A FINITE-DIFFERENCE ALTERNATING METHOD FOR A COST-EFFECTIVE DETERMINATION OF WEIGHT-FUNCTIONS FOR ORTHOTROPIC MATERIALS IN MIXED-MODE FRACTURE

K.-L. CHEN† and S. N. ATLURI‡
Center for Computational Mechanics, Georgia Institute of Technology, Atlanta, GA 30332-0356, U.S.A.

Abstract—In this paper, we present a general two-dimensional finite element alternating method for the determination of weight functions for isotropic or orthotropic cracked structures, subjected to mixed-mode loading. The numerical results show that the weight functions are indeed load-independent, and are insensitive to a wide range of virtual crack extensions of \( \Delta a \) (\( 10^{-3}a \sim 10^{-9}a \)). The present alternating procedure (which involves a stress analysis of only the uncracked structure) is indeed a most effective technique for engineering fracture and life estimation analyses.

1. INTRODUCTION

The problems of two-dimensional fracture mechanics have received much attention due to the fact that the actual flaw from which fracture is initiated in a structural component, can be approximated, often by an embedded or edge crack. There are several numerical methods for evaluating the fracture parameters such as stress intensity factors. The hybrid finite element method[1–3], boundary integral equation [4], and alternating method[5] are some examples of such numerical schemes. Nishioka and Atluri[5] have presented a general alternating procedure to handle three-dimensional structures with elliptical surface cracks subjected to arbitrary loading.

In the practical application of fracture mechanics, the determination of weight functions[6] is often more advantageous than the calculation of stress intensity factors alone: in life prediction studies it is indispensable to analyse the cyclic load changes on a flawed structure. The use of weight functions can obviate the repeated computer calculations for given structural and crack geometries. With the weight function concept, the stress intensity factor is expressed as a sum of a worklike product between the applied load and the weight function at their point of application.

In this paper an analysis of fracture parameters (such as \( K \)-factors and weight functions) for two dimensional elastic structures with straight embedded or edge cracks, by alternating finite element method, will be presented. The governing equations and the solution procedure[7, 8] for an embedded crack subjected to arbitrary normal and shear loading on the crack face, in an infinite isotropic or orthotropic domain, are presented. In general these solutions involve the computation of awkward definite integrals. An orthogonal polynomial numerical method[9] is employed to calculate displacements and stresses.

A general finite element alternating solution procedure for multiple embedded cracks is presented. This procedure can be applied to a single crack as well. The necessary modification to consider an edge crack is presented next. Two accurate methods of weight functions evaluation (by finite difference method and by analytical differentiation respectively) for arbitrary mixed mode loading is also presented. Numerical examples for certain illustrative problems, which encompass the above mentioned areas, are presented.

2. ANALYTICAL SOLUTION FOR AN INFINITE PLANE (ISOTROPIC OR ORTHOTROPIC) WITH AN EMBEDDED LINE CRACK

In the alternating method[5], the tractions at the location of the crack in the uncracked body are erased, so as to create the traction free crack as in the given loaded cracked structure. The key

†Ph.D candidate.
‡Director, Regents' Professor of Mechanics.
step in this iterative approach[5] is the analytical solution for an embedded crack, subjected to arbitrary crack-face tractions, in an infinite plane. This solution procedure is briefly discussed below.

The basic equations of plane elasticity have the general solution:

(a) (isotropic,[7])

\[
2\mu(u + iv) = k\Omega(z) - z\Omega'(z) - \omega(z)
\]

\[
\tau_{xx} + \tau_{yy} = 2\Omega'(z) + 2\Omega(z)
\]

\[
\tau_{xy} - i\tau_{yx} = \Omega'(z) + \Omega(z) + z\Omega'(z) + \omega'(z)
\]

(1)

where the complex potentials \(\Omega'(z), \omega'(z)\) are holomorphic functions in the region occupied by the body, and \(\mu, k\) are elastic constants \((k = 3 - 4v\) for plane strain, \(k = (3 - v)/(1 + v)\) for general plane stress). Moreover, \(u, v\) are displacements and \(\tau_{xx}, \tau_{yy}, \tau_{xy}\) are stresses at the point \((x, y), z = x + iy\).

(b) (anisotropic,[8])

\[
\tau_{xx} = 2 \text{Re}[s_1^2 \phi'(z_1) + s_2^2 \psi'(z_2)]
\]

\[
\tau_{yy} = 2 \text{Re}[\phi'(z_1) + \psi'(z_2)]
\]

\[
\tau_{xy} = -2 \text{Re}[s_1 \phi'(z_1) + s_2 \psi'(z_2)]
\]

\[
u = 2 \text{Re}[p_1 \phi(z_1) + p_2 \psi(z_2)]
\]

(2)

where

\[
s_1 = \mu_1 = \alpha_1 + i\beta_1, \quad s_2 = \mu_2 = \alpha_2 + i\beta_2
\]

\[
\mu_3 = \mu_1, \quad \mu_4 = \mu_2
\]

\[
z_1 = x + s_1 y, \quad z_2 = x + s_2 y, \quad s_1 \neq s_2
\]

(3)

where \(\alpha_j, \beta_j (j = 1, 2)\) are real constants and \(\mu_j (j = 1, 2, \ldots 4)\) are the roots of the characteristic equation

\[
a_{11} \mu_1^4 - 2a_{16} \mu_1^2 + (2a_{12} + a_{66})u_j^2 - 2a_{26} \mu_j + a_{22} = 0
\]

