Analysis of Embedded and Surface Elliptical Flaws in Transversely Isotropic Bodies by the Finite Element Alternating Method

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The general analytical solution to the problem of a flat elliptical crack embedded in an infinite, transversely isotropic solid, oriented perpendicular to the axis of elastic symmetry, is derived along the lines of Vijayakumar and Atluri’s solution procedure for the isotropic case. The prior work of Kassir and Sih on this problem is limited to some constant and linear variations of normal and shear tractions on the crack face. The generalized solution is employed in conjunction with the finite element method. Such a method of analysis is shown to be an efficient way to evaluate the stress intensity factors along the flaw border.

1 Introduction

The analysis of surface flaws (of a semi-elliptical shape) in laminated composite plates, or even in unidirectionally reinforced composites, is of paramount importance in the integrity and life prediction analysis of such composite structures. A direct computational modeling of such surface flaws is prohibitively expensive. An efficient analysis of elliptical surface flaws in laminates involves the use of the analytical solution for an embedded crack in an infinite solid of transversely isotropic material, subjected to arbitrary crack face loading in conjunction with the Schwarz-Neumann alternating method. In this method a direct analysis of the uncracked structure is first performed and the analytical solution for a pressurized embedded elliptical crack in a transversely isotropic solid is then used to erase the stresses in an uncracked solid, at the location of the crack, to create a crack.

The analytical solution, for a flat elliptical crack embedded in an infinite solid of transversely isotropic material, where the axis normal to the ellipse is oriented along the axis of elastic symmetry, and where the crack face is subjected to arbitrary normal as well as shear loads, is considered in detail in this paper. The prior work on this problem (Kassir and Sih, 1968, 1969) is limited to the case of constant and linear variations of normal and shear tractions on the crack face. In the present analysis, the results of an embedded elliptical crack in an infinite isotropic solid subjected to arbitrary crack face loading (Vijayakumar and Atluri, 1981; Nishioka and Atluri 1983) are extended to a transversely isotropic case, wherein the crack axis is perpendicular to the axis of elastic symmetry.

The latter sections of this paper are concerned with the application of the finite element alternating technique to analyze embedded elliptical flaws or part-elliptical flaws in transversely isotropic bodies of finite dimension where the crack axis is perpendicular to the material axis.

As a first attempt, the transversely isotropic material properties are taken to be very close to those of an isotropic body (since the present analytical solution for the transversely isotropic case does not reduce to the isotropic case without an elaborate limiting process) and the stress intensity factors for a finite cracked body are compared with those of the isotropic formulation, as a check for the solution algorithm. The materials considered in the present paper are \( \beta \)-quartz, Zn, and a typical composite laminate (Christensen, 1975). The characteristic equation governing material moduli (equation (6)) yields real roots for \( \beta \)-quartz and complex roots for Zn. As for the composite laminate, the ratio of \( E_T \) and \( E_L \) was taken to be 25. \( (E_T \) is the Young’s modulus in the fiber direction and \( E_L \) is the Young’s modulus in the transverse direction.) Both embedded and surface elliptical flaws were considered in the present analysis. Situations where the body is subjected to uniform tensile and bending loads were considered. Comparisons were made with the isotropic case in all situations considered. Convergence studies for certain selected examples are also presented. It is found that there is no appreciable difference in the qualitative nature of the K-factor variation for the case of the embedded flaw, along the flaw border, with changes in material properties. This was not the case for the part elliptical surface flaw.
2 Potential Function Representation

A material is said to be "transversely isotropic" when it possesses an axis of elastic symmetry such that the material is isotropic in the plane normal to this axis. Such a material can be described by five independent elastic constants (see Lekhnitski, 1981). Let z be in the axis of elastic symmetry. Then the stress-strain relations in the (x, y, z) system could be written as

\[ \sigma_{11} = C_{11} \frac{\partial u_1}{\partial x} + C_{12} \frac{\partial u_2}{\partial y} + C_{13} \frac{\partial u_3}{\partial z} \]

\[ \sigma_{22} = C_{12} \frac{\partial u_1}{\partial x} + C_{11} \frac{\partial u_2}{\partial y} + C_{13} \frac{\partial u_3}{\partial z} \]

\[ \sigma_{33} = C_{13} \left( \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y} \right) + C_{33} \frac{\partial u_3}{\partial z} \]

\[ \sigma_{32} = C_{44} \left( \frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x} \right) \]

\[ \sigma_{12} = \frac{1}{2} \left( C_{11} - C_{12} \right) \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right). \]  

(1)

