FINITE ELEMENT ANALYSIS OF STATIC AND DYNAMIC FRACTURE OF BRITTLE MICROCRACKING SOLIDS
Part 1: Formulation and Simple Numerical Examples

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Abstract—The continuum constitutive modeling for rate-dependent fracture of brittle microcracking solids is discussed. The rate-type constitutive equation that is proposed takes into account the rate effect on microcracking and plastic deformation. In order to best the validity of the proposed modeling, numerical studies are conducted on a bar under uniaxial tension, a beam under pure bending, and on the phenomenon of microcracking around the tip of a macro-crack under mode-I loading.

I. INTRODUCTION

The present study is concerned with the continuum constitutive modeling of the rate-dependent behavior of brittle microcracking solids, and its applications to the finite element analysis of structural and fracture mechanics problems involving brittle microcracking materials. The final purpose is to establish a finite element method for the study of dynamic fracture of brittle materials based on the approach of continuum damage mechanics (Kachanov [1980,1986], Krückinovic [1985] and Charalambides & McMeeking [1987a,b]). However, only static problems will be treated in this Part 1, as a first step. The outlines of each section are described below, referring to the existing related works.

Section 2 contains a description of the constitutive modeling of the rate-dependent behavior of brittle microcracking solids, which is used in numerical calculations performed in the present study. As a prelude, the rate-independent constitutive modeling adopted in Charalambides and McMeeking [1987a,b], which is based on the work of Fu and Evans [1985], Evans and Fu [1985] and Budiansky and O'Connell [1976] is summarized first in Section 2.1. Charalambides and McMeeking expressed their constitutive equation in terms of total stresses and strains as for a hyper-elastic body, and employed the Newton's iteration scheme to solve the resulting nonlinear algebraic equations. However, rate type of constitutive equations are preferable in general, considering that they can be used in arbitrary loading/unloading situations, can be easily extended to include various kinds of material nonlinearities such as creep and viscoplasticity, and bring out the true inelastic nature of microcracking. Furthermore, they can be directly implemented in the existing finite element analysis codes most of which are based on the incremental solution procedures (Zienkiewicz [1977]; Owen & Hinton [1980]). For those reasons, the constitutive equation used in Charalambides and McMeeking [1987a,b] is transformed into the rate form in the following Section 2.2.

The presently proposed rate-dependent constitutive modeling for brittle microcracking solids, and the attendant computational procedure in the finite element analysis, are described in the latter half (Sections 2.3 and 2.4) of Section 2, where the natural
and simple extension to the rate-dependent problem of the rate-form expression of the modeling by Charalambides and McMeeking [1987a,b] is combined with the simplified viscoplastic constitutive relation based on the modeling by Perzyna [1986].

It should be mentioned here that Perzyna [1986] proposes the elasto-viscoplastic model of damaged solids based on the works by Mackenzie [1959], Seaman, Curran and Shocky [1976] and Seaman, Curran and Murri [1985], in which both microcracking and plastic deformation are taken into account with their rate effects. Although his equations are complicated and include too many material constants from a practical point of view, they seem to be complete in their mathematical forms, so that the application of Perzyna's constitutive modeling to the finite element analysis will be an interesting subject, if adequate experimental data are available to determine the material constants contained in his theory.

Section 3 contains three numerical examples, in which the proposed constitutive modeling is applied to the finite element analysis of some standard problems of structural and fracture mechanics under static loading.

The uniaxial tension problem (Owen & Hinton [1980]), the simplest one-dimensional problem, is solved first to confirm the validity of the proposed rate-form formulation, and to observe the physics of the typical behavior of materials obeying the proposed constitutive equation. In advance of this analysis, the one-dimensional form of the constitutive equation for brittle microcracking solids, which is a special case of the general three-dimensional theory given in Section 2, is briefly described.

The second example is that of the pure bending of a beam composed of brittle microcracking material (Krajcinovic [1979]), which is one of the fundamental problems in structural mechanics of such materials. The results for material behavior with and without plastic deformations are described, along with the formulation of the solution procedure.

The last example involves the finite element analysis of the microcracking around the tip of a macro-crack under mode-I static loading (tensile opening), which is one of standard problems in fracture mechanics. In this analysis we employ the two-dimensional (plane strain) specialization of the general three-dimensional theory described in Section 2. The calculated result involving the presently assumed rate-dependency of microcracking density is checked against the rate-independent solution given by Charalambides and McMeeking [1987a,b]. Section 4 contains some concluding remarks.