(4)

where \(a_{ij} (i, j = 1, 2, \ldots 6)\) are the material constants of generalized Hooke’s law

\[
\epsilon_x = a_{11} \tau_{xx} + a_{12} \tau_{yy} + a_{16} \tau_{xy}
\]

\[
\epsilon_y = a_{12} \tau_{xx} + a_{22} \tau_{yy} + a_{26} \tau_{xy}
\]

\[
\gamma_{xy} = a_{16} \tau_{xx} + a_{26} \tau_{yy} + a_{66} \tau_{xy}
\]

(5)

(Note that the values of \(a_{ij}\) in (5) are different between plane strain and generalized plane stress. Nevertheless, the constitutive laws are of the same type. As an arbitrary choice the constants \(a_{ij}\) will be used in the following derivatives.)

\[
p_1 = a_{11} s_1^2 + a_{12} - a_{16} s_1, \quad p_2 = a_{11} s_2^2 + a_{12} - a_{16} s_2
\]

\[
q_1 = -\frac{a_{12} s_1^2 + a_{26} s_1}{s_1}, \quad q_2 = \frac{a_{12} s_2^2 + a_{26} s_2}{s_2}
\]

(6)

Suppose a line crack on \(y = 0, |x| \leq a\) in an infinite plane is inflated by equal and opposite tractions, over the faces of the crack, given by

\[
\tau_{yy} - i\tau_{xy} = -[p(t) + i\delta(t)], \quad |t| \leq a
\]

(7)
with zero tractions at infinity. Then the potential functions can be written as below:

(a) (isotropic[7])

\[
\Omega'(z) = \frac{X(z)}{2\pi i} \int_{-a}^{a} \frac{p(t) + is(t)\,dt}{[X(t)]^+(t-z)}
\] (8)

where

\[
X(z) = (z + a)^{-1/2}(z - a)^{-1/2}
\] (9)

and (+) implies the upper crack face.

\[
\omega(z) = \overline{\Omega(z)} - z\Omega(z).
\] (10)

(b) (anisotropic[8])

\[
\phi'(z) = \phi'_1(z_1) + \phi'_2(z_2)
\]

\[
\psi'(z) = \psi'_1(z_1) + \psi'_2(z_2)
\] (11)

where

\[
\left(\frac{s_2 - s_1}{s_2}\right)\phi'_1(z_1) = -\frac{X(z_1)}{2\pi i} \int_{-a}^{a} \frac{p(t)\,dt}{[X(t)]^+(t-z_1)}
\]

\[
\left(\frac{s_1 - s_2}{s_1}\right)\psi'_1(z_2) = -\frac{Y(z_2)}{2\pi i} \int_{-a}^{a} \frac{p(t)\,dt}{[Y(t)]^+(t-z_2)}
\] (12)

where

\[
X(z_1) = (z_1 + a)^{-1/2}(z_1 - a)^{-1/2}
\]

\[
Y(z_2) = (z_2 + a)^{-1/2}(z_2 - a)^{-1/2}
\] (13)

and

\[
(s_2 - s_1)\phi'_2(z_1) = -\frac{X(z_1)}{2\pi i} \int_{-a}^{a} \frac{s(t)\,dt}{[X(t)]^+(t-z_1)}
\]

\[
(s_1 - s_2)\psi'_2(z_2) = -\frac{Y(z_2)}{2\pi i} \int_{-a}^{a} \frac{s(t)\,dt}{[Y(t)]^+(t-z_2)}
\] (14)

We approximate the applied crack-face tractions in the form[9].

\[
p(t) + is(t) = -\sum_{n=1}^{N} b_n U_{n-1}(t) \quad |t| \leq a
\] (15)

where \( U_{n-1}(t) \) is the Chebyshev polynomials of the second kind and is defined as

\[
U_n = \sin((n+1)\theta)/\sin \theta \quad t = a \cos \theta.
\] (16)

It could be easily shown that:

(a) (isotropic[7])

\[
2\Omega(z) = \sum_{n=1}^{N} b_n G_{n-1}(z)
\]

\[
2\Omega(z) = \sum_{n=1}^{N} b_n R_n(z)/n
\] (17)

where \( R_n(z) \) and \( G_{n-1}(z) \) are defined as follows:

\[
R_n(z) = a\{z^* - (z^*^2 - 1)^{1/2}\}^n
\]

\[
G_{n-1}(z) = -(z^* - a^2)^{-1/2} R_n(z)
\] (18)

where \( z^* = z/a \).
The stresses on \( y = 0, |x| \geq a \) are given by
\[
\tau_{xy} - i\tau_{yx} = -\text{sgn}(x)(x^2 - a^2)^{-1/2} \sum_{n=1}^{N} b_n R_n(x).
\] (19)

The stress intensity factor \( K_I \) and \( K_{II} \) at \( x = +a \) are defined as
\[
K_I = \lim_{x \to a} \left[ \sqrt{2\pi(x - a)} \tau_{yy} \right]
\]
\[
K_{II} = \lim_{x \to a} \left[ \sqrt{2\pi(x - a)} \tau_{xy} \right]
\] (20)
making use of eq. (19) in eq. (20) we get
\[
K_I - iK_{II} = -\sqrt{\pi a} \sum_{n=1}^{N} b_n.
\] (21)

Similarly at \( x = -a \) the stress intensity factors are
\[
K_I - K_{II} = \sqrt{\pi a} \sum_{n=1}^{N} (-1)^n b_n.
\] (22)

