based yield to the threshold shear yield behaviour.

III.2. Kinematic effects

The kinematics of the shear band formation can be examined using a vector plot depicting the orientation of the shear plane in each element, as shown in Fig. 9a. At the very beginning of the tensile deformation of the specimen, at almost all the locations in the specimen, the principal direction of the shear strain is found to be around 45° and well aligned throughout the specimen. With the progress of the tensile elongation (Fig. 9b–d), the principal directions of the strain rotate by different angles at different

![Fig. 9a](image)

![Fig. 9b](image)

Fig. 9. Vector plot showing the orientation of the shear plane in each element. (a) At the very beginning of the tensile deformation, the principal direction of the shear strain is around 45°. (b) With the progress of the tensile elongation, the principal directions of the strain rotate by different angles at different locations.

![Fig. 9c](image)

![Fig. 9d](image)

Fig. 9. Vector plot showing the orientation of the shear plane in each element. (c) At the very beginning of the tensile deformation, the principal direction of the shear strain is around 45°. (d) With the progress of the tensile elongation, the principal directions of the strain rotate by different angles at different locations.

![Fig. 9e](image)

![Fig. 9f](image)

Fig. 9. Vector plot showing the orientation of the shear plane in each element. (e) At the very beginning of the tensile deformation, the principal direction of the shear strain is around 45°. (f) With the progress of the tensile elongation, the principal directions of the strain rotate by different angles at different locations.

![Fig. 9g](image)

![Fig. 9h](image)

Fig. 9. Vector plot showing the orientation of the shear plane in each element. (g) At the very beginning of the tensile deformation, the principal direction of the shear strain is around 45°. (h) With the progress of the tensile elongation, the principal directions of the strain rotate by different angles at different locations.

![Fig. 9i](image)

![Fig. 9j](image)

Fig. 9. Vector plot showing the orientation of the shear plane in each element. (i) At the very beginning of the tensile deformation, the principal direction of the shear strain is around 45°. (j) With the progress of the tensile elongation, the principal directions of the strain rotate by different angles at different locations.
locations, essentially leading to heterogeneity in deformation. When the principal directions at two adjacent material points align favorably parallel to one another, the material can shear easily from one point to another. Let us refer to this phenomenon as "kinematic docking." On the other hand, when they are misaligned, the material-shear from one point to another is restrained to the extent of misalignment, which can be referred to as "kinematic locking." Therefore, the nucleation of the shear band can be visualized as a type of kinematic docking of the principal shear plane at every material location with that of the adjacent locations along the plane of the shear band. Then, the shear flow can take place easily across the entire specimen. Though the kinematic locking and the kinematic docking processes are the results of a pure continuum analysis, they seem to offer an explanation for the microscopic observation of the shear bands formed across several grains without any deviation in certain polycrystalline materials.

III.3. Instability analysis

Many investigators employ instability based procedures to locate or orient the shear band in their simulation whereas, in this study, the constitutive equation used and the consideration of imperfections are such that they obviate the need for any such insta-
bility analysis as a necessary step for the simulation; the band appears more naturally as a solution to the boundary value problem.

Nevertheless, to investigate the local instability features associated with this deformation as predicted by the dual yield model, we used the function presented in appendix-2 of Ortiz et al. [1987]. This function was derived to detect jump discontinuity in strain-rate space arising as a result of local constitutive breakdown. The function is expressed in terms of the constitutive tensor and the orientation of the plane across which the discontinuity is exhibited. That is,

$$\kappa = f(Z_{ijkl}, \phi)$$

(27)
where $Z_{ijkl}$ is constitutive tensor and $\phi$ is the angle. Let us designate $\kappa$ as O-L-N parameter and $\phi$ as O-L-N angle. When the value of $\kappa$ becomes negative at any material point for a given $Z_{ijkl}$ and $\phi$, the jump discontinuity is indicated. When we substituted the present constitutive tensor ($Z_{ijkl} = E_{ijkl}$) in eqn (27), $\kappa$ remained always positive. Therefore we obtained the nontrivial $\phi$ for which $\kappa$ becomes minimum in each element. The vector plot of these angles is shown in Fig. 10. The clear delineation of the shear band from the rest of the material points in the specimen amply proves the usefulness of O-L-N parameter. While ORTIZ et al. [1987] assumed $J_2$-softening material, we have considered hardening material and this perhaps explains why $\kappa$ dipped down to negative values in their case but not in ours. Also, it is interesting to note that the nontrivial $\phi$ in the shear band locations turns out to be the angle of orientation of the plane of principal shear strain itself (Fig. 10).