II. CONSTITUTIVE MODELING OF BRITTLE MICROCRACKING SOLIDS

II.1. Constitutive modeling for rate-independent fracture

The constitutive modeling by Charalambides and McMeeking [1987a,b] is summarized here. The relation between the microcrack density and the equivalent stress is assumed as

\[
\begin{align*}
\xi &= 0 \quad \text{when } \sigma_e < \sigma_c \\
\xi &= \lambda (\sigma_e - \sigma_c) \quad \text{when } \sigma_c \leq \sigma_e \leq \sigma_m \\
\xi &= \lambda (\sigma_m - \sigma_c) = \xi_s \quad \text{when } \sigma_e > \sigma_m
\end{align*}
\]

(1)

where \(\sigma_e = (\sigma_y \sigma_d)^{1/2}\). The microcracking is governed by three relations given in eqns (1) depending on the value of equivalent stress. The first and the third equations
indicate the absence of microcracking, and the saturation of microcrack density (with saturation level $\xi_s$), respectively. And in the region governed by the second equation, the microcrack density increases linearly with the equivalent stress, provided that eqns (1) are applicable only in the region where the maximum principal stress is tensile and increases monotonically. The relation between the reduced elastic moduli and the microcrack density is given by the following equation according to BUDIANSKY and O'CONNELL [1976] (which is a very rough approximation):

$$\frac{E}{E} = \frac{\dot{\varepsilon}}{\nu} = \{1 - (16/9)\xi\} = 1/f.$$  \hfill (2)

By replacing the elastic constants in the three-dimensional stress-strain equations of isotropic linear elastic bodies with the reduced moduli as given in eqns (2), the following expressions can be obtained:

$$\varepsilon_{ij} = \left[\frac{(f + \nu)/E}{\nu/E}\right]_{\sigma_{kk}} \delta_{ij}$$  \hfill (3a)

or

$$\sigma_{ij} = \left\{\frac{E}{(f + \nu)}\right\} \left\{\varepsilon_{ij} + \left[\frac{\nu/(f - 2\nu)}{\nu}\right]_{\varepsilon_{kk}} \delta_{ij}\right\}.$$  \hfill (3b)

Charalambides and McMeeking formulated the problem as the nonlinear elastic (hyperelastic) problem by using the above constitutive relationship (3a) or (3b) and applied an iterative technique to solve the resulting highly nonlinear algebraic equations. The hyperelastic type constitutive law employed by Charalambides and McMeeking [1987a,b] is applicable only in the case of monotonic loading, and precludes any unloading. However, there is another way of formulating the problem as shown below, in which the rate-form (or incremental form) constitutive equation is employed. This rate-form of continuum modeling of microcracking effects is similar, in mathematical structure, to that of continuum plasticity and is amenable to a very easy implementation in piecewise linear finite element computations; furthermore, this rate form clearly brings out the true inelastic nature of microcrack formulation in a loading/unloading process.

II.2. Rate-form expression of constitutive modeling for rate-independent fracture

Defining the equivalent stress as $\sigma_e = (\sigma_{ij}\sigma_{ij})^{1/2}$, eqns (1) can be rewritten as follows:

$$\xi = 0 \quad \text{when} \quad \sigma_e < \sigma_c$$

$$\xi = \lambda (\sigma_e - \sigma_c) \quad \text{when} \quad \sigma_c \leq \sigma_e \leq \sigma_m$$

$$\xi = \xi_s \quad \text{when} \quad \sigma_e > \sigma_m.$$  \hfill (4)

The differentiation of eqns (4) with respect to time leads to the following expression for the microcracking rate (or evolution equation for microcracking)

$$\dot{\xi} = 0 \quad \text{when} \quad \sigma_e < \sigma_c$$

$$\dot{\xi} = \lambda \sigma_{ij} \sigma_{ij}/\sigma_e \quad \text{when} \quad \sigma_c \leq \sigma_e \leq \sigma_m; \text{ and } \sigma_{ij} \delta_{ij} > 0$$

$$\dot{\xi} = 0 \quad \text{when} \quad \sigma_{ij} \delta_{ij} \leq 0$$

$$\dot{\xi} = 0 \quad \text{when} \quad \sigma_e > \sigma_m.$$  \hfill (5)
The microcracking internal variable is defined as

\[ f = \frac{9}{(9 - 16\xi)} \]  

and then, the evolution equation for the internal variable is calculated as

\[ \dot{f} = \left[ \frac{144}{(9 - 16\xi)^2} \right] \dot{\xi}. \]  

The differentiation of eqn (3a) with respect to time leads to the following rate-type constitutive relationship:

\[ \dot{\varepsilon}_{ij} = \frac{\dot{f}}{E} \varepsilon_{ij} + \frac{(f + \nu)}{E} \dot{\sigma}_{ij} - \frac{\nu}{E} \dot{\sigma}_{kk} \delta_{ij}. \]  

Using eqn (6) for \( f \), eqn (7) for \( \dot{f} \), eqns (4) for \( \xi \) and eqns (5) for \( \dot{\xi} \) in eqn (8), the following constitutive equations in the rate form can be finally obtained:

\[ \dot{\varepsilon}_{ij} = C_{ijkl} \sigma_{kl} \]

\[ = \left[ \frac{(1 + \nu)}{E} \varepsilon_{ij} - \frac{\nu}{E} \varepsilon_{kk} \delta_{ij} \right] \]

when \( \sigma_e < \sigma_c \)

\[ \dot{\varepsilon}_{ij} = C_{ijkl}(\sigma_{ij}) \sigma_{kl} + \dot{\varepsilon}_{ij}^{\text{visc}}(\sigma_{ij}, \dot{\sigma}_{ij}) \]

\[ = \left[ \frac{1}{E} \left\{ \frac{f/e}{(\sigma_c - \sigma_c)} + \nu \right\} \varepsilon_{ij} - \frac{\nu}{E} \varepsilon_{kk} \delta_{ij} \right] \]