(b) (anisotropic[8])
\[
2\phi'(z_1) = \left( \frac{z_1}{s_1} \right) \sum_{n=1}^{N} c_n G_{n-1}(z_1) + \left( \frac{1}{s_1 - s_2} \right) \sum_{n=1}^{N} d_n G_{n-1}(z_1)
\]
\[
2\psi'(z_2) = \left( \frac{s_1}{s_1 - s_2} \right) \sum_{n=1}^{N} c_n G_{n-1}(z_2) + \left( \frac{1}{s_2 - s_1} \right) \sum_{n=1}^{N} d_n G_{n-1}(z_2)
\] (23)

where
\[
c_n = \text{real}(b_n),
\]
\[
d_n = i(\text{imag}(b_n))
\]
\[
2\phi(z_1) = \left( \frac{z_1}{s_2 - s_1} \right) \sum_{n=1}^{N} c_n R_n(z_1) - \left( \frac{1}{s_1 - s_2} \right) \sum_{n=1}^{N} d_n R_n(z_1)
\]
\[
2\psi(z_2) = \left( \frac{s_1}{s_1 - s_2} \right) \sum_{n=1}^{N} c_n R_n(z_2) + \left( \frac{1}{s_2 - s_1} \right) \sum_{n=1}^{N} d_n R_n(z_2). \] (24)

The stress intensity factors \( K_j (j = I, II) \) have been defined in a manner consistent with those for isotropic materials.
\[
K_I = 2\sqrt{2\pi} \left( \frac{s_2 - s_1}{s_2} \right) \lim_{z_1 \to a} (z_1 - a)^{1/2} \phi'_1(z_1)
\]
\[
K_{II} = 2\sqrt{2\pi} (s_2 - s_1) \lim_{z_1 \to a} (z_1 - a)^{1/2} \phi'_2(z_1)
\] (25)

It is easy to show that
\[
K_I - iK_{II} = -\sqrt{\pi a} \sum_{n=1}^{N} b_n \quad \text{for} \ x = a
\]
\[
K_I - iK_{II} = \sqrt{\pi a} \sum_{n=1}^{N} (-1)^n b_n \quad \text{for} \ x = -a.
\] (26)

If we examine eqs (21), (22) and (26), we see that the stress intensity factors for both the isotropic and anisotropic materials are identical for similar self-equilibrating loads in infinite planes as had been mentioned in[11].
3. FINITE ELEMENT ALTERNATING METHOD

By employing the Schwartz–Neumann alternating method, it is possible to obtain the stress intensity factors for an embedded or edge-crack in a finite solid. Since no general solutions exist for a finite body subjected to arbitrary externally applied tractions, a finite element solution is used to evaluate the stresses at the location of the crack. These stresses must be erased in order to create a traction-free crack as in the given problem. The infinite body with an embedded crack has a solution which is valid for an arbitrary distribution of tractions on the crack face. The present finite element alternating method requires the following steps (without loss of generality, we only discuss the case of two cracks in a body).

(1) Solve the uncracked body under the given external loads by FEM. The uncracked body has the same geometry as in the given problem except the crack(s).

(2) Using FEM, calculate the stresses in the uncracked body at the locations of the two given cracks.

(3) Reverse the residual stress at the first crack. Determine the coefficient, $b_1$. Calculate the S.I.F. at the first crack.

(4) Calculate the residual stress on boundaries due to the tractions on the faces of the first crack. Reverse them and calculate equivalent nodal forces $T_1$.

(5) Calculate the residual stresses at the location of the second crack due to the tractions in the faces of the first crack.

(6) Add the stresses from step (2) and step (5) of the second crack together, they are the residual stresses at crack 2. Reverse them. Determine $b_2$. Calculate the S.I.F. at the second crack.

(7) Calculate the residual stress on boundaries due to the tractions in the faces of the second crack. The equivalent nodal forces are called $T_2$.

(8) Calculate the residual stresses at the location of the first crack due to the tractions in the faces of the second crack.

(9) Sum up $T_1$ (step 4) and $T_2$ (step 7). Now these nodal forces are the external loads in step (1).

(10) After evaluating the residual stresses at crack 1 due to $T_1 + T_2$, add these stresses and the stresses of step (8) together. Now, these are the residual stresses at crack 1. Go to step (3) and repeat steps (3)–(10) until a convergence criterion is satisfied (ex. S.I.F. convergence). To obtain the final solution, add the S.I.F. of all iterations.

In step (4) and step (7), the nodal forces $T_1$ and $T_2$ can be expressed in terms of coefficients of $b_n$:

$$\{T_i\} = -[G]_m \{b_n\}$$  \hspace{1cm} (27)

and

$$[G]_m = \int_{\Sigma_m} [N]^T [n] [p] \, d\sigma$$  \hspace{1cm} (28)

where $m$ denotes the number for the finite element, $[N]$ is the matrix of isoparametric element shape functions, $[n]$ is the matrix of the direction cosines for a normal, and $[p]$ is the basis function matrix for stresses and can be derived from the previous section. In order to save computational time, the matrix $[G]_m$ is calculated prior to the start of iteration.