IV. SUMMARY

In this investigation, a material undergoing plastic deformation is hypothesized to satisfy not only the $J_2$-based hardening flow rule but also a threshold shear stress-based
flow on the plane of principal shear strain. The second yield function accounts for the directionally preferred strains characteristic of the shear band. The total plastic velocity strain is additively decomposed into two components, and these are related to the respective yield potentials through scalar constants. Neither of the components affects the volume-constancy condition. Basically, this type of decomposition subdivides the total irreversible distortional energy into two parts corresponding to two different yield functions. Implicitly, this manipulation can be shown to be a particular type of non-proportional loading leading to vertex formation in the yield surface. It is shown that both the types of incremental strain are normal to the respective yield surfaces, but the normality of the overall plastic strain may not be satisfied. Employing these potentials, yield functions, and the velocity strain components, we established a constitutive equation relating an objective stress rate and velocity strain. The constitutive tensor is symmetrical and is explicitly expressed in terms of elastic constants, deviatoric stress state, orientation of the plane of principal shear strain and material constants. Also, the objective stress rate is expressed in terms of Jaumann stress rate and a correction stress rate to account the difference between the continuum and the structural spins exclusively associated with the directionally preferred deformation.

Fig. 10. Vector plot of Ortiz-Leroy-Needleman (O-L-N) instability parameter for a tensile specimen strained to (a) 0.02; (b) 0.04; (c) 0.06; (d) 0.08.
We incorporated this constitutive equation in a Finite Element Method capable of handling finite strains and rotations and numerically simulated the tensile test conditions published by Anand and Spitzig [1980]. The shear band emerges at a reasonable overall tensile strain as observed in the experiments and the numerically predicted orientation of the shear band agrees with the experimental result satisfactorily. The numerical simulation shows the simultaneous development of the shear band and the diffused necking as observed in the experiment. At the shear band locations, there seems to be a perfect alignment of the plane of principal shear strains (kinematic docking) allowing the material to flow easily by shear, and in all the other locations such an alignment is missing (kinematic locking). This offers a possible explanation for the microscopically observed phenomenon of shear band running across several grains without any deviation in polycrystalline material.

Though the directionally preferred strain is restricted to be planar, the model appears to have the merit of being simple and ideally suitable for plane-strain or plane-stress conditions. Also, it does not demand an a priori knowledge of the orientation of the shear band. There is no need for invoking any bifurcation analysis to locate or orient the shear bands, and the bands appear more naturally as a solution to the boundary value problem. Therefore, we hope that the procedure can be easily upgraded for analyzing the shear band formation in more general cases with complex geometries and cases involving complex boundary conditions.

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NOMENCLATURE

Representation of different coordinate systems

\( ^\circ \) = associated with a cartesian coordinate system with its \( x_3^\circ \) axis placed along the direction of the maximum shear velocity strain (at an angle \( \theta \) to the global \( x_1 \) measured in anticlockwise direction) and \( x_1^\circ \) placed along \( x_3 \)

\( ^\circ \) = a coordinate system with its \( x_3^\circ \) axis placed along an arbitrary direction, making an angle \( \phi \) with the global \( x_1 \) and \( x_2^\circ \) along \( x_3 \)

\( ^\circ \) = coordinate system corresponding to the principal directions of Cauchy stress

\( ^\# \) = coordinate system with its \( x_3^\# \) along a direction making an angle of \( (\pi/4 - \theta) \) with \( x_1 \) and \( x_2^\# \) placed along \( x_3 \)
Subscripts

e = elastic
J = Jaumann rate
m = material parameter or experimental equivalent parameter
p = plastic strain associated with Mises flow
s = plastic strain associated with directionally preferred deformation introduced in the present study
y = yield condition

General

A
Cijkl = isotropic elastic constitutive tensor
Dijkl = velocity strain tensor; appears with subscripts e, p, and s
Eijkl = dual yield constitutive tensor
Fp, Fp = yield functions
Hp, Hs = tangent modulus of experimental stress–plastic strain variation
k = (i) constants appearing in power-law equations (with subscripts p or s)
(ii) constant parameters appearing in the constitutive equation (with subscripts 0, 1, 2, or 3)