\[ + \left[ \frac{144}{(9 - 16\xi)} \sigma_{ij} \dot{\sigma}_{ij} / \sigma_e \right] / \sigma_e \]

when \( \sigma_c \leq \sigma_e \leq \sigma_m \), and \( \sigma_{ij} \dot{\sigma}_{ij} > 0 \)

\[ \dot{\varepsilon}_{ij} = C_{ijkl} \sigma_{kl} \]

\[ = \left[ \frac{f/e}{(9 - 16\xi)} + \nu \right\} \varepsilon_{ij} - \frac{\nu}{E} \varepsilon_{kk} \delta_{ij} \right] \]

when \( \sigma_e > \sigma_m \).

Equation (9b) consists of two components, which behave like a nonlinear elastic strain rate, and a viscoelastic strain rate, respectively. Equations (9) can be directly used in the conventional load-incremental finite element method. The inelastic nature of the rate form of eqns (9) is made clear through the example of uniaxial tension in Section 3.1 of this paper (Part 1).

II.3. Rate-dependent constitutive equations for brittle microcracking solids

In this section the constitutive modeling of brittle microcracking materials is discussed, taking into account the rate effect of microcracking as well as the effect of viscoplastic deformation.

Equations (5) are extended so as to take into account the rate effect on microcracking as follows:

\[ \dot{\xi} = 0 \quad \text{when} \quad \sigma_e < \sigma_c \]  

\[ \dot{\xi} = \frac{1}{\eta} \left( \left[ \frac{\sigma_e}{\sigma_c + \xi/\lambda} \right] - 1 \right) \quad \text{when} \quad \sigma_e \geq \left[ \sigma_c + \left( \xi/\lambda \right) \right] \]  

\[ \dot{\xi} = 0 \quad \text{when} \quad \xi_s \leq \xi \]
which is the natural and the simplest extension from eqns (5). Equation (10b) for the time-rate of microcrack density is analogous to that for plastic strain-rate in strain-rate-hardening elastic-plastic materials. The saturating condition for microcrack density in eqn (10c) is expressed in terms of the microcrack density value itself instead of the equivalent stress in eqn (5), because the equivalent stress can be larger than \( \sigma_m \) due to the rate effect here. Substituting eqns (6), (7), and (10) into eqns (8), the following new form of rate-dependent constitutive equations for brittle microcracking solids can be derived:

\[
\dot{\epsilon}_{ij} = C_{ijkl} \dot{\sigma}_{kl} \\
= [(1 + \nu)/E] \dot{\epsilon}_{ij} - [\nu/E] \dot{\sigma}_{kk} \dot{\epsilon}_{ij}
\]

when \( \sigma_e < \sigma_c \)

\[
\dot{\epsilon}_{ij} = C_{ijkl} (\xi) \dot{\sigma}_{kl} + \dot{\epsilon}_{ij}^p (\sigma_e, \xi)
\]

\[
= [(9/9 - 16\xi) + \nu]/E] \dot{\epsilon}_{ij} - [\nu/E] \dot{\sigma}_{kk} \dot{\epsilon}_{ij}
+ [(144/(9 - 16\xi)^2) (1/\eta) \sigma_e/(\sigma_e + \xi/\lambda) - 1)]/E \sigma_{ij}
\]

when \( \sigma_c \leq \sigma_e \)

\[
\dot{\epsilon}_{ij} = C_{ijkl} \dot{\sigma}_{kl}
\]

\[
= [(9/9 - 16\xi_s) + \nu]/E] \dot{\epsilon}_{ij} - [\nu/E] \dot{\sigma}_{kk} \dot{\epsilon}_{ij}
\]

when \( \xi_s \leq \xi \).

As shown in eqn (11b), the strain rate in the microcracking region before saturation, as in eqn (9b), consists of two components, which are the nonlinear elastic strain rate and the viscoelastic strain rate respectively, and the latter includes the rate effect on microcracking.

On the other hand, the viscoplastic component of the strain-rate is assumed as follows, according to the conventional formulation (Owen & Hinton [1980]):

\[
\dot{\epsilon}_{ij}^p = \gamma \left\langle \Phi \left[ \frac{f(\cdot)}{f_0} - 1 \right] \right\rangle \frac{\partial f}{\partial \sigma_{ij}}
\]

where

\[
\left\langle \Phi [x] \right\rangle = \begin{cases} 
0 & \text{if } x \leq 0 \quad \{f(\cdot) \leq f_0(\dot{\epsilon}_p, \xi)\} \\
\Phi [x] & \text{if } x > 0 \quad \{f(\cdot) \geq f_0(\dot{\epsilon}_p, \xi)\}
\end{cases}
\]

\[
\Phi [\cdot] = \left[ \frac{f(\cdot)}{f_0} - 1 \right]^n, \quad n = 1, 3, 5, \ldots
\]