The crack in the infinite body has stresses defined over its entire surface. It is then necessary in edge crack problems, for stresses to be defined over the entire crack, including the fictitious portion of the crack which lies outside the finite body. According to[5], if we choose the stresses in the fictitious part to be the same as those at the free end of the edge crack (Type C of Fig. 13 in[5]), we can obtain monotonic convergence. Moreover, this type of fictitious stress is the easiest for numerical implementation. Therefore this type of fictitious stress is used in this analysis. After the stresses on the full crack are determined, the rest of the algorithm is the same as before.
4. WEIGHT FUNCTION OF ELASTIC FRACTURE

The concept of weight functions for elastic crack problems dates back to the work of Bueckner[14] and Rice[6] (see also Bortman and Banks Sills[15]). The "weight function" may generally be viewed as the appropriately normalized rate of change of displacements (at the surface where tractions are applied, or in the domain where body forces are applied) due to a unit change in the crack length for a reference state of loading. The practical importance of the concept of the weight functions lies in the fact that, if the weight functions are evaluated for a (perhaps simple) reference state of loading, then the stress-intensity factors for any arbitrary state of loading can be computed by using an integral of the worklike product between the applied tractions at a point on the surface in the arbitrary state of loading and the weight function for the reference state at the same point.

The energy-release rate due to a unit crack-extension in a cracked elastic body, subject to a system of surface tractions, (We assume, for simplicity, that the surface \( S_u \) where non-zero displacements are prescribed, is zero. One can easily generate the ensuing discussion to the situation when \( S_u \) is nonzero.), and body forces, is given by:

\[
\mathcal{G} = \int_{S_u} t_i \frac{du}{da} \, dS + \int_V f_i \frac{du}{da} \, dV - \frac{d}{da} \int_V W \, dV
\]  

(29)

where \( t_i \) are tractions applied at the surface \( S_u \); \( f_i \) are body forces in the domain \( V \); \( u \) are displacements, and \( W \) is the strain-energy density (internal energy in mechanical work). Equation (29) may be written as:

\[
\mathcal{G} + \int_{S_u} t_i \frac{du}{da} \, dS + \int_V f_i \frac{du}{da} \, dV = \frac{d}{da} \left\{ \int_{S_u} t_i \, dS + \int_V f_i \, dV - \int_V W \, dV \right\}
\]

(30)

or

\[
\mathcal{G} \, da + \int_{S_u} t_i \, dS + \int_V f_i \, dV = -d\pi.
\]

(31)

Let the reference load state be characterized by a parameter \( \lambda \). Thus,

\[
\mathcal{G} \, da + d\lambda q(\lambda) = -d\pi
\]

(32)

where

\[
q(\lambda) = \int_{S_u} \hat{t}_i \, dS + \int_V \hat{f}_i \, dV
\]

(33)

where, \( dt_i = d\lambda \hat{t}_i \); \( df_i = d\lambda \hat{f}_i \); and, in general, in a nonlinear elastic problem, the generalized displacement \( q \) is a function of \( \lambda \). Equation (31) implies that:

\[
\frac{d\mathcal{G}}{d\lambda} = \frac{dq(\lambda)}{da}.
\]

(34)

Consider a linear-elastic homogeneous solid, that is in general anisotropic, and consider the case when the crack is at an arbitrary angle to the material directions, and under a general mixed mode loading. The energy release rate, \( \mathcal{G} \), for a mixed-mode crack in an anisotropic solid may be written as:

\[
\mathcal{G} = AK_1^2 + BK_2^2 + CK_1K_2
\]

(35)

where

\[
A = -\frac{\pi}{2} a_{22} \text{Im}\left(\frac{\mu_1 + \mu_2}{\mu_1 \mu_2}\right)
\]

\[
B = \frac{\pi}{2} a_{11} \text{Im}(\mu_1 + \mu_2)
\]

\[
C = \frac{\pi}{2} \left\{ -a_{22} \text{Im}\left(\frac{1}{\mu_1 \mu_2}\right) + a_{11} \text{Im}(\mu_1 \mu_2) \right\}
\]

(36)
where \( a_{ij} \) are material constants in the relation \( \epsilon_i = a_{ij} \sigma_j \) \((i,j = 6)\) and \( \mu_j \) are the complex roots of the characteristic equation, \( a_{11} \mu_j^2 - 2a_{16} \mu_j^1 + (2a_{12} + a_{56}) \mu_j^1 - 2a_{26} \mu_j + a_{22} = 0 \). (See Sih and Liebowitz[8] for further details.) In the case of isotropy (35) reduces to

\[
A = \frac{1}{H}; \quad B = \frac{1}{H}; \quad C = 0
\]

\( H = E/(1 - \nu^2) \) plane strain; \( H = E \) plane stress. \( \text{(37)} \)

We now consider the simultaneous action of 2 load-systems on the cracked body. The load system is:

\[
(\lambda^1 \tilde{\tau}_1^1 + \lambda^2 \tilde{\tau}_1^2) \text{ at } S_1; \quad \text{and} \quad (\lambda^1 \tilde{\tau}_1^3 + \lambda^2 \tilde{\tau}_1^3) \text{ in } V. \\text{(38)}
\]

When (37) is used in (32) one obtains:

\[
\mathcal{G} \, da + d \lambda^R C_{\rho a} \lambda^a = -d \pi; \quad R, n = 1, 2 \quad \text{(39)}
\]

or

\[
\frac{d \mathcal{G}}{d \lambda^R} = \frac{d C_{\rho a}}{d \lambda^R} \lambda^R \quad \text{(40)}
\]

and

\[
\frac{d C_{\rho a}}{d \lambda^a} = \int_{S_1} \tilde{\tau}_1^1 \frac{d \tilde{\tau}_1^1}{d \lambda^a} \, dS + \int_V \tilde{\pi}_1^1 \frac{d \tilde{\pi}_1^1}{d \lambda^a} \, dV, \quad n, R = 1, 2 \text{ load cases, } \quad i = 1, 2, 3. \quad \text{(41)}
\]