Mijkl = defined in eqn (9)
n1, n2 = hardening exponents appearing in power-law equations
Qp, Qs = flow potentials
sij = deviatoric stress tensor
Rij = a tensor appearing in eqn (17)
Uij = a tensor defined in eqn (4)
u = displacement
Wp, Ws = work done

\alpha_p, \alpha_s = binary switch pertaining to yield conditions
\beta = angle subtended by the directions of principal shear stress and shear strain
\epsilon_m = uniaxial effective strain

\gamma_m = effective engineering shear strain

\phi = an arbitrary angle; O-L-N angle wherever applicable

\kappa = O-L-N parameter

\eta = angle of orientation of the principal stress system

\lambda = Lame's constant

\lambda_p, \lambda_s = scalar proportionality parameters

\nu = Poisson's ratio

\mu = shear modulus

\tau_{ij} = Kirchhoff stress tensor

\tau_m = equivalent shear stress

\sigma_m = uniaxial equivalent stress

\theta = angle of inclination of the plane of shear band with respect to x_1-axis of the global reference coordinate system, measured in anticlockwise direction in x_1-x_2 plane or the direction of the principal shear velocity strain

APPENDIX

As discussed in section II.1, the total velocity strain is additively decomposed into elastic, plastic (Mises), and directionally preferred plastic components as,

\[ D_{ij} = (D_{ij})_e + (D_{ij})_p + (D_{ij})_s, \]  

(A1)
where $D_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i})$. The plastic potentials and the yield functions as described in subsection II.2 are,

$$Q_p = F_p = \frac{1}{2} \tau_{ij} s_{ij} - \left( \frac{1}{2} \right) (\sigma_m)^2 = 0 \tag{A2}$$

$$Q_s = F_s = \frac{1}{2} \left[ (\tau_{12}^2) + (\tau_{21}^2) \right] - (\tau_m)^2 = 0, \tag{A3}$$

where $\tau_{ij}$ is Kirchhoff stress tensor; $s_{ij}$ is the deviatoric stress tensor; $\sigma_m$ and $\tau_m$ are uniaxial and shear equivalent stresses obtained experimentally. $\tau_{12}'$ and $\tau_{21}'$ are shear stress components in the coordinate system (comprising of the axis of maximum shear velocity strain and its normal) and are equal.

$$\tau_{12}' = \tau_{21}' \tag{A4}$$

It is known that,

$$(D_{ij})_p = \lambda_p \frac{\partial Q_p}{\partial \tau_{ij}}, \tag{A5}$$

and it has been shown in subsection II.2 that,

$$(D_{ij})_s = \lambda_s \frac{\partial Q_s}{\partial \tau_{ij}}. \tag{A6}$$

Let us denote the objective stress rate in this case be ($\dot{\tau}$); then,

$$(\dot{\tau}_{ij})_0 = \lambda \delta_{ij} D_{kk} + 2\mu (D_{ij})_e, \tag{A7}$$

where $\lambda$ and $\mu$ are elastic constants.

Consider the $J_2$-flow and the special shear yield functions to be functions of certain variables:

$$F_p(\tau_{ij}, W_p) = 0 \tag{A8}$$

$$F_s(\tau_{ij}, W_s, \theta) = 0. \tag{A9}$$

$W_p$ and $W_s$ represent the work done in producing the corresponding strains. The total differentials of $F_p$ and $F_s$ result in,

$$\dot{F}_p = (\partial F_p / \partial \tau_{ij}) (\dot{\tau}_{ij})_0 + \partial F_p / \partial W_p \cdot \dot{W}_p = 0 \tag{A10}$$

$$\dot{F}_s = (\partial F_s / \partial \tau_{ij}) (\dot{\tau}_{ij})_0 + \partial F_s / \partial W_s \cdot \dot{W}_s + \partial F_s / \partial \theta \cdot \dot{\theta} = 0. \tag{A11}$$

Substituting (A.5) and (A.6) in (A.1) and combining it with (A.7) and also using $(D_{kk})_p = (D_{kk})_s = 0$, we obtain,

$$(\dot{\tau}_{ij})_0 = \lambda \delta_{ij} D_{kk} + 2\mu \left[ D_{ij} - \lambda_p (\partial F_p / \partial \tau_{ij}) - \lambda_s (\partial F_s / \partial \tau_{ij}) \right]. \tag{A12}$$
The rate of work done is given by,

\[ \dot{W}_p = \lambda_p \tau_{ij} \left( \partial F_p / \tau_{ij} \right) \]  
\[ \dot{W}_s = \lambda_s \tau_{ij} \left( \partial F_s / \tau_{ij} \right). \]