Here, \( \sigma_{ij}, i=1,2,3; j=1,2,3 \) and \( u_{ij}, i=1,2,3 \) are stresses and displacement components in the (x, y, z) system and \( C_{ij} \) are elastic constants for transverse isotropy defined in (Lekhnitski, 1981). Equation (1), when substituted into the equilibrium equations, yields a system of three equations for \( u_j, (j=1,2,3) \) as:

\[ C_{11} \frac{\partial^2 u_1}{\partial x^2} + \frac{1}{2} \left( C_{11} - C_{12} \right) \frac{\partial^2 u_1}{\partial y^2} + C_{44} \frac{\partial^2 u_1}{\partial z^2} + \frac{\partial}{\partial x} \left[ \frac{1}{2} \left( C_{11} - C_{12} \right) \frac{\partial u_1}{\partial y} + \left( C_{13} + C_{44} \right) \frac{\partial u_3}{\partial z} \right] = 0 \]

\[ = \frac{1}{2} \left( C_{11} - C_{12} \right) \frac{\partial^2 u_2}{\partial x^2} + C_{11} \frac{\partial^2 u_2}{\partial y^2} + C_{44} \frac{\partial^2 u_2}{\partial z^2} + \frac{\partial}{\partial y} \left[ \frac{1}{2} \left( C_{11} - C_{12} \right) \frac{\partial u_1}{\partial x} + \left( C_{13} + C_{44} \right) \frac{\partial u_3}{\partial z} \right] = 0 \]

\[ C_{44} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_3 + C_{33} \frac{\partial^2 u_3}{\partial z^2} + \left( C_{13} + C_{44} \right) \frac{\partial}{\partial z} \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) = 0. \]  

(2)

The displacement field \( u_1, u_2, \) and \( u_3 \), which satisfies the above equations identically, is represented in terms of potential functions \( \phi_j (j=1,2,3) \) (see Elliot (1949) and Hu (1952)), as:

\[ u_1 = \frac{\partial}{\partial x} (\phi_1 + \phi_2) - \frac{\partial \phi_3}{\partial y} \]

\[ u_2 = \frac{\partial}{\partial y} (\phi_1 + \phi_3) - \frac{\partial \phi_2}{\partial x} \]

\[ u_3 = \frac{\partial}{\partial z} (m_1 \phi_1 + m_2 \phi_2) \]  

(3)

where the potential functions \( \phi_j (j=1,2,3) \) satisfy the harmonic equation in the modified\(^1\) coordinates \( x_j, y_j \) and \( z_j (j=1,2,3) \) as follows:

\[ \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2} \right) \phi_j = 0 \quad (j=1,2,3 \text{ no sum on } j) \]  

(4)

where \( x_j = x, y_j = y, z_j = \frac{z}{\sqrt{n_j}} \) (\( j=1,2,3 \)).

The constants \( m_1 \) and \( m_2 \) in equation (3) are defined as

\[ m_j = \frac{C_{11} n_j - C_{12} (C_{13} + C_{44}) n_j}{C_{11} + C_{44} - C_{12} n_j} \quad (j=1,2,3). \]  

(5)

The quantities \( n_1 \) and \( n_2 \) are the roots of the quadratic equation

\[ C_{11} C_{44} n^2 + \left[ C_{13} (C_{13} + 2 C_{44}) - C_{11} C_{33} \right] n + C_{13} C_{44} = 0 \]  

(6)

and \( n_3 \) is defined as:

\[ n_3 = \frac{2 C_{44}}{C_{11} - C_{12}}. \]  

(7)

In general, \( n_1 \) and \( n_2 \) are either real or complex conjugate quantities. The constants \( m_1 \) and \( m_2 \) behave the same way. When \( n_1 \) and \( n_2 \) are complex conjugates, care must be taken to take the principal values of \( n_1^2 \) and \( n_2^2 \). The same applies to \( m_1 \) and \( m_2 \). In the case of complex \( n_1 \) and \( n_2 \), the potentials \( \phi_1 \) and \( \phi_2 \) turn out to be complex conjugate and hence, the displacements and stresses turn out to be single valued and real (Elliot, 1949).

The stresses from equation (1) and (3) in the \( x, y, z \) system take the following form:

\[ \sigma_{11} = (C_{11} \delta x^2 + C_{12} \delta y^2) (\phi_1 + \phi_2) + C_{13} \delta z (m_1 \phi_1 + m_2 \phi_2) \]

\[ + C_{12} \delta y (C_{13} n_1 \phi_1 + (1 + m_1) n_2 \phi_2) \]

\[ \sigma_{12} = C_{44} \delta x (1 + m_1) \phi_1 + (1 + m_2) \phi_2 \]

\[ \sigma_{13} = C_{44} \delta y (1 + m_1) \phi_1 + (1 + m_2) \phi_2 \]

\[ \sigma_{22} = C_{12} \delta y (\delta x (C_{13} n_1 \phi_1 + (1 + m_1) n_2 \phi_2) + \delta \phi_3) \]

\[ \sigma_{33} = C_{13} \delta z (\delta x (m_1 \phi_1 + m_2 \phi_2) + \delta \phi_3) \]

\[ \sigma_{31} = 2 C_{44} \delta z (\delta x (\phi_1 + \phi_2) + \frac{1}{2} (C_{11} - C_{12}) (\delta \phi_3 - \delta \phi_3)). \]  