The \( K \)-factors under the combined mode loading are

\[
K_i = \hat{K}_i^m \lambda^m; \quad K_{ii} = \hat{K}_{ii}^m \lambda^m \quad \text{(42)}
\]

[sum on \( m = 1, 2 \)] and

\[
\mathcal{G} = A \hat{K}_1^m \hat{K}_1^m \lambda^m \lambda^m + B \hat{K}_{11}^m \hat{K}_{11}^m \lambda^m \lambda^m + C \hat{K}_{11}^{m+1} \hat{K}_{11}^{m+1} \lambda^m \lambda^m \quad \text{(43)}
\]

where \( \hat{K}_1^m \) is the mode I \( K \)-factor of load state \( m \) due to \( \lambda^m = 1 \). Using (42) in (39), one obtains:

\[
2A \hat{K}_1^m \hat{K}_1^m + C \hat{K}_1^m \hat{K}_1^m + 2B \hat{K}_{11}^m \hat{K}_{11}^m + C \hat{K}_{11}^{m+1} \hat{K}_{11}^{m+1} = \frac{d C_{\rho a}}{d \lambda^a} \quad \text{(44)}
\]

Let \( R \) be a known reference load-state, for which the solution, i.e. \( \hat{K}_1^R, \hat{K}_{11}^R, \) and \( (d \lambda^R/da) \) are known, and \( n \) is an arbitrary load-state for which the mixed mode factors \( K_i^m \) and \( K_{ii}^m \) are to be computed. Equation (44) is thus a single equation governing the two unknowns \( K_i^m \) and \( K_{ii}^m \). By writing
Eq. (44) for two known reference states, two equations for two unknowns \( K_1 \) and \( K_2 \) can be obtained. The solution of these equations can be seen to be:

\[
K_1 = \frac{\hat{R}_1 R^2 \lambda^m}{K^R(2A + C)} \frac{dC_{mR1}}{da} - \frac{\hat{R}_2 R^2 \lambda^m}{K^R(2A + C)} \frac{dC_{mR2}}{da} \quad \text{(no sum on } m) \tag{45}
\]

\[
K_2 = \frac{\hat{R}_1 R^2 \lambda^m}{K^R(2B + C)} \frac{dC_{mR1}}{da} - \frac{\hat{R}_2 R^2 \lambda^m}{K^R(2B + C)} \frac{dC_{mR2}}{da} \quad \text{(no sum on } m) \tag{46}
\]

and

\[
K^R = \hat{R}_1 \hat{R}_2 - \hat{R}_2 \hat{R}_1 \tag{47}
\]

where \((dC_{mR1}/da)\) etc. are defined from (40) by replacing \( R \) by \( R_1 \) etc. It is seen that for \( K^R \) to be nonzero, \( R_1 \) and \( R_2 \) cannot be both pure mode I or pure mode II. Furthermore, in a general anisotropic body with an arbitrarily oriented crack, the reference states \( R_1 \) and \( R_2 \) can be taken to either loads on external surfaces, or tractions on the crack-faces themselves.

Thus, to evaluate the mixed-mode load factors for any reference state \( m \), one only needs the appropriately normalized weight-functions, \((du_{m1}/da)\) and \((du_{m2}/da)\). In the following we discuss some computational methods for these weight functions, for anisotropic or isotropic materials.

5. DETERMINATION OF WEIGHT FUNCTION BY ALTERNATING F.E.M.

In this section a finite difference method and an analytical difference method are presented to calculate weight functions.
5.1. Finite difference method

As discussed in the previous section, the most important data for weight function evaluations are $K_{R1}^i$, $K_{R2}^i$, $K_{R1}^s$, $K_{R2}^s$, $du_{R1}^{(R1)}/d(2a)$ and $du_{R2}^{(R2)}/d(2a)$. Because two different reference loading states are needed, some modifications of the procedure of alternating method in Section 2.3 are necessary. The following solution procedure is adopted to compute the weight functions for an embedded or edge crack in a general anisotropic, finite dimensional structure, when the crack is oriented arbitrarily with respect to the material axes of anisotropy. Only a single crack is considered here.

1. Solve the uncracked body under the given external load using the finite element method. It is same as step (1) in Section 3.

2. Calculate the stresses at the location of the given crack in the uncracked body. Consider the normal pressure on the crack-face as reference loading states $R1$, the shear traction on the crack-face as $R2$ respectively. Care must be exercised in the choosing of the given external load in step (1) so that it is a mixed-mode loading (i.e. neither $R1$ or $R2$ is trivial). Henceforth it is understood that the following steps are carried out for states $R1$ and $R2$ respectively.

3. Proceed with the alternating procedure until convergence is obtained. Record $K_{R1}^i$, $K_{R2}^i$, $K_{R1}^s$, $K_{R2}^s$, $u_{R1}^{(R1)}(2a)$ and $u_{R2}^{(R2)}(2a)$ ($2a$ is the crack length.)

4. Extend the crack length by a small quantity $\Delta a$ in the crack axis direction. Go back to step (2) and repeat steps (2) and (3). Step (1) is neglected because the first iteration will yield the same results for crack length $2a$ and $2a + \Delta a$.

5. Calculate the differential of displacement by finite difference scheme

$$\frac{d u_{R1}^{(R)}}{d(2a)} = \frac{u_{R1}^{(R)}(2a + \Delta a) - u_{R1}^{(R)}(2a)}{\Delta a} \quad (48)$$

$R = R_1, R_2$.