The quasi-static yield functions can be assumed, for instance, as follows, referring to Perzyna [1986]:

\[
f(\cdot) = (3J_2)^{1/2}(1 - m\xi) \quad \text{von Mises}
\]

\[
f(\cdot) = [\alpha J_1 + (J_2)^{1/2} - k](1 - m\xi) \quad \text{Drucker-Prager}
\]
and the material work-hardening and -softening function can be given by

$$f_0 = f_0(\dot{\varepsilon}^p, \xi) = (\sigma_0 + H' \dot{\varepsilon}^p)(1 - n \xi^{1/2}).$$

(16)

By combining eqns (11) with eqn (12), the total strain rate, which is composed of nonlinear elastic, viscoelastic and viscoplastic components, can be expressed as follows:

$$\dot{\epsilon}_{ij} = C_{ijkl}(\xi) \dot{\sigma}_{kl} + \dot{\epsilon}_{ij}^{ve}(\sigma_{ab}, \xi) + \dot{\epsilon}_{ij}^{vp}(\sigma_{ab}, \xi)$$

(17a)

eqn (17a) can be also written as

$$\dot{\sigma}_{ij} = D_{ijkl}(\xi) \{ \dot{\sigma}_{kl} - \dot{\epsilon}_{i}^{ve}(\sigma_{ab}, \xi) - \dot{\epsilon}_{j}^{vp}(\sigma_{ab}, \xi) \}.$$  

(17b)

II.4. Computational procedure in the finite element analysis

The computational procedure in the finite element analysis using the constitutive equations derived above is explained in the following.

Consider the equilibrium state at time $t_{n+1}$ (i.e., $t_n + \Delta t_n$) assuming that all state variables are known at time $t_n$. The discretized equilibrium equations at time $t_{n+1}$ are expressed as follows:

$$\int_{V} [B]^T \sigma_{n+1} dV + \{ f_{n+1} \} = 0.$$  

(18)

Another set of nonlinear algebraic equations can be derived as follows, applying Euler's formula for numerical time integration to the constitutive eqn (17b):

$$\{ \sigma_{n+1} \} - \{ \sigma_n \} - [D(\xi_n)][B]\{ \Delta u_n \} + [D(\xi_n)]\Delta t_n (\{ \dot{\epsilon}_{n}^{ve} \} + \{ \dot{\epsilon}_{n}^{vp} \}) = 0$$

(19)

where

$$\{ \dot{\epsilon}_{n}^{ve} \} = \{ \dot{\epsilon}_{n}^{ve}(\sigma_n, \xi_n) \}$$

$$\{ \dot{\epsilon}_{n}^{vp} \} = \{ \dot{\epsilon}_{n}^{vp}(\sigma_n, \xi_n) \}.$$  

(20)

Substituting eqn (19) into eqn (18), the following incremental stiffness equation to obtain the displacement increments can be derived:

$$\int_{V} [B]^T [D(\xi_n)][B] dV \cdot \{ \Delta u_n \} + \int_{V} [B]^T \{ \sigma_n \} dV$$

$$- \int_{V} [B]^T [D(\xi_n)] \Delta t_n (\{ \dot{\epsilon}_{n}^{ve} \} + \{ \dot{\epsilon}_{n}^{vp} \}) dV + \{ f_n \} = 0.$$  

(21)

In the above-mentioned solution procedure the effect of reduction of elastic moduli due to microcracking is included in the tangential stiffness, and the viscoelastic and the viscoplastic components are treated as initial strains. Although an alternate approach using a full tangential stiffness, based on the implicit time marching schemes of state variables (Yoshimura, Chen, & Atluri [1987]) might be devised, and may yield an improvement of numerical stability; however, it remains as an object of future work.
III. FINITE ELEMENT ANALYSIS OF BRITTLE MICROCRACKING SOLIDS

III.1. Analysis of a bar under uniaxial tension

The one-dimensional constitutive modeling for brittle microcracking materials is briefly discussed, based on the general, three-dimensional theory described in the preceding section.

The relation between the microcrack density rate and uniaxial stress is expressed as

\[ \dot{\xi} = \begin{cases} 0 & \text{when } \sigma < \sigma_c \\ \frac{1}{\eta} \left( \frac{\sigma}{\sigma_c + \xi/\lambda} \right) & \text{when } \sigma > \sigma_c + \xi/\lambda \\ 0 & \text{when } \sigma < \sigma_c + \xi/\lambda \\ 0 & \text{when } \xi_s \leq \xi. \end{cases} \]  

(22)

The microcracking internal variable and its rate are given by eqns (6) and (7), respectively. The constitutive relation is expressed in the rate form as follows:

\[ \dot{\varepsilon} = \frac{[f/E]}{\varepsilon} + \frac{[f/E]}{\dot{\varepsilon}}. \]  

(23)