(8)

3 Potential Functions for Transverse Isotropy

Let the region of discontinuity in the transversely isotropic material be bounded by an ellipse, which in the \( x, y, z \) system, takes the following form:

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{in} \quad z = 0. \]  

(9)

The axis of elastic symmetry in this case is along the \( z \)-axis. The arbitrary loading on the crack face can be decomposed into symmetric and skew symmetric stress distributions. As in Kassir and Sih (1968, 1975) the potential functions for the symmetric case are related to the solution of the equations:

\[ \nabla_j^2 f_j (x_j, y_j, z_j) = 0 \quad (j=1,2,3 \text{ no sum on } j) \]  

(10)

where \( x_j = x, y_j = y \) and \( z_j = \frac{z}{\sqrt{n_j}} \)

and

\[ \nabla_j^2 = \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2}. \]

The potential functions for the skew symmetric case are related to the solutions of the equations:
\[ \nabla^2 p_j = 0 \quad \text{and} \quad \nabla^2 q_j = 0 \quad (j = 1, 2, 3 \text{ no sum on } j). \] (11)

Hence, the potentials \( \phi_j (j = 1, 2, 3) \) for arbitrary loading on the crack face can be defined as

\[ \phi_1 = \frac{n_{1/2}^j}{1 + m_j} f_1(x_j, y_j, z_j) + \frac{1 + m_j}{m_j} F_1(x_j, y_j, z_j) \]
\[ \phi_2 = \frac{n_{1/2}^j}{1 + m_j} f_2(x_j, y_j, z_j) + \frac{1 + m_j}{m_j} F_2(x_j, y_j, z_j) \]
\[ \phi_3 = F_3(x_j, y_j, z_j) \] (12)

where

\[ F_1(x_j, y_j, z_j) = \frac{\partial p_j}{\partial x_j} \frac{\partial q_j}{\partial y_j} \quad (j = 1, 2 \text{ no sum on } j) \]
and

\[ F_3(x_j, y_j, z_j) = \frac{\partial p_j}{\partial x_j} \frac{\partial q_j}{\partial y_j} \] (13)

Substituting the above-mentioned expressions for \( \phi_j (j = 1, 2, 3) \) in equation (8), we have

\[ \sigma_{11} = \left( C_{11} \frac{\partial^2}{\partial x_j^2} + C_{12} \frac{\partial^2}{\partial y_j^2} \right) \left[ (-1)^{j-1} \frac{n_{1/2}^j}{1 + m_j} f_1(x_j, y_j, z_j) + \frac{1 + m_j}{m_j} F_1(x_j, y_j, z_j) \right] \]
\[ + \frac{1 + m_j}{m_j} \frac{\partial^2}{\partial y_j^2} \left[ (-1)^{j-1} \frac{n_{1/2}^j}{1 + m_j} f_3(x_j, y_j, z_j) \right] \]
\[ + C_{13} \frac{\partial^2}{\partial z_j^2} \left[ (-1)^{j-1} \frac{n_{1/2}^j}{1 + m_j} f_3(x_j, y_j, z_j) \right] \]
\[ - (C_{11} - C_{12}) \frac{\partial^2 F_1(x_j, y_j, z_j)}{\partial x_j \partial y_j} \quad (j = 1, 2) \]
\[ \sigma_{22} = \left( C_{12} \frac{\partial^2}{\partial x_j^2} + C_{11} \frac{\partial^2}{\partial y_j^2} \right) \left[ (-1)^{j-1} \frac{n_{1/2}^j}{1 + m_j} f_3(x_j, y_j, z_j) \right] \]
\[ + \frac{1 + m_j}{m_j} \frac{\partial^2}{\partial x_j^2} \left[ (-1)^{j-1} \frac{n_{1/2}^j}{1 + m_j} f_1(x_j, y_j, z_j) \right] \]
\[ + C_{13} \frac{\partial^2}{\partial z_j^2} \left[ (-1)^{j-1} \frac{n_{1/2}^j}{1 + m_j} f_3(x_j, y_j, z_j) \right] \]
\[ + (C_{11} - C_{12}) \frac{\partial^2 F_3(x_j, y_j, z_j)}{\partial x_j \partial y_j} \quad (j = 1, 2) \]
\[ \sigma_{33} = C_{44} \frac{\partial^2}{\partial z_j^2} \left[ (-1)^{j-1} f_3(x_j, y_j, z_j) \right] \]
\[ + (C_{11} - C_{12}) \frac{\partial^2 F_3(x_j, y_j, z_j)}{\partial x_j \partial y_j} \quad (j = 1, 2) \]

