Finite Elasticity Solutions Using Hybrid Finite Elements Based on a Complementary Energy Principle

The possibility of deriving a complementary energy principle for the incremental analysis of finite deformations of nonlinear-elastic solids, in terms of incremental Piola-Lagrange (unsymmetric) stress alone, is examined. A new incremental hybrid stress finite-element model, based on an incremental complementary energy principle involving both the incremental Piola-Lagrange stress, and an incremental rotation tensor which leads to discretization of rotational equilibrium equations, is presented. An application of this new method to the finite strain analysis of a compressible nonlinear-elastic solid is included, and the numerical results are discussed.

Introduction

Most of the work in finite elasticity is based on the simplifying assumption of incompressibility, and very few analytical solutions to boundary-value problems for compressible elastic solids can be found. Even the numerical solutions based, notably, on the finite-element method are few. As summarized in the book by Oden [1], almost all of the finite-element solutions for finite elasticity problems are based on the principle of stationary potential energy. In the finite-element terminology, these can be said to be based on the so-called "compatible displacement finite-element model." However, to the best of authors' knowledge, no studies exist on the convergence of such finite-element solutions for finite elasticity. From this standpoint, as well as from that of the related question of studying solution bounds, it is of interest to construct finite-element solutions for such finite elasticity problems based, if possible, on the complementary energy principle.

Finite-element formulations for finite elasticity problems based on multifield variational principles, i.e., those involving stresses, strains, and displacements all as independent variables were discussed by Nemat-Nasser and his coworkers [2, 3]. General incremental variational principles, using alternate measures of stress and conjugate measures of strain for finite elasticity, and their modifications which also allow for the a priori relaxation of constraints of interelement displacement continuity and traction reciprocity, were summarized recently by the authors [4]. Various hybrid finite-element models were shown [4] to be derivable as special cases of these general principles, and both stationary Lagrangean as well as updated Lagrangean-type incremental formulations were presented.

In the present paper we are concerned primarily with the development and application of incremental finite-element formulations for finite elasticity, using a complementary energy principle. The considerations of complementary energy principles for finite elasticity are of very recent origin, as seen from the recent works of Zubov [5], the late Francis de Veubeke [6], Koeberl [7, 8], Christoffersen [9], Dill [10], and [4]. In the present paper, we start with a summary of these works, and examine the possibility of casting them into incremental form, to facilitate the construction of piecewise-linear incremental finite-element solutions for a finite strain problem. Thus we first treat the question whether the incremental complementary energy principle can be cast entirely in terms of the unsymmetric incremental Piola-Lagrange (or what is also referred to as the first Piola-Kirchhoff) stress. Second, we consider the incremental version of the complementary principle stated in [6], which involves both the incremental Piola-Lagrange stress and the incremental rotation tensor, and which yields the incremental angular momentum balance equations as its Euler equations. We then develop a "hybrid" finite-element formulation based on this principle, which, in addition, allows for the a priori relaxation of traction reciprocity condition at the interelement boundaries (this modification to the principle is the reason for labeling the finite-element model a "hybrid stress model," consistent with the general theory of hybrid models as discussed in [4, 11]). The developed finite-element procedure is applied to solve the problem of stretching to twice its original length ("the biaxial strip problem") of a sheet made of a nonlinear elastic compressible material of the Blatz-ko [12] type; and the results are discussed.

In order, however, to make the present paper reasonably self-con-
tained, we begin with some preliminaries of kinematics, definition of various stress measures and their conjugate strain measures, the field equations as expressed in these alternate measures, and various forms of constitutive relations, pertaining to the considered problem of finite elasticity.

1 Basic Formulation

For the initial, as well as deformed configurations of the solid are referred to the same Cartesian frame. The undeformed and deformed position vectors are \( \mathbf{r} \) and \( \mathbf{y} \), respectively. The gradient of \( \mathbf{y} \) is the tensor \( F \), such that,

\[
d_\mathbf{y} = \frac{d\mathbf{y}}{d\mathbf{x}}; \quad F = \sum_i F_i = \gamma_{ij} \tag{1}
\]

The nonsingular \( F \) has the polar-decomposition,

\[
F = \mathbf{q} \cdot (\mathbf{I} + \mathbf{h})
\]

where \( (\mathbf{I} + \mathbf{h}) \) is a symmetric, positive-definite tensor, herein called the stretch tensor; \( \mathbf{I} \) is the identity tensor; and \( \mathbf{q} \) is the orthogonal rotation tensor, such that \( \mathbf{q}^T = \mathbf{q}^{-1} \). The gradient of the displacement vector \( \mathbf{u} \) is defined by

\[
\mathbf{e} = \nabla \mathbf{y}; \quad \mathbf{e}_i = \mathbf{u}_{ij} \tag{3}
\]

The deformation tensor, \( G \), is defined by

\[
G = F^T \cdot F = (h + I)^2 \tag{4}
\]

The Green-Lagrange strain tensor is defined by

\[
\dot{e} = \frac{1}{2} (G - I) = \frac{1}{2} \left[ \Sigma \mathbf{u} + (\Sigma \mathbf{u})^T + (\Sigma \mathbf{u}) \cdot (\Sigma \mathbf{u}) \right] \tag{5}
\]