Note that \( \dot{\xi} > 0 \) when \( \sigma \geq [\sigma_c + (\xi/\lambda)] \) during the loading, while \( \dot{\xi} = 0 \) [and hence \( \dot{\varepsilon} = 0 \)] during unloading even when \( \sigma \geq \sigma_c \). Substitution of eqns (6), (7), and (22) into eqn (23) leads to the final form of constitutive equations given by

\[ \dot{\varepsilon} = C\dot{\varepsilon} \]
\[ = \left[ \frac{1}{E} \right] \dot{\varepsilon} \]
\[ \text{when } \sigma < \sigma_c \]
\[ \dot{\varepsilon} = C(\xi)\dot{\varepsilon} + \dot{\varepsilon}_{\text{tr}}(\sigma, \xi) \]
\[ = \left[ \frac{9/(9 - 16\xi)}{E} \right] \dot{\varepsilon} \]
\[ + \sigma \left( \frac{144/(9 - 16\xi)^2}{(1/\eta) \left[ \frac{\sigma}{(\sigma_c + \xi/\lambda)} - 1 \right]} \right) / E \]
\[ \text{when } \sigma \geq [\sigma_c + (\xi/\lambda)] \]
\[ \dot{\varepsilon} = C\dot{\varepsilon} \]
\[ = \left[ \frac{9/(9 - 16\xi_s)}{E} \right] \dot{\varepsilon} \]
\[ \text{when } \xi_s \leq \xi. \]

The viscoplastic strain rate is expressed as

\[ \dot{\varepsilon}^{\text{vp}} = \gamma \left( \Phi \left[ \frac{f(\cdot)}{f_0} - 1 \right] \right) \partial f / \partial \sigma \]  

(25)

where

\[ \langle \Phi[x] \rangle = \left\{ \begin{array}{ll} 0 & \text{if } x \leq 0 \\ \Phi[x] & \text{if } x > 0 \end{array} \right. \left\{ \begin{array}{ll} f(\cdot) \leq f_0(\varepsilon_p, \xi) \\ f(\cdot) > f_0(\varepsilon_p, \xi) \end{array} \right. \]  

(26)
The quasi-static yield function and the material work-hardening and softening function are given by eqn (28) and eqn (29), respectively.

\[
f(\cdot) = \sigma(1 - m\xi) \tag{28}
\]

\[
f_0 = f_0(\epsilon^p, \xi) = (a_0 + H'\epsilon^p)(1 - n\xi^{1/2}). \tag{29}
\]

By adding the viscoplastic component given by eqn (25) to eqns (24), the final form of one-dimensional constitutive equation can be obtained as follows:

\[
\dot{\varepsilon} = C(\xi)\sigma + \dot{\varepsilon}^{ve}(\sigma, \xi) + \dot{\varepsilon}^{vp}(\sigma, \xi) \tag{30a}
\]

or

\[
\dot{\sigma} = D(\xi)\{\dot{\varepsilon} - \dot{\varepsilon}^{ve}(\sigma, \xi) - \dot{\varepsilon}^{vp}(\sigma, \xi)\}. \tag{30b}
\]

The input data used in the analysis of a bar under tensile loading are as follows:

- Loading function: \( f(t) = 0.2t \)
- Duration time of loading: \( t_f = 10 \)
- Length of a bar: \( L = 10 \)
- Sectional area of a bar: \( A = 1 \)
- Young's modulus: \( E = 10000 \)
- Critical stress for microcrack initiation: \( \sigma_c = 0.75 \)
- Microcracking rate with stress: \( \lambda = 2.0 \)
- Viscous coefficient for microcrack density: \( \eta = 0.1, 1.0, 3.0 \)
- Static yield stress: \( \sigma_0 = 1.2, 1.6 \)
- Work hardening coefficient: \( H' = 0 \)
- Material constants for microcracking effect on plastic deformation: \( m = 0, n = 0 \)
- Viscosity constant for plastic deformation: \( \gamma = 0.01, 0.001, 0.0001 \).

The following three cases are analyzed by using one-element idealization and a time increment of \( \Delta t = 0.1 \) (100 steps):

**Case 1:** microcracking analysis

- \( \eta = 0.1 \) \( \gamma = 0 \)
- \( \eta = 1.0 \) \( \gamma = 0 \)
- \( \eta = 3.0 \) \( \gamma = 0 \)

**Case 2:** microcracking-viscoplastic analysis

- \( \eta = 0.1 \) \( \gamma = 0.01 \) \( \sigma_0 = 1.6 \)
- \( \eta = 1.0 \) \( \gamma = 0.001 \) \( \sigma_0 = 1.6 \)
- \( \eta = 3.0 \) \( \gamma = 0.0001 \) \( \sigma_0 = 1.6 \)
Case 3: microcracking-viscoplastic analysis

\[ \eta = 0.1 \quad \gamma = 0.01 \quad \sigma_0 = 1.2 \]

\[ \eta = 1.0 \quad \gamma = 0.001 \quad \sigma_0 = 1.2 \]

\[ \eta = 3.0 \quad \gamma = 0.0001 \quad \sigma_0 = 1.2 \].

Figure 1 shows the calculated results for Case 1 in which the plastic deformation is not considered. Figure 1a is the stress–strain relationship and Fig. 1b shows the rela-

Fig. 1. Microcracking analysis of a bar under tensile loading (a) stress–strain curves, (b) microcrack density–stress relations.
tion between microcrack density and stress. From Fig. 1a, the loading/unloading paths in the uniaxial stress-strain space clearly demonstrate the inelastic nature of the process of microcracking. For \( \eta = 0 \), during loading, the material is linear when \( \sigma < \sigma_c \); is nonlinear in the region \( \sigma_c < \sigma < \sigma_m \); and is again linear for \( \sigma > \sigma_m \). However all unloading paths from any given stress-state lead to the origin in the stress-strain space. In these figures the solutions given by Charalambides and McMeeking's rate-independent modeling are also drawn in dotted lines. It is noted that the present results using the smallest value of viscosity coefficient almost coincide with the rate-independent solutions of Charalambides and McMeeking [1987a,b]. Figures 2a and 2b are the calculated stress-strain curves for Cases 2 and 3, respectively, in which different values of yield stress are assumed. Both results can be considered reasonable from a qualitative point of view.