The stress boundary conditions to be satisfied on the crack face in the plane \( z = 0 \) (i.e., \( z_j = 0 (j = 1, 2, 3) \)) takes much simpler forms

\[ \sigma_{33} = C_{44} \frac{\partial^2}{\partial z_j^2} \left[ (-1)^{j-1} f_3(x_j, y_j, z_j) \right] \]
\[ + (C_{11} - C_{12}) \frac{\partial^2 F_3(x_j, y_j, z_j)}{\partial x_j \partial y_j} \quad (j = 1, 2) \] (14)

Making use of equation (13) and defining \( f_j (j = 1, 2, 3) \) as

\[ f_j(x_j, y_j, z_j) = \frac{\partial p_j(x_j, y_j, z_j)}{\partial z_j} \]
\[ f_3(x_j, y_j, z_j) = \frac{\partial q_j(x_j, y_j, z_j)}{\partial z_j} \] (16)

we could transform the boundary conditions in equation (15) as

\[ \sigma_{33} = \gamma_1 \left[ \frac{\partial^2 f_3(x_j, y_j, z_j)}{\partial z_j^2} \right] \quad (z = 0) \]
\[ \sigma_{33} = \gamma_2 \left[ \frac{\partial^2 f_3(x_j, y_j, z_j)}{\partial z_j^2} \right] \quad (z = 0) \]
\[ \sigma_{33} = \gamma_3 \left[ \frac{\partial^2 f_3(x_j, y_j, z_j)}{\partial z_j^2} \right] \quad (z = 0) \] (17)

where

\[ \gamma_1 = C_{44} \frac{n_{1/2}^j - n_{1/2}^{j-1}}{n_1} \]
\[ \gamma_2 = C_{44} \frac{(1 + m_j)(1 + m_j)}{(m_j - m_j)n_{1/2}^j} \left[ \frac{1}{n_1^{1/2}} - \frac{1}{n_2^{1/2}} \right] \]
\[ \gamma_3 = \frac{C_{44}}{n_3^{1/2}} \]
Also, it should be noted that from equation (16), $f_j(j=1,2,3)$ should be harmonic since $p_j, q_j(j=1,2,3)$ are harmonic functions.

4 Embedded Elliptical Crack in an Infinite Transversely Isotropically Solid

Let the region of discontinuity in the transversely isotropic material be bounded by an ellipse, which in the $(x, y, z)$ system takes the following form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{in} \quad z = 0. \quad (18)$$

The axis of elastic symmetry in this case is along the $z$-axis. The representation of the ellipse remains unchanged in the modified coordinates $(x_j, y_j, z_j)(j=1,2,3)$ for a point in the $(x, y, z)(j=1,2,3)$ coordinate system are given by the roots of the cubic equation

$$\omega_j(\xi^j) = 0 \quad (j=1,2,3 \text{ no sum on } j) \quad (19)$$

They are connected to the Cartesian coordinates, by the relations

$$\alpha^2(a^2 - b^2)y_j^2 = (a^2 + \xi^j)(a^2 + \xi^j)(a^2 + \xi^j) \quad (j=1,2,3) \quad (20)$$

The expression for $\omega_j(\xi^j)$ in equation (21) may be written in alternate form

$$\omega_j(\xi^j) = \frac{P_j(\xi^j)}{Q_j(\xi^j)} \quad (j=1,2,3) \quad (22)$$

where

$$P_j(\xi^j) = (\xi^j - \xi^j)(\xi^j - \xi^j)(\xi^j - \xi^j) \quad (j=1,2,3)$$

and

$$Q_j(\xi^j) = \xi^j(\xi^j + \alpha^2\xi^j + \alpha^2) \quad (j=1,2,3) \quad (23)$$

