A NEW APPROACH TO DETERMINE WEIGHT FUNCTIONS FROM BUECKNER'S FUNDAMENTAL FIELD BY THE SUPERPOSITION TECHNIQUE

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The theory of weight functions as an elegant method of calculating the stress intensity factors for a crack in a linear elastic body was first introduced by Bueckner [1]. Once the weight functions $h_N(x,a)$ are known for a specific crack configuration, the stress intensity factors $K_N$ ($N=I, II, III$ refer to the crack opening modes) for any other loading of the body can easily be obtained from the integral

$$K_N = \int_S t_i(x) h_{N}(x,a) \, ds$$

where $S$ is the boundary of the body and $t_i$ are the tractions acting on $S$. For the sake of brevity, we restrict ourselves to the case that $S$ is the crack face $S$, only, because any external loading can be transformed into equivalent crack face tractions.

An efficient approach to weight functions is based on the so-called Bueckner fundamental field [1]. It is generated by a pair of forces $P$, acting in opposite directions on both crack faces with distances $c$ to the crack tip, see Fig. 1. (For mode $N=I$, $P$ acts in $i=2$ - direction. In an analogous manner fundamental fields for modes $II$ and $III$ can be defined by appropriate directions of $P$.) The fundamental field is the solution obtained by keeping $P/vc$ constant and shrinking $c$ to zero. The corresponding displacement field is denoted by $u_N$. It exhibits a singularity of the type $r^{-1/2}$ at the crack. The associated stress field has the stronger singularity $r^{-3/2}$. The strength of these "Bueckner singularities" is characterized by the factor

$$B_N = P \sqrt{c} / \pi$$

As has been derived in [1] and [2], the Bueckner displacement field for a cracked
structure is simply related to the weight function by

\[ h_N(x,a) = \frac{E' u_N(x,a)}{4\sqrt{2\pi} B_N} \]

(3)

Here \( E' = E \) holds for plane stress and \( E' = E(1-v^2) \) for plane strain with Young’s modulus \( E \) and Poisson’s ratio \( v \).

Paris et al. [2] were the first to propose the use of the Bueckner fundamental field in combination with a numerical technique, the finite element method, to determine weight functions for two-dimensional crack problems. To incorporate the Bueckner singularity into the numerical model, a small hole was cut out around the crack tip, on which the displacements \( u_N \) were prescribed according to the fundamental field for the crack in the infinite plane. Later on, in [3] the boundary element method was used to compute weight functions by the same procedure. The disadvantage of this approach is that it requires a relatively fine discretization around the hole. Moreover, the quality of the solution is very sensitive to the size of that hole.

The new approach suggested here [4] for computing the fundamental field for a finite cracked body is the superposition of the Bueckner solution for the crack in the infinite region with a regular solution compensating the tractions on the boundary. This simple idea has obviously been overlooked up to now, although the decomposition of the fundamental field into a singular and a regular one was already discussed by Bueckner [5]. The principle is illustrated in Fig. 1. The necessary presumption is the knowledge of the fundamental field for the considered crack geometry in the infinite region.

Then the algorithm consists of the following steps:

(a) Find the tractions \( t_N \) on the boundary \( S \) of the body according to the fundamental field for the infinite region.

(b) Solve the regular crack problem of the body, with the opposite tractions \(-t_N\) being imposed on the boundary. This can be done by any numerical up-to-date tool.

(c) Evaluate the regular displacements of \( u_{Nn} \) on the crack faces from the finite body solution of (b).

(d) Compute the analytically known displacements \( u_{\infty} \) on the crack faces from the finite body solution of (b).

(e) Add the displacement fields of (c) and (d) to obtain the total displacements \( u_N \) (or of any other point) and compute the weight function by (3).

Thus the procedure requires essentially the solution of the regular crack problem (b). Note that the displacements \( u_{Nn} \) behave regularly at the crack tip, whereas the \( u_{\infty} \) of (d) are singular as \( r^{1/2} \).

The suggested approach was implemented for plane crack problems of mode I. The configurations of an edge crack and an internal crack were considered. To compute the finite body correction term in (b), the boundary element alternating technique [6] (BEM-A) was used with its outstanding feature that no explicit crack tip modelling is required and that the results are given in a quasi-analytical
form by means of the coefficients of Chebyshev-polynomials. The code reported in [6] was modified to incorporate the traction boundary conditions $t_\gamma$ of item (a) and superimposed the calculated regular displacement field with the singular fundamental field.