Following [6, 13] we define the Piola-Lagrange (unsymmetric) stress tensor \( \mathbf{g} \) and the symmetric Kirchhoff-Trefftz stress \( \mathbf{S} \), in terms of the true (Euler) stress \( \mathbf{\sigma} \) in the deformed body, through the following relations:

\[
\mathbf{g} = \frac{1}{2} (F^{-1} \cdot \mathbf{r}) = \frac{1}{2} F \cdot \mathbf{S} \cdot F^T \tag{6a}
\]

or, inversely,

\[
\mathbf{r} = J (F^{-1} \cdot \mathbf{y}); \quad \mathbf{S} = J (F^{-1} \cdot \mathbf{r}) \cdot F^{-T} \tag{6b}
\]

and

\[
\mathbf{g} = \mathbf{S} \cdot F^T \tag{6c}
\]

where \( J \) is the determinant of the matrix \( (y_{ij}) \). If \( W \), the strain energy per unit initial volume, is expressed symmetrically in terms of the six independent strain measures, \( (1/2)(g_{ij} + g_{ji}) \), the stress-strain relations can be summarized [6] as:

\[
\mathbf{S} = \frac{\partial W}{\partial \mathbf{e}} \quad \text{(Symm.)}; \quad \mathbf{g} = \frac{\partial W}{\partial \mathbf{e}^T} \quad \text{(Unsym.)} \tag{7}
\]

The Jaumann stress tensor, which we label as \( \mathbf{r} \), is defined, following [6], as

\[
\mathbf{r} = \frac{\partial \mathbf{g}}{\partial \mathbf{e}} \quad \text{(Symm.)}
\]

which leads to the relations

\[
\mathbf{r} = \frac{1}{2} (\mathbf{r} \cdot \mathbf{Q} + \mathbf{Q} \cdot \mathbf{r}) = \frac{1}{2} \left[ (\Sigma \mathbf{I} \cdot (h + I)) \right] \tag{8}
\]

A general three-field variational principle involving \( \mathbf{S}, \mathbf{g}, \) and \( \mathbf{u} \), whose Euler equations and natural boundary conditions (b.c.) are the full set of field equations and b.c. for the finite elasticity problem, has been stated by Washizu [14]. The impossibility of deriving a single-field complementary energy principle for finite elasticity, involving

\[\text{Transactions of the ASME}\]

[4, 14]. If, on the other hand, one uses the measures \( \mathbf{r}, \mathbf{S}, \) and \( \mathbf{u} \) in formulating the finite elasticity problem, it is shown in [6, 4] that the stationarity conditions of the functional,

\[
\chi_W (\mathbf{r}, \mathbf{S}, \mathbf{u}) = \int_\Omega \left[ W (\mathbf{g} (\mathbf{e})) + \mathbf{r}^T \left[ \frac{\partial W}{\partial \mathbf{e}^T} \right] - \mathbf{p}^T \cdot \mathbf{u} \right] d\Omega
\]

(10)

lead to the equations,

\[
\nabla \cdot \mathbf{r} + \mathbf{p}^T = 0 \tag{11}
\]

\[
\mathbf{e} = \nabla \mathbf{u} \tag{12}
\]

\[
\mathbf{r}^T = \frac{\partial \mathbf{g}}{\partial \mathbf{e}} \tag{13}
\]

\[
\mathbf{u} = \mathbf{u} \quad \text{on} \quad S_\infty \tag{14}
\]

\[
\mathbf{u} = \mathbf{u} \quad \text{on} \quad S_{\partial \infty} \tag{15}
\]

and finally, the rotational equilibrium condition

\[
\mathbf{F} \cdot \mathbf{r} = \mathbf{T}^T \cdot \mathbf{F} \tag{16}
\]

The fact that the rotational equilibrium equations (16) become integrated in the principle of equation (10) provided \( W \) has a special structure (namely, that \( W \) is expressed in terms of \((1/2)(g_{ij} + g_{ji})\)) is noted in [6]; for, in that case,

\[
\mathbf{g}_{ij} = \frac{\partial W}{\partial e_{ij}} = \frac{\partial W}{\partial e_{i\alpha}} \frac{\partial e_{\alpha j}}{\partial \mathbf{e}} = \mathbf{S}_{\alpha i} F_{i\alpha} \tag{17}
\]

Thus, by definition,

\[
\mathbf{r} = \mathbf{S} \cdot \mathbf{F} \tag{18}
\]

The definition of equation (15) and the fact that \( \mathbf{S} \) is symmetric, reduce the rotational equilibrium equation (16) to a trivial identity.