The partial derivatives of $\omega_j(\xi^j)$ with respect to $x_j, y_j$, and $z_j$ when $\beta$ takes values 1, 2, and 3 respectively, $\delta_j$, corresponds to the $\beta$th partial derivative of $\beta_j$ and $P_j$ indicates the partial derivative w.r.t. $\xi^j$. The elliptic boundary in the plane $z = 0$ (i.e., $z_j = 0 (j=1,2,3)$) corresponds to the curves $\xi^j = \xi^j = 0 (j=1,2,3, \text{ no sum on } j)$. The stress surface, in the region inside the ellipse in the plane $z = 0$ is given by $\xi^j = \xi^j = 0 (j=1,2,3, \text{ no sum on } j)$. The three potential functions $\omega_j(\alpha = 1,2,3)$ which satisfy the stress boundary conditions as suggested by Segedin (1967) can be taken as

$$f_\alpha(x_j, y_j, z_j) = \sum_{k} \sum_{l} C_{\alpha, k, l} F_{k l} \quad (\alpha = 1,2,3) \quad (24)$$

where

$$F_{k l} = \oint_{\xi^j} \frac{\partial^{k+l} \omega_j(s)}{\partial x_j^k \partial y_j^l} ds \quad \sqrt{Q_j(s)} \quad (25)$$

$C_{\alpha, k, l}$ are the undetermined coefficients (in equation (24)) which are to be evaluated through applied tractions on the crack face. The abbreviated version of equation (25) can be represented as:

$$F_{k l} = \int_{\xi^j} \frac{\partial^{k+l} \omega_j(s)}{\partial x_j^k \partial y_j^l} \frac{ds}{\sqrt{Q_j(s)}} \quad (26)$$

The notation $\delta_j$ was used in the above equation to denote the $\alpha$th partial derivative with respect to $\beta = 1,2,3$ which denotes $x_j, y_j$, and $z_j$ respectively. By successive differentiation of equation (26) and using the fact that $\omega_j(\xi^j) = 0$, one can easily show that

$$F_{k l, \beta} = \int_{\xi^j} \frac{\partial^{k+l} \omega_j^{(k+l)}(s)}{\partial x_j^k \partial y_j^l} \frac{ds}{\sqrt{Q_j(s)}} \quad (27)$$

where $\delta_{\beta \beta}, \text{ etc.,}$ are defined as Kronecker deltas. Further,

$$F_{k l, \beta} = \int_{\xi^j} \frac{\partial^{k+l} \omega_j^{(k+l)}(s)}{\partial x_j^k \partial y_j^l} \frac{ds}{\sqrt{Q_j(s)}} \quad (28)$$

Evaluation of the generic integral of the type given in equation (27) and (28) can be derived along the lines of Vijayakumar and Atluri (1981) by expanding $\omega_j^{(k+l)}(s)$.

$$\int_{\xi^j} \frac{\partial^{k+l} \omega_j^{(k+l)}(s)}{\partial x_j^k \partial y_j^l} \frac{ds}{\sqrt{Q_j(s)}}$$

\begin{align*}
&= (k + l + 1)! \sum_{p=0}^{k} \sum_{q=0}^{l} \sum_{r=0}^{(k + l + 1 - p)!} \frac{(-1)^p (2p - 2q)!}{(p - q)!} \times (2q - 2r)! (2r)! \frac{\xi^{2q - 2r - 1}}{2r - 1} \frac{\xi^{2q - 2r - 1}}{2r - 1} J_{p-q,a-r}(\xi^j) \quad (29)
\end{align*}

It should be noted here that the integral in equation (31) becomes singular when $r = 1$. As noted by Shah and Kobayashi (1971), the singular portion cancels out when equation (28) is considered as a whole. A systematic procedure for the evaluation of the above-mentioned elliptic integrals can be found in Nishioka and Atluri (1983) for real $\xi^j$. As for complex $\xi^j$, the formulation procedure is similar to that of real $\xi^j$ (Nishioka and Atluri, 1983), except for the fact that the elliptic integrals

\Admittedly, this is not a very precise notation; but is used here for the sake of simplicity.
defined here have complex amplitudes. The procedure for evaluating such elliptic integrals can be found in Bird and Friedman (1971) or in Abramowitz and Stegun (1971).

5 Representation of Crack Face Tensions

The tractions on the crack surface can be expressed in the form

$$\sigma_{3a}^{(\alpha, m-n,n)} = \sum_{i=0}^{1} \sum_{j=0}^{i} \sum_{k=0}^{i-j-1} M_{k} \sum_{l=0}^{i-k-1} (-1)^{k} A_{(4, m-n,n)}^{(i,j)} F_{i}^{(2-2i+2l+1)} F_{j}^{(2-2j+2l+1)}$$

so that the values of $(i,j)$ specify the symmetry of the load with respect to the axes of the ellipse. Here, the $(x, y)$ are the global coordinate system which is aligned along the major and minor axis of the ellipse. $A_{(4, m-n,n)}^{(i,j)}$, in general, is given or can be determined once the tractions on the crackface is prescribed. Taking symmetry into consideration, the potential functions in the modified coordinates $(x_{f}, y_{f}, z_{f}) (j = 1, 2, 3)$ are assumed as

$$f_{0}(x_{f}, y_{f}, z_{f}) = \frac{1}{2} \sum_{i=0}^{1} \sum_{j=0}^{i} \sum_{k=0}^{i-j-1} \sum_{l=0}^{i-k-1} C_{(4, m-n,n)}^{(i,j)} F_{i}^{(2-2i+2l+1)} F_{j}^{(2-2j+2l+1)}$$

Here, $C_{(4, m-n,n)}^{(i,j)}$ are undetermined coefficients to be evaluated through the traction boundary conditions on the crack face.