III.2. Analysis of a beam under pure bending

Here we consider the pure bending of a beam made of a microcracking material. The width of the beam is taken to be unity, and only a unit length of the beam is analyzed, primarily to obtain an understanding of the variation of the stresses through the depth of the beam, in the presence of material nonlinearity induced by microcracking damage.

The one-dimensional constitutive equation used in the present numerical study has been given in the preceding section, therefore only the computational procedure will be described here.

We consider the equilibrium state at time \( t_{n+1} = t_n + \Delta t_n \) under the assumption that all state variables at time \( t_n \) are known. The discretized equilibrium equations at time \( t_{n+1} \) are expressed as follows:

\[
\int_A \sigma_{n+1} dA - P_{n+1} = 0
\]  

(31a)

Fig. 2. Microcracking and viscoplastic analysis of a bar under tensile loading (a) \( \sigma_0 = 1.6 \), (b) \( \sigma_0 = 1.2 \).
Equation (31a) is the equilibrium equation in the axial direction and eqn (31b) represents the equilibrium condition for bending moment. The application of Euler's time integration formula to eqn (30b) leads to the following nonlinear equation:

\[ \sigma_{n+1} - \sigma_n - D(\xi_n)\Delta \epsilon_n + D(\xi_n)\Delta t_n(\dot{\epsilon}^{\text{ur}} + \dot{\epsilon}^{\text{up}}) = 0 \]  

(32)

where

\[ \Delta \epsilon_n = \Delta \epsilon_n^0 + y\Delta \kappa_n \]

\[ D(\xi_n) = [(9 - 16\xi_n)/9]E \]

\[ \dot{\epsilon}^{\text{ur}} = \dot{\epsilon}^{\text{ur}}(\sigma_n, \xi_n) \]

\[ = [(144/(9 - 16\xi_n)^2)(1/\eta)\langle\sigma_n/(\sigma_c + \xi_n/\lambda)\rangle - 1]/E\sigma_n \]

\[ \dot{\epsilon}^{\text{up}} = \dot{\epsilon}^{\text{up}}(\sigma_n, \xi_n) \]

\[ = \gamma [\sigma_n(1 - m\xi_n)/((\sigma_0 + H\epsilon^p)(1 - n\xi_n^{1/2})) - 1]. \]  

Substituting eqn (32) into eqns (31a) and (31b), the following set of incremental stiffness equations can be derived:

\[ \int_A D(\xi_n)dA \cdot \Delta \epsilon_n^0 + \int_A yD(\xi_n)dA \cdot \Delta \kappa_n \]

\[ - \int_A D(\xi_n)\Delta t_n(\dot{\epsilon}^{\text{ur}} + \dot{\epsilon}^{\text{up}})dA + \int_A \sigma_n dA - P_{n+1} = 0 \]  

(34a)

\[ \int_A yD(\xi_n)dA \cdot \Delta \epsilon_n^0 + \int_A y^2D(\xi_n)dA \cdot \Delta \kappa_n \]

\[ - \int_A yD(\xi_n)\Delta t_n(\dot{\epsilon}^{\text{ur}} + \dot{\epsilon}^{\text{up}})dA + \int_A y\sigma_n dA - M_{n+1} = 0. \]  

(34b)

By solving eqns (34a) and (34b) simultaneously, the increments of uniform axial strain and curvature change can be obtained.

The pure bending behavior of a beam with a square cross-section is simulated according to the computational procedure described above.

The input data used in the analysis are as follows:

Loading function: \( P(t) = 0 \) (i.e. pure bending)

\[ M(t) = 0.05t \]

Duration time of the loading: \( t_f = 1 \)

Breadth of the cross-section: \( b = 1.0 \)

Depth of the cross-section: \( d = 1.0 \)

Young's modulus: \( E = 10000 \)
Critical stress for microcrack initiation: $\sigma_c = 0.75$

Microcracking rate with stress: $\lambda = 2.0$

Saturated value of microcrack density: $\xi_s = 0.5$

Viscosity coefficient for microcrack density: $\eta = 0.1$

Static yield stress: $\sigma_0 = 0.75$

Work hardening coefficient: $H' = 0$

Material constants for microcracking effect on plastic deformation: $m = 0, n = 0$

Viscosity constant for plastic deformation: $\gamma = 0.01$

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Fig. 3. Microcracking analysis of a beam under pure bending, (a) bending moment-curvature curve, (b) stress variations in the depth direction, (c) stress-strain relations, (d) development of microcracking region in the depth direction.
The following two cases are analyzed using the time increment $\Delta t = 0.01$ (100 steps):

**Case 1: microcracking analysis**

$\eta = 0.1 \quad \gamma = 0 \quad \sigma_0 = \infty$

**Case 2: microcracking-viscoplastic analysis**

$\eta = 0.1 \quad \gamma = 0.01 \quad \sigma_0 = 0.75$

The numerical integration to calculate stiffness matrices is conducted by using the trapezoidal rule with eleven integration points in the depth direction.