6. By applying eq. (44) and (45), one can obtain the weight functions of the given cracked structure.

5.2. Analytical differentiation method

Recall that the analytical solution for displacements in an infinite domain for an anisotropic situation is

$$2G(u + iv) = k\Omega(z) - \Omega(\bar{z}) + (\bar{z} - z)\overline{\Omega(z)}$$

$$= \sum_{n=0}^{M} \frac{b_n}{2n} \left( kR_n(z) - R_n(\bar{z}) + (\bar{z} - z)G_{n-1}(z) \right) \quad (49)$$
and for anisotropic situation is

\[ u = 2 \text{Re}[p_1 \phi(z_1) + p_2 \psi(z_2)] \]

\[ = \text{Re} \left[ p_1 \left( \frac{s_2}{s_2 - s_1} \right) \sum_{n=1}^{M} c_n \frac{R_n(z_1)}{n} + p_2 \left( \frac{1}{s_1 - s_2} \right) \sum_{n=1}^{M} d_n \frac{R_n(z_1)}{n} \right] \]

\[ + p_2 \left( \frac{s_1}{s_1 - s_2} \right) \sum_{n=1}^{M} c_n \frac{R_n(z_2)}{n} + p_2 \left( \frac{1}{s_2 - s_1} \right) \sum_{n=1}^{M} d_n \frac{R_n(z_2)}{n} \]

\[ v = 2 \text{Re}[q_1 \phi(z_1) + q_2 \psi(z_2)] \]

\[ = \text{Re} \left[ q_1 \left( \frac{s_2}{s_2 - s_1} \right) \sum_{n=1}^{M} c_n \frac{R_n(z_1)}{n} + q_2 \left( \frac{1}{s_1 - s_2} \right) \sum_{n=1}^{M} d_n \frac{R_n(z_1)}{n} \right] \]

\[ + q_2 \left( \frac{s_1}{s_1 - s_2} \right) \sum_{n=1}^{M} c_n \frac{R_n(z_2)}{n} + q_2 \left( \frac{1}{s_2 - s_1} \right) \sum_{n=1}^{M} d_n \frac{R_n(z_2)}{n} \] \[ \quad \text{(50)} \]

All the above mentioned expressions have been defined previously. Consider the isotropic case first. Differentiating eq. (48) w.r.t. crack length \(2a\), one can obtain

\[ \frac{d(u + iv)}{d(2a)} = \frac{1}{2G} \left\{ \left( k \sum_{n=1}^{M} b_n \frac{\partial R_n(z)}{\partial (2a)} \right) - \left( \sum_{n=1}^{M} b_n \frac{\partial R_n(z)}{\partial (2a)} \right) + (z - \bar{z}) \sum_{n=1}^{M} b_n \frac{\partial G_{n-1}(z)}{\partial (2a)} \right\} \]

\[ + \left\{ \left( k \sum_{n=1}^{M} \frac{\partial b_n}{\partial (2a)} \frac{R_n(z)}{2n} \right) - \left( \sum_{n=1}^{M} \frac{\partial b_n}{\partial (2a)} \frac{R_n(z)}{2n} \right) + (z - \bar{z}) \sum_{n=1}^{M} \frac{\partial b_n}{\partial (2a)} \frac{G_{n-1}(z)}{2} \right\} \]. \[ \quad \text{(51)} \]

To calculate \(du/d(2a)\), one has to know \(\partial R_n(z)/\partial (2a), \partial G_{n-1}(z)/\partial (2a), \text{and } \partial b_n/\partial (2a)\). The first two data can be obtained by differentiating eq. (18) w.r.t. \(2a\) as:

\[ \frac{\partial R_n(z)}{\partial (2a)} = \frac{1}{2} \frac{\partial R_n(z)}{\partial a} \]

\[ = \frac{1}{2} \left\{ 1 + n(1 + z^*) (z^* - 1)^{1/2} \right\} \frac{R_n(z)}{a} \]

\[ \quad \text{(52)} \]

where \(\partial z^*/\partial a = -(1 + z^*)/a\) is used.

\[ \frac{\partial G_{n-1}(z)}{\partial (2a)} = \frac{1}{2} (z + a)(z^2 - a^2)^{-1} G_{n-1}(z) - (z^2 - a^2)^{-1/2} \frac{\partial R_n(z)}{\partial (2a)} \]

\[ \quad \text{(53)} \]

where \(\partial z/\partial a = -1\) is used.

To calculate \(\partial b_n/\partial (2a)\) is a more involved task. First, recall that

\[ \sigma_y = -i \sigma_{xy} = \sum_{n=1}^{M} b_n U_{n-1}(t), \quad \text{if } \quad |t| \leq a. \]

(54)

Differentiating eq. (53) w.r.t. \(2a\), one obtains

\[ \frac{\partial (\sigma_y - i \sigma_{xy})}{\partial (2a)} = \frac{\partial U_{n-1}(t)}{\partial \theta} \frac{\partial \theta}{\partial (2a)} \]

\[ = \frac{(n \cos n \theta \sin \theta - \sin n \theta \cos \theta) (\cos \theta + 1)}{2a \sin^3 \theta} \]

\[ \text{where } \frac{\partial \theta}{\partial (2a)} = (\cos \theta + 1)/(2a \sin \theta) \text{ is used.} \]
The value of \( \frac{\partial b_n}{\partial (2a)} \) can be calculated from eq. (54) by utilizing orthogonality of \( U_{n-1} \):

\[
\frac{\partial b_n}{\partial (2a)} = \frac{2}{\pi a^2} \int_{-a}^{a} \left\{ \frac{\partial (\sigma_x - i\sigma_y)}{\partial (2a)} - \sum_{n=1}^{M} b_n \frac{\partial U_{n-1}(t)}{\partial (2a)} \right\} U_{n-1}(t)(a^2 - t^2)^{1/2} dt
\]

\[
= \frac{2}{a^2} \sum_{i=1}^{M} \frac{a^2 - t_i^2}{n + 1} \left\{ \frac{\partial (\sigma_x(t_i) - i\sigma_y(t_i))}{\partial (2a)} - \sum_{n=1}^{M} b_n \frac{\partial U_{n-1}(t_i)}{\partial (2a)} \right\}
\]