Substituting equation (33) into equation (26) the crack face stresses in terms of $f_{0}$ $(\alpha = 1, 2, 3)$ (on the crack face $x_{j} = x, y_{j} = y$, and $z_{j} = z = 0$) can be expressed

$$\sigma_{3a}^{(\alpha, m-n,n)} = \sum_{i=0}^{1} \sum_{j=0}^{i} \sum_{k=0}^{i-j-1} (-1)^{k} A_{(4, m-n,n)}^{(i,j)} F_{i}^{(2-2i+2l+1)} F_{j}^{(2-2j+2l+1)}$$

Substituting equation (33) into equation (34) and equating coefficients of like powers of $\alpha$ in the resulting expression, with those in equation (33), the coefficients $\{A_{f}\}$ for the Mode I problem are obtained.

$$A_{(3, m-n,n)}^{(i,j)} = \frac{1}{(2m - 2n + i)!(2n - j)!} \sum_{k=0}^{i-j} (-1)^{k}$$

$$\times \{H_{i}^{(i,j)}(2j - 1)(2j - 2)\} C_{(4, m-n,n)}^{(i,j)} F_{i}^{(2-2k+2j+1)} F_{j}^{(2-2j+2l+1)}$$

$$m = 0, 1, \ldots, M, \quad n = 0, 1, \ldots, m$$

Owing to the skew symmetric nature of the mode II and III problems, they could be decomposed in the following two problems $P_{1}$ and $P_{2}$ (Vijayarajkumar and Atluri, 1981): (i) problem $P_{1}$: $\sigma_{3a}^{(1)}$ is symmetric and $\sigma_{3a}^{(2)}$ is antisymmetric in both $x_{1}$ and $x_{2}$; (ii) problem $P_{2}$: $\sigma_{3a}^{(3)}$ is symmetric in $x_{1}$ and antisymmetric in $x_{2}$ and symmetric in $x_{2}$.

The coefficients for problem $P_{2}$ can be shown to be the following

$$A_{(3, m-n,n)}^{(i,j)} = \frac{1}{(2m - 2n + i)!(2n - j)!} \sum_{k=0}^{i-j} (-1)^{k}$$

$$\times \{H_{i}^{(i,j)}(2j - 1)(2j - 2)\} C_{(4, m-n,n)}^{(i,j)} F_{i}^{(2-2k+2j+1)} F_{j}^{(2-2j+2l+1)}$$

$$m = 0, 1, \ldots, M, \quad n = 0, 1, \ldots, m$$

Similar expressions could be derived along the same line for the $P_{1}$ problem (Vijayarajkumar and Atluri, 1981). Here,

6 Stress Intensity Factors

As shown by Kassir and Sih (1968, 1975), the ellipsoidal coordinates in the vicinity of the periphery of the elliptical crack on the plane $z = 0$ becomes

$$\xi_{1} = -(a^{3} \sin \theta + b^{3} \cos \theta)$$

$$\xi_{2} = 0$$

$$\xi_{3} = 2abr(a^{3} \sin \theta + b^{3} \cos \theta)$$

$$x = a \cos \theta \quad y = b \sin \theta.$$}

The coefficients of the square matrix $[B]$ are complete elliptic integrals defined above. By solving the system of equations the unknown coefficients, $\{C\}$ can be put in matrix form as follows:

$$[A] = [B] \{C\}.$$
Table 1  Elastic constants for transversely isotropic material (consistent with (1))

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<thead>
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<td>tending to isotropy (E=1.0, ν=0.3)</td>
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Fig. 1 Cracked configuration analyzed for uniform loading; (a) embedded crack, (b) surface crack

Fig. 2 Crack surface area used in residual pressure fit (rectangular model)

Fig. 3 Fictitious tractions used for the surface flaw problem

Fig. 4 Comparison of the isotropic situation and transverse isotropy tending to isotropy for the case of embedded flaw

Fig. 5 Comparison of the isotropic situation and transverse isotropy for the case of surface flaw

of an elliptical flaw in a finite transversely isotropic solid. Since no general solution exists for a finite body subjected to arbitrary externally applied tractions, a finite element solution is used to evaluate the stresses at the location of the crack in the transversely isotropic solid. The infinite body with a crack has a closed-form solution which is valid for any degree polynomial applied crack face tractions. Hence, the two main ingredients for the effective implementation of the alternating method for an elliptical crack embedded in a transversely isotropic solid are:

Solution 1: the elliptical crack subjected to arbitrary normal and shear loading on the crack surface in a transversely isotropic infinite solid.