The results for Case 1 are shown in Figs. 3a–d and the results for Case 2 are seen in Figs. 4a–d. Each set of figures shows the bending moment–curvature relation, the

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**Fig. 4.** Microcracking and viscoplastic analysis of a beam under pure bending, (a) bending moment–curvature curve, (b) stress variations in the depth direction, (c) stress–strain relations, (d) development of microcracking region in the depth direction.
stress variations in the depth direction as the loading progresses, the stress–strain relations at three reference points, and the development of microcracking (and viscoplastic in Case 2) regions in the depth direction. In Case 1, the fracture, which is due to microcracking, occurs only in the tension side of a beam, therefore the behaviors on the tension and the compression sides are extremely different as shown in Figs. 3b,c. On the other hand in Case 2, where the yield stress is assumed equal to the critical stress for microcrack initiation, the fracture takes place both in the tension and the compression side and the nearly symmetrical behavior with respect to the original neutral axis can be observed. The present results for both cases can be considered reasonable from a qualitative point of view.

III.3. Analysis of microcracking around the tip of a macro-crack under mode-I static loading

Figure 5 shows the microcracking boundary value problem around the tip of a macro-crack under mode-I loading. The tractions $T_i$ on the outer boundary of the analyzed region are determined by the following singular crack tip stress field for mode-I (tensile opening).

$$
\sigma_x = \frac{K_I}{2\pi r} \cos\left(\frac{\theta}{2}\right) [1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)]
$$

$$
\sigma_y = \frac{K_I}{2\pi r} \cos\left(\frac{\theta}{2}\right) [1 - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)]
$$

$$
\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right).
$$

The material constants are as follows:

Young's modulus: $E = 21000$

Poisson's ratio: $\nu = 0.25$

Critical stress for microcrack initiation: $\sigma_c = 9$

Microcracking rate with stress: $\lambda = 0.28125$

Saturated value of microcrack density: $\xi_s = 0.4$

Viscous coefficient for microcrack density: $\eta = 0.1$

Critical stress intensity factor: $K^f = 45$.

![Fig. 5. Boundary conditions used in the finite element analysis of the boundary value problem of microcracking around the tip of a macro-crack under mode-I static loading.](image-url)
The developed code is, of course, applicable to the analysis considering also the viscoplastic deformation, however only microcracking is taken into account here by assuming a large yield stress value in order to check the results against Charalambides and McMeeking's solutions [1987a,b] in which plastic deformation is neglected.

The input data for the finite element analysis are as follows:

- Total number of elements = 1004
- Total number of nodes = 1046
- Total number of degrees of freedom = 2039
- Type of analysis = Plane strain analysis
- Number of nodes per element = 4 (Bilinear quadrilateral)
- Order of integration formula = 2 (Two point Gauss quadrature rule)
- Number of increments in which the total loading is applied = 100.

The finite element modeling and results are described in Figs. 6-8. Figure 6 shows the finite element mesh subdivisions. This mesh is almost the same as that used by Charalambides and McMeeking [1987a,b]. Figure 7 shows the development of the zone of microcracks near the major crack tip, where the microcrack density level is marked for each element (of the near-tip mesh in Fig. 6) using the maximum value on four Gaussian integration points. The results at $K_f = K_{fc}$ [(4$/\sigma_c = 5$)] agrees well with the Charalambides and McMeeking result given [1987a], although the area of microcracking zone is a little smaller in the present solution due to the rate effect on microcracking. Figure 8 shows the microcrack density variation along the y-direction at different positions along the crack line ahead of the crack-tip. These are plotted using the average value on the integration points in each element. The nonlinearity in the density profile for the stationary microcrack can be clearly observed as in Charalambides and McMeeking [1987a,b]. The implication of these results in the toughening effect on the macro-crack due to the presence of the microcracking are discussed in detail in Part 2 of this paper.

IV. CONCLUDING REMARKS

In the present paper a continuum constitutive modeling for rate-dependent fracture of brittle microcracking solids has been presented, and it has been applied to the finite element analyses of some standard structural and fracture mechanics problems.

The proposed constitutive relation, which involves an extension of Charalambides and McMeeking's work [1987a,b] to the rate-dependent problem including viscoplasticity, is expressed in the rate form; therefore it is easier in the present modeling than in the original theory to implement the resulting equations in the conventional finite element codes based on the incremental procedure, and to take into account various kinds of material nonlinearities such as creep and viscoplasticity as well as unloading. This rate form also clearly reveals the inelastic nature of microcrack formation. However, further studies, especially based on experiments, will be necessary in order to confirm the validity of the proposed modeling from a physical point of view.

As numerical examples the static finite element analyses of three basic problems in the field of structural and fracture mechanics have been conducted.