(\text{A13})

(\text{A14})

\[ \theta, \] being a parameter not known a priori, is to be established iteratively; nevertheless an approximate seed value facilitates the convergence process. The rate at which the plane of maximum velocity strain rotates is equal to the shear velocity gradient perpendicular to it and therefore,

\[ \dot{\theta} = v_{z,1} = (v_{z,1})_p + (v_{z,1})_s \approx \tau_{12} (\lambda_p + \lambda_s). \]

(\text{A15})

The derivatives of \( F_p \) and \( F_s \) with respect to \( W_p \) and \( W_s \) are,

\[ \partial F_p / \partial W_p = (-\frac{3}{2}) \left( d\sigma_m / d\epsilon_p \right) = (-\frac{3}{2}) H_p \]

and

\[ \partial F_s / \partial W_s = -2 \left( d\tau_m / d\gamma_s \right) = -2 H_s, \]

(\text{A16})

(\text{A17})

where \( H_p \) and \( H_s \) have to be determined experimentally. Partial differentiation of \( F_s \) in eqn (11) with respect to \( \theta \) results in,

\[ \partial F_s / \partial \theta = \{ \partial (A_{1i} A_{2j} A_{1k} A_{2l}) / \partial \theta \} \tau_{ij} \tau_{kl} \]

\[ = \{ \partial (A_{1k} A_{2l}) / \partial \theta \} \tau_{kl} (A_{1i} A_{2j} \tau_{ij}) \]

\[ + \{ \partial (A_{1i} A_{2j}) / \partial \theta \} \tau_{ij} (A_{1k} A_{2l} \tau_{kl}) \]

\[ = 4 \tau_{12} (\tau^2)_{12}, \]

(\text{A18})

where superscript \# corresponds to a coordinate system with its \( x^\# \) placed at an angle of \( (\pi/4 - \theta) \) with \( x_1 \) of the global system and \( x_3^\# \) along \( x_3 \). Substituting (A.12) to (A.15) in (A.10) and (A.11) we obtain a set of scalar equations:

\[ \Phi_{pp} \lambda_p + \Phi_{ps} \lambda_s = \Psi_p \]

(\text{A19})

\[ \Phi_{sp} \lambda_p + \Phi_{ss} \lambda_s = \Psi_s. \]

(\text{A20})

Using equations (A.16) to (A.18) along with certain minor simplifications, it can be shown that,

\[ \Phi_{pp} = (\frac{4}{3}) \sigma_m^2 (3\mu + H_p) \]

(\text{A21})

\[ \Phi_{ps} = 4\mu \tau_m^2 \]

(\text{A22})

†This is for a special case where the spin \( \omega_{ij} \) are zero. The rotations of stresses due to \( \dot{\theta} \) are accounted for in eqn (22), by using \( \omega^*_ij \).
\[
\Phi_{sp} = 4 \tau_m^2 (\mu - \tau_{12}^t) \\
= 4 \mu \tau_m^2 \quad \text{(for } \mu \gg \tau_{12}^t) \quad \text{(A23)}
\]
\[
\Phi_{ss} = 4 \tau_m^2 (\mu + H_s - \tau_{12}^t) \quad \text{(A24)}
\]
\[
\Psi_p = 2 \mu s_{ij} D_{ij} \quad \text{(A25)}
\]
\[
\Psi_s = 2 \mu \tau_m R_{ij} D_{ij}, \quad \text{(A26)}
\]

where

\[
R_{ij} = A_{1i} A_{2j} + A_{2i} A_{1j} \quad \text{(A27)}
\]

(A_{ij} is the coordinate transformation matrix appearing in eqn (5) of section II.1).

By solving the simultaneous equations (A.19) and (A.20) for the variables \(\lambda_p\) and \(\lambda_s\) and substituting them in equation (A.12), we obtain

\[
(\tau_{ij})_0 = E_{ijkl} D_{kl}. \quad \text{(A28)}
\]

The constitutive tensor \(E_{ijkl}\) is given in eqn (16) of section II.3.