Solution 2: a finite element solution for the uncracked transversely isotropic body: The following iterative solution procedure is adopted for the alternating method.
1 Solve the uncracked transversely isotropic body under the given external loads by the finite element method. The uncracked body has the same geometry with the given problem except the crack.

2 The stresses at the location of the original crack in the uncracked structure is computed using the finite element method.

3 Compare the residual stresses calculated in step (2) with a permissible stress magnitude. In the present study one percent of the maximum external applied stress is used for the permissible stress magnitude.

4 To create a crack, reverse the residual stresses at the location of the crack and determine the coefficients \( A \) of equation (39) useful for solution 1 by least squares method.

5 Determine the coefficients \( C \) of equation (39).

6 Compute \( K \)-factors for the current iteration by substituting coefficients \( C \) into equations (42)-(44).

7 Calculate the residual stress from equation (14) on external surfaces of the body due to loads in step (4).

8 Reverse the residual surface stresses at the boundary and the solution 2 is made use of to obtain the stresses at the uncracked location as in step (1).

Repeat all steps in the iteration process until the residual stresses on the crack surface become negligible. It should be noted here that the stiffness matrix of solution 2 remains the same during every iteration process and thus needed to be computed only once during the entire solution process. Also, such a solution procedure obviates the need for any singular elements at the crack front, since solution 2 involves the uncracked body only. In the case of the surface flaw, fictitious tractions are introduced to the imaginary portion of the crack (so that the analytical solution for the embedded crack can be employed) before proceeding to step (4) in the solution algorithm. The fictitious tractions could be chosen arbitrarily and will only affect the convergence of the alternating method.

8 Numerical Results

8.1 Isotropic Finite Body Solution. As a check on the correctness of the analytical solution for the transversely isotropic case, embedded as well as surface elliptical flaws in a finite body, subjected to remote uniform tensile loading, were considered. In all the cases, the elastic constants for transverse isotropy were chosen such that they were close to isotropic behavior (Table 1). It should be noted here that the analytical solution for the isotropic case cannot be recovered from the present transversely isotropic case unless an elaborate limiting process is performed analytically. Hence, as an overall check
for the solution algorithm the limiting case for isotropy is taken numerically by taking elastic constants for transverse isotropy to be close to isotropy. The two example problems considered are shown in Fig. 1. The ratio of \( a_1/a_2 \) was taken to be 4 for the embedded flaw and 2 for the surface flaw (\( a_1 \) and \( a_2 \) are the semimajor and semiminor axis of the ellipse, respectively). The specimen and flaw dimensions are the same as in the case of transverse isotropy and will be discussed in the forthcoming section. The highest order polynomial variation (\( M \)) in the analytical solution was taken to be 5. Rectangular finite element models were considered (Raju et al., 1988) throughout the present analysis as in Fig. 2. As for the part elliptical surface flaw, the stress distribution in the fictitious region was taken to be a constant, as in Fig. 3. This was found to give the best convergence of all the other variations considered (Nishioka and Atluri, 1983). Twenty noded brick elements were considered throughout the present analysis. As for the embedded elliptical crack, a 756-noded graded mesh with 125 elements was considered. In the case of the part elliptical surface flaw, a 1856 noded graded mesh (with a finer mesh refinement near the region where the crack is supposed to be located) with 343 elements was considered. The above-mentioned meshes are the final mesh discretizations after a systematic convergence study. Such a convergence study would be presented for transversely isotropic materials in the forthcoming section. The nondimensionalized parameter \( K_r/S\sqrt{a_2/Q} \) is plotted against the parametric angle \( \theta \). Here, \( K_r \) is the mode I stress intensity factor, \( S \) is the uniform remote tensile loading of the finite body, \( a_2 \) is the semiminor radial distance of the elliptical flaw, and \( Q \) is the shape factor of the ellipse (which is the square of the complete integral of the second kind). As seen in Figs. 4 and 5 the agreement between the transversely isotropic algorithm (when the elastic constants were taken to be close to the isotropic situation) and the exact isotropic formulation was good for both the embedded and surface elliptical flaws.