The fundamental field for a semi-infinite edge crack in the infinite plane is represented by the complex Westergaard stress function (see [7] as follows [2]:

$$Z(z) = B_1 z^{-3/2}; \quad \bar{Z}(z) = -B_1 z^{-1/2}$$

(4)

Accordingly the crack opening displacements are:

$$u_r^\infty = 4 \ B_1 r^{-1/2} E'$$

(5)

and the tractions can be computed from the normal vector on $S$ and the stress state derived from (4):

$$\sigma_{11} = B_1 r^{-3/2} \left( \cos^2 \theta - \frac{3}{2} \sin \theta \sin \frac{5}{2} \theta \right)$$

$$\sigma_{22} = B_1 r^{-3/2} \left( \cos^2 \theta + \frac{3}{2} \sin \theta \sin \frac{5}{2} \theta \right)$$

$$\sigma_{12} = \frac{3}{2} \sin \theta \cos \frac{5}{2} \theta$$

(6)

As an example the method was applied to the single edge crack specimen of width $b=10.0$, height $2h=4b$ and various crack lengths. Figure 2 shows a typical discretization with boundary elements of quadratic shape functions used in the BEM-Alternating analysis. At first the crack face weight functions were calculated following the above algorithm. Then these weight functions were used to integrate the $K_f$-factor from (1) assuming a constant tensile stress $t_\gamma = 1.0$ or a linear bending stress distribution $t_\gamma = 5.0 \ x/b$. Furthermore, the $K_f$-factors were directly obtained for this crack configuration by means of the BEM-A, imposing the tensile or bending stresses on the upper and lower edge of the mesh in Fig. 2. The results are summarized in Table 1, with the reference values from the $K_f$-handbook [7] being also included. The comparison shows good agreement between all three results.

As the next testing example, the same rectangular sheet with a central inner crack was chosen (width $2b=10.0$, $2h=4b$, crack length $2a$). The Bueckner field needed for the internal crack in the infinite plane is given in [2] by the Westergaard stress functions:
\[ Z(z) = 2B_i \sqrt{2a z/(z^2 - a^2)^{3/2}} \] (7)

\[ \overline{Z}(z) = -2B_i \sqrt{2a l/(z^2 - a^2)^{1/2}} \] (8)

The resulting opening displacements along the crack \(-a < x < +a\) are:

\[ u_{12} = 4\sqrt{2a B_i/((x - a)^*(x + a))^{1/2}}/E' \] (9)

The longer the expressions for the stress state of the fundamental field for the internal crack are omitted here.

Again, the stress intensity factors for this tensile sheet were determined by calculating and integrating the weight functions and additionally by means of an explicit BEM-A analysis. The comparison with the \(K_i\)-handbook \[7\] revealed an accuracy better that 2 percent, see Table 2.

A special advantage of the BEM-alternating technique \[6\] is that the results are represented by a series expansion of Chebyshev-polynomials. Thus the regular crack face displacements \(u_{12}^f\) obtained in items b) and c) can be expressed by a few normalized coefficients \(c_n = b_n a^6/B_i^2/\) :

\[ u_{12}^f = \frac{2}{E\sqrt{a}} \sum_{n} c_n \text{Im} \left\{ \frac{x/a - (x^2/a^2 - 1)^{1/2}}{n} \right\} \] (10)

After superimposing it with the singular term for the infinite region (3) or (9), the weight functions are given in an analytical form! For example, the coefficients \(c_0 = 0.524737\) and \(c_1 = 0.017934\) (all other are negligible) are sufficient to describe with high accuracy the \(u_{12}^f\)-displacements for the considered internal crack problem \(a = 0.5\) b, and the \(K_i\)-factor evaluated in this manner amounts to 3.269 (compare Table 2).

The aim of the present paper was to verify the outlined new procedure, i.e. to find weight functions by calculating the Bueckner fundamental field for a finite crack geometry by the superposition technique. The following conclusions can be drawn:

- The superposition technique avoids any numerical difficulties connected with the numerical treatment of the (inadmissible) Bueckner singularity as encountered in [2,3]. The numerical effort is reduced to one single analysis of a regular crack problem with a \(K_i\)-singularity, which is feasible by contemporary FEM- or BEM-codes.

- As the results for edge and internal cracks demonstrate, weight functions with good accuracy can be achieved.

- The extension of the method to mode II, mode III, or mixed mode analyses is straightforward.

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REFERENCES


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Figure 1. Superposition of the finite body solution with Bueckner-dipole from the infinite body solution and a finite body correction term.
Figure 2. Typical discretization of a single edge crack specimen used in the boundary element alternating method.