(56)

where \( t_i = a \cos \left[ \pi / (n + 1) \right], i = 1, 2, \ldots M. \)

Note that, in the derivation of weight functions, the reference load state \((R1 \text{ or } R2)\) is invariable during crack extension. It is obvious that \( \frac{\partial (\sigma_x - i\sigma_y)}{\partial (2a)} = 0 \) in the first iteration. But in the subsequent iterations \( \frac{\partial (\sigma_x - i\sigma_y)}{\partial (2a)} \) are not zero, because the differential of residual external boundary forces \( \frac{\partial T}{\partial (2a)} \) are not zero. To calculate \( \frac{\partial T}{\partial (2a)} \), one has to have the differential of stresses on external boundaries.

\[
\frac{\partial (\sigma_x + \sigma_y)}{\partial (2a)} = 2 \text{Re} \left\{ \sum_{n=1}^{M} \left( \frac{\partial b_n}{\partial (2a)} G_{n-1}(z) + b_n \frac{\partial G_{n-1}(z)}{\partial (2a)} \right) \right\}
\]

\[
\frac{\partial (\sigma_x - i\sigma_y)}{\partial (2a)} = \sum_{n=1}^{M} \left\{ \frac{1}{2} \frac{\partial b_n}{\partial (2a)} G_{n-1}(z) + \frac{b_n}{2} \frac{\partial G_{n-1}(z)}{\partial (2a)} + \frac{1}{2} \frac{\partial b_n}{\partial (2a)} G_{n-1}(z) + \frac{b_n}{2} \frac{\partial G_{n-1}(z)}{\partial (2a)} \right\}
\]

\[
+ (\varepsilon - \bar{\varepsilon}) \left( \frac{1}{2} \frac{\partial b_n}{\partial (2a)} G_{n-1}(z) + \frac{b_n}{2} \frac{\partial G_{n-1}(z)}{\partial (2a)} \right)
\]

(57)

Fig. 4(a)  
Fig. 4(b)  
Fig. 4(c)

Fig. 4. (a) An edge crack near a hole: tensionless. (b) An edge crack near a hole: bending load. (c) An edge crack in a plate: shear loading.
where

\[ G_{n-1}(z) = \frac{R_n}{(z^2 - a^2)}(z(z^2 - a^2)^{1/2} + n) \]

\[ \frac{\partial G_{n-1}(z)}{\partial (2a)} = \frac{\partial R_n(z)}{\partial (2a)} \frac{(z^2 - a^2) + R_n(z)(z + a)}{(z^2 - a^2)^2} (z(z^2 - a^2)^{1/2} + n) \]

\[ + \frac{R_n(z)}{2(z^2 - a^2)} \left( \frac{z(z + a)}{(z^2 - a^2)^{3/2}} - (z^2 - a^2)^{-1/2} \right). \] (58)

By the same token, one can derive the necessary equations for anisotropic domain. The alternating method for analytical weight function evaluation is presented below:

1. Solve the uncracked problem by finite element method. Separate crack-face loading into two linear-independent reference states. This step is exactly the same as step (1) in Section 5.1.
2. Set \( \partial(\sigma_x - i\sigma_y)/\partial (2a) = 0 \).
3. Calculate \( \partial b_j/\partial (2a) \) by using eq. (56). Calculate \( d\psi_\alpha/d(2a) \) by using eq. (50).
4. Calculate stress differential on the boundary by using eq. (57). Using these differentials, calculate the residual force differential \( \partial T_j/\partial (2a) \).
5. Calculate \( \partial(\sigma_x - i\sigma_y)/\partial (2a) \) on crack-face by using F.E.M. and treating \( \partial T_j/\partial (2a) \) as a pseudo-load.
6. Go back to step (3), until convergence is obtained.
7. Sum up \( d\psi_\alpha/d(2a) \) of each iteration. By applying eqs (44) and (45), one can evaluate weight functions on the crack faces.

It is important to note that the crack is not numerically modelled at all. The dependence of crack-plane displacements on the crack-length as evaluated in steps (3), for infinite domains, is explicitly known; and thus the weight-functions are evaluated in the above alternating method, in an analytical sense without using numerical differentiations.

6. NUMERICAL EXAMPLES

All isotropic materials considered here have the properties, \( E = 1.0, \nu = 0.3 \). All calculations were performed on a Micro Vax station II.