8.2 Embedded Elliptical Crack in a Finite Body. Three different materials were considered in the present paper. They were \( \beta \)-quartz, zinc, and a typical graphite epoxy composite laminate (Christensen, 1979). The properties of the above-mentioned materials are listed in Table 1. The characteristic equation (4) yields real roots for \( \beta \)-quartz and the composite laminate and complex roots for Zn. A convergence study was carried out for the FEM idealization of the uncracked body. An embedded elliptical crack with \( a_1 = 1.0 \) and \( a_2 = 0.25 \) in a \( 8 \times 0.67 \times 8 \) solid subjected to a uniaxial load as shown in Fig. 1 was considered. The highest order polynomial considered for a crack face traction variation was \( M = 5 \). A quarter of the ellipse was modeled due to the symmetry of the problem. A systematic convergence study was undertaken starting with a 425-noded mesh for the uncracked body, for the case of the
embedded crack for all the materials considered. As shown in Fig. 6, the convergence was excellent for β-quartz with a 725-noded mesh. The mode I K-factor variation was considered for β-quartz, Zinc, and the composite laminate (E2/E1 = 25) with a 725-noded graded mesh. Due to the nature of the symmetry present, the K-factor variation of only quarter of the ellipse was considered (Fig. 7). The results are compared to that of the isotropic material with $E = 1.0$ and $\nu = 0.3$.

8.3 Surface Elliptical Crack. A surface semi-elliptical flaw in a $2.5 \times 2.5 \times 10$ body under uniaxial tension was considered for the present analysis as seen in Fig. 1. The dimensions of the ellipse considered was $a_1 = 1.25$ and $a_2 = 0.625$. Due to the symmetry present, only a half the body was modeled. The highest order polynomial considered for the traction variation was $M = 5$. The extended tractions on the fictitious part of the crack was taken to be a constant as shown in Fig. 3. A systematic convergence study was performed starting from a 725-noded mesh for all the above-mentioned materials considered. The results of such a convergence study are shown for Zinc in Fig. 8. The 1856-noded graded mesh displayed adequate convergence for the present problem. Due to the nature of the symmetry present in the problem, the K-factor variation was presented for half of the elliptical flaw. Figure 9 shows the K-factor variation of β-quartz, Zinc, and the composite laminate (with $E_2/E_1 = 25$) after convergence has been established in each case. The results for the above-mentioned materials are compared with those for an isotropic elastic material with $E = 1.0$ and $\nu = 0.3$. The qualitative variation of K-factors for the present problem with the above-mentioned materials was different when compared to that of the embedded crack. The results for the surface flaw in a finite body clearly demonstrate the effect of the degree of transverse isotropy on the K-factors as compared to the case of isotropy.

8.4 Bending Loads. An embedded elliptical flaw in a $8 \times 0.67 \times 8$ solid with $a_1 = 1.0$ and $a_2 = 0.25$ and a semi-elliptical surface flaw in a $2.5 \times 2.5 \times 10$ solid with $a_1 = 1.25$ and $a_2 = 0.625$ were considered when subjected to bending loads as shown in Fig. 10. Due to the nature of the symmetry present in the problem, only quarter of the ellipse was modeled. A convergence study was undertaken for both problems considered. The non-dimensionalized parameter $K_f/S[=a_2/Q]^{1/2}$ is plotted against the parametric angle of the ellipse. Here, $K_f$ is the mode I stress intensity factor, $S$ is the remote out of fiber loading of the finite body (Fig. 10), $a_2$ is the semi-minor radial distance of the elliptical flaw, and $Q$ is the shape factor of the ellipse.

The materials considered were β-quartz, Zinc, and the composite laminate (Christensen, 1975). Comparisons are made against the isotropic case with $E = 10$ and $\nu = 0.3$. A 725-noded mesh was used for the uncracked solid in the case of the embedded flaw and the results are presented in Fig. 11. As for the semi-elliptical surface flaw a 1856-noded graded mesh was employed for modeling the uncracked body. The K-factor variations are given in the tensile portion of the ellipse in Fig. 12.

9 Conclusions

The solution for an embedded elliptical crack in an infinite transversely isotropic solid (when the crack axis is perpendicular to the axis of elastic symmetry) subjected to arbitrary crack face tractions is derived. This analytical solution, in conjunction with the finite element method, is employed in the Schwarz-Neumann alternating method to solve for K-factors in a finite solid subjected to arbitrary loading. Mode I solutions are obtained for some typical transversely isotropic materials. It was observed that the maximum K-factors for the surface flaw problems were sensitive to material properties of the transversely isotropic body. On the other hand, as is to be expected, the solutions for embedded crack problems in transversely isotropic bodies did not deviate much from the isotropic situation. The ability of the present method to obtain K-factors for arbitrary loading of cracked transversely isotropic bodies using the infinite body solution renders such a technique powerful and versatile.

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References