The question of constructing the complementary energy principle based on \( \mathbf{r} \) alone has been a subject of much recent study. If equations (11) and (14) are satisfied \textit{a priori} (which is simple to accomplish), it is seen that \( \mathbf{u} \) can be eliminated as a variable in equation (10). Further \( \mathbf{r} \) can be eliminated from equation (10) provided the contact transformation

\[
W - \mathbf{r}^T \cdot \mathbf{g} = -T (\mathbf{r}) \tag{19}
\]

exists. This contact transformation depends on finding the inverse of the relation,

\[
\mathbf{r}^T = \frac{\partial W}{\partial \mathbf{e}} \tag{20}
\]

That there is no unique inverse relation in general for \( \mathbf{r} \) in terms of \( \mathbf{r} \) has been noted by Truesdell and Noll [13], and more recently in [10]. In the case of isotropic "semilinear" materials, Zubov [5] attempts to establish such an inverse relation, however his arguments were refuted by Dill [10] who shows that the inverse relation must be multiple-valued. Further, Fraeijs de Veubeke [6] has earlier discussed the difficulties associated with evaluating \( T \) in terms of \( \mathbf{r} \) alone, such that the rotational equilibrium equations, for general materials, may be reflected in the special structure for \( T \); the rotational equilibrium equations, equation (16) are expected to be built in the complementary energy principle, and the stress \( \mathbf{r} \) is subject to only the constraint of linear momentum balance equation, equation (11). Thus even though some progress has been made, much remains to be done in order to derive a truly complementary energy principle involving \( \mathbf{r} \) alone (with rotational equilibrium conditions being inherently embedded in the principle) which is in a practically applicable form.

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[1] In the following we use the notation; \( \mathbf{\cdot} \) under symbol denotes a tensor; \( \mathbf{\cdot} \) under symbol denotes a vector; \( \Sigma = \sum_i \frac{\partial (\partial \mathbf{e}_{\alpha i})}{\partial \mathbf{e}_{\alpha j}} \) denotes the gradient in undeformed metric; \( \mathbf{\cdot} \) denotes \( \frac{\partial (\partial \mathbf{X})}{\partial \mathbf{Y}} \). \( \mathbf{A} \) denotes product of two tensors such that \( \mathbf{A} \cdot \mathbf{B} = \mathbf{A}_{ij} B_{kj} \); \( \mathbf{A} \cdot \mathbf{B} \) is trace \( (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A}_{ij} B_{ij} \), i.e. the inner product of two tensors, \( \mathbf{u} \cdot \mathbf{e} = \mathbf{u}_{ij} e_{ij} \), i.e. dot product of two vectors; and \( \mathbf{A} \cdot \mathbf{B} = \mathbf{A}_{ij} B_{ij} \), i.e. \( \mathbf{A} = \mathbf{A}_{ij} \). The expression \( \Sigma \mathbf{e} \) does not mean the tensor product of \( \Sigma \) with \( \mathbf{e} \) but is defined by equation (1).

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[2] This is not to deny forever the existence of such a principle. Recently Washizu [13] succeeded in deriving a complementary energy principle involving \( \mathbf{r} \) alone for a special class of finite deformation problems. These cases can be described as small strain, small rotation, linear elastic but finite deformation problems of beams and plates wherein certain plausible deformation hypotheses of the Kirchhoff-type are invoked; thus only a few displacement gradients enter the nonlinear strain-displacement relations.
Finally, it is shown in [6, 4] that the functional

$$\pi_{W}[\dot{f} (t); \dot{u}; \dot{u}] = \int_{V_0} \left[ \int_{T} \left( \sum_{j=1}^{n} T_{ij} \dot{u}_j \right)^2 - \int_{S_{00}} \left( \sum_{i=1}^{m} T_{ij} \right) \dot{u}_i \right] \, dv - \int_{V_0} \rho \dot{u}_g \cdot \dddot{u}_g \, dv - \int_{S_{00}} \dot{\tau} \cdot \dddot{u}_g \, ds \tag{21}$$

where $W$ is expressed symmetrically as a function of the engineering strain tensor $\tilde{e}$, has its Euler equations and natural b.c.

$$\nabla \cdot \tilde{e} + \rho \dot{u}_g = 0 \tag{22}$$

$$\left( \frac{h + \alpha}{2} \right) \cdot \tilde{e} = \text{Symmetric} \tag{23}$$

$$\dot{\alpha} W / \dot{h} = \frac{1}{2} \left( \alpha \cdot \tilde{e} + \alpha \cdot \tilde{e}^T \right) \tag{24}$$

$$\left( \tilde{e} + \sum_{j=1}^{n} T_{ij} \right) = \left( \tilde{e} + \frac{h + \alpha}{2} \right) \tag{25}$$

$$\dot{\alpha} = \dot{\tau} \left( \sum_{i=1}^{m} T_{ij} \right) \text{ on } S_{00} \tag{26}$$

$$\dot{\tau} = \dot{\alpha} \text{ on } S_{00} \tag{27}$$

Thus the rotational equilibrium equations, equation (23), follow, unambiguously, as the Euler equations corresponding to the variation of the functional with respect to the rotation tensor $\alpha$. As discussed in [6] the stress-strain relations, equations (24), can be inverted (again, this is in the nature of a physical assumption) and the constant transformation,

$$\tau \frac{\partial}{\partial x} - W(h) = R(r) \tag{