6.1. Square plate with two offset parallel cracks

To demonstrate the accuracy of the algorithm in Section 3, we consider a square plate with two offset parallel cracks subjected to uniform tension loading \( (\sigma_y = 1) \) (See Fig. 1a for details), \( a = 2 \). Figure 1(b) shows the finite element mesh. Comparative values were found from [13].

\[
\begin{array}{c|c|c|c}
\hline
\text{Location} & \Delta u & \Delta v & \hline
1 & 1.726 & 1.177 & 1.867 & 1.867 \\
2 & -0.198 & -0.216 & -0.0692 & -0.0813 \\
\hline
\end{array}
\]

6.2. Square plate with a slanted embedded crack

A square plate with a slanted crack is considered here (Fig. 2a), with \( \theta = 45^\circ \). Figure 2(b) shows the mesh used. To find the appropriate range of the virtual crack extension \( \Delta a \) in the numerical differentiation \( \partial \psi_\alpha/\partial a \), we considered this structure subjected to uniform uniaxial tension. The weight functions for variable virtual crack extensions in some selected points are shown below.

\[
\begin{array}{c|c|c|c|c}
\hline
(h_i, \text{ of some points}) & \Delta a = 10^{-6} & \Delta a = 10^{-5} & \Delta a = 10^{-4} \\
\hline
\text{Location} & \Delta u & \Delta v & \Delta u & \Delta v & \Delta u & \Delta v \\
\hline
(5.0, 5.0) & 0.02376 & 0.03378 & 0.3378 & \\
(10.0, 5.0) & 0.10927 & 0.10925 & 0.10925 & \\
(0.0, 5.0) & 0.18629 & 0.18621 & 0.18621 & \\
(5.0, 5.0) & 0.13533 & 0.13528 & 0.13528 & \\
\hline
\end{array}
\]
We can see that the weight functions are independent of $\Delta a$ for $\Delta a = 10^{-3}a \sim 10^{-9}a$. It is therefore quite straightforward for an analyst to use the present method to calculate weight functions. From here on, we chose $\Delta a = 10^{-6}a$. With the weight function concept, the stress intensity factor is expressed as a sum of a worklike product between the applied load and the weight function at their points of application as

$$K_{i(II)} = \int t \times h_{i(II)} \, ds$$

(59)

where $t = $ surface tractions, $s = $ surface with applied tractions, $h_{i(II)} = $ weight functions for mode I or mode II and $K_{i(II)} = $ stress intensity factor for mode I or mode II.

To check whether the calculated weight functions are correct, we compare the stress intensity factors calculated from eq. (33) with the stress intensity factors calculated from alternating method in the following problems.

6.3. Orthotropic square plate with a slanted embedded crack

The geometry is identical to the previous problem except the material here is orthotropic with $E_{xx} = 30 \times 10^6$ psi, $E_{yy} = 2.7 \times 10^6$ psi, $G_{xy} = 0.65 \times 10^6$, $v_{xy} = 0.21$. The material axes are coincident with the global axes. Uniform uniaxial tension is considered.

<table>
<thead>
<tr>
<th></th>
<th>Eq. (33)</th>
<th>Alternating</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{i}$</td>
<td>1.1932</td>
<td>1.1758</td>
<td>1.4%</td>
</tr>
<tr>
<td>$K_{ii}$</td>
<td>-1.0947</td>
<td>-1.1097</td>
<td>1.3%</td>
</tr>
</tbody>
</table>

This result shows that the algorithm for decomposing the mixed-mode problem into two single mode problems is correct. Figure 2(c) shows weight functions of the upper loading face.

6.4. Rectangular plate with a single edge crack

The problem geometry and loading are shown in Fig. 3. The comparison between $K_{i}$'s is given below.

<table>
<thead>
<tr>
<th></th>
<th>Eq. (33)</th>
<th>Alternating</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{i}$</td>
<td>20.5</td>
<td>20.211</td>
<td>1.4%</td>
</tr>
</tbody>
</table>

Orthotropic: $E_{xx} = 30 \times 10^6$, $E_{yy} = 2.7 \times 10^6$

$G_{xy} = 0.65 \times 10^6$, $v_{xy} = 0.21$

<table>
<thead>
<tr>
<th></th>
<th>Eq. (33)</th>
<th>Alternating</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{i}$</td>
<td>19.731</td>
<td>19.723</td>
<td>0.04%</td>
</tr>
</tbody>
</table>

6.5. Single edge crack near a hole

This is a more practical problem which illustrates that the proposed method is appropriate for the analysis of real engineering structures. Refer to Fig. 4 for problem definition. To demonstrate the load-independent characteristics of weight functions, we compare $h_{i}$ at some selected points. Figure 4 shows the loading for each case.

<table>
<thead>
<tr>
<th>Location</th>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 20)</td>
<td>0.4642</td>
<td>0.46308</td>
<td>0.46306</td>
</tr>
<tr>
<td>(10, 20)</td>
<td>0.25767</td>
<td>0.25707</td>
<td>0.25706</td>
</tr>
<tr>
<td>(20, 20)</td>
<td>0.1107</td>
<td>0.11045</td>
<td>0.11044</td>
</tr>
<tr>
<td>(30, 20)</td>
<td>0.027497</td>
<td>0.02744</td>
<td>0.027439</td>
</tr>
<tr>
<td>(40, 20)</td>
<td>-0.037469</td>
<td>-0.037358</td>
<td>-0.037356</td>
</tr>
</tbody>
</table>

The $K_{i}$ for each case is given on the next page.
The general procedure of the alternating method in conjunction with the 2-D finite body analytical solution and the finite element method was developed for the analysis of cracked structures. The present method leads to a very efficient and accurate evaluation of stress intensity factors for multiple cracks in mixed-mode fracture situations. Furthermore, the present alternating method lends itself to a very cost-effective evaluation of weight functions. The accuracy and load-independent nature of these weight functions are numerically demonstrated.

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REFERENCES


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