The first numerical example, which is the analysis of a bar under tensile loading,
Fig. 6. Finite element mesh subdivisions, (a) mesh for the area except for the crack tip neighborhood, (b) mesh for the region A, (c) mesh for the region B.
indicates the feasibility of the proposed rate-form formulation by its comparison with the rate-independent solution, and illustrates the typical rate-dependent constitutive behaviors of brittle microcracking solids, especially the loading/unloading behavior of the uniaxial stress-strain curve.

Fig. 7. Development of the zone of microcracks near the crack tip ($0 \leq \xi/(K_f/\sigma_c)^2 \leq 0.8, 0 \leq \eta/(K_f/\sigma_c)^2 \leq 0.4$).

Fig. 8. Microcracking density variation in the $y$ direction at different positions in the $X$ direction ahead of the macro-crack tip.
The problem of pure bending of a beam has been analyzed as the second numerical example, by the use of one-dimensional constitutive equation derived from the general, three-dimensional theory described in Section 2. The obtained results are reasonable from a qualitative point of view.

A finite element analysis of the stationary crack tip problem under mode-I static loading has been conducted, by using the two-dimensional, rate-dependent constitutive equation for brittle microcracking solids and the calculated result has been checked against the solution given by Charalambides and McMeeking [1987a,b] for the same problem. The rate effect on the development of microcracks, which was excluded in Charalambides and McMeeking's analysis, has caused a slight discrepancy between the results of these two analyses. However, as a whole, both solutions correspond reasonably well with each other.

It should be noted here that the finite element code used in the crack-tip analysis was developed by implementing the proposed microcracking constitutive relation in the two-dimensional elasto-viscoplastic analysis program contained in Owen and Hinton [1980]. The developed program is applicable to a wide variety of problems concerning brittle microcracking solids such as ceramics (Charalambides & McMeeking [1987a,b]; Suresh [1987]) and concrete (Krajinovic [1986]) in the fields of solid, fracture, and structural mechanics. The extension to dynamic problems focusing on dynamic fracture (Atluri & Nishihara [1985]) will be presented in the accompanying Part 2 of this paper. The discussion of numerical stability, convergency and accuracy of the finite element analysis is left to a future work.

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NOTE

1. The premise of saturation of microcracking, upon which eqns (1) are based, is justifiable only for a certain class of ceramic materials under certain specific conditions. See Charalambides and McMeeking [1987a] for further details.

REFERENCES

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NOMENCLATURE

\( A \) = cross-sectional area of a bar or a beam  
\( \{B\} = \) strain-displacement matrix  
\( C_{ijkl} = \) compliance for a microcracking solid  
\( \{C\} = \) compliance matrix for a microcracking solid  
\( D_{ijkl} = \) elastic constants for a microcracking solid  
\( \{D\} = \) stress-strain matrix for a microcracking solid  
\( E = \) Young's modulus of unmicrocracked solids  
\( \bar{E} = \) Young's modulus of microcracked solids  
\( f = \) microcracking internal variable  
\( \{f\} = \) external force vector  
\( f(\cdot) = \) quasi static yield function  
\( f_0 = \) material work hardening and softening function  
\( H' = \) work hardening coefficient  
\( J_1 = \) first invariant of the Cauchy stress tensor  
\( J'_i = \) second invariant of the stress deviator  
\( K_f = \) stress-intensity factor for mode-I  
\( K_f^{cr} = \) critical stress-intensity factor for mode-I  
\( M = \) bending moment on a beam  
\( m = \) material constant  
\( n = \) material constant  
\( (\cdot)^n = \) indicates the value at \( n \)-increment (or time) step  
\( P = \) axial force on a bar or a beam  
\( t = \) time  
\( \{u\} = \) nodal displacement vector  
\( y = \) depth coordinate on a cross-section of a beam  
\( \gamma = \) viscosity constant for plastic deformation  
\( \Delta = \) indicates increments of variables  
\( \delta_{ij} = \) Kronecker delta  
\( \varepsilon = \) macroscopic strain tensor  
\( \varepsilon^{\circ} = \) uniform axial strain in a beam  
\( \varepsilon^{p} = \) equivalent plastic strain  
\( \{\varepsilon\} = \) total strain vector  
\( \{\varepsilon^{vis}\} = \) viscoelastic strain vector due to microcracking
\( \{ \varepsilon^{vp} \} \) = viscoplastic strain tensor
\( \eta \) = viscous coefficient for microcracking
\( \kappa \) = curvature change in a beam
\( \lambda \) = microcracking rate with stress
\( \nu \) = Poisson's ratio of unmicrocracked solids
\( \rho \) = Poisson's ratio of microcracked solids
\( \xi \) = microcrack density
\( \xi_s \) = saturated value of microcrack density
\( \sigma_{ij} \) = macroscopic stress tensor
\( \sigma_e \) = equivalent stress (= \( \sqrt{\sigma_{ij} \sigma_{ij}} \))
\( \sigma_c \) = critical stress for microcrack initiation
\( \sigma_m \) = critical stress for microcrack saturation
\( \sigma_0 \) = uniaxial yield stress
\( \{ \sigma \} \) = stress vector
(\( \cdot \)) = indicates differentiation with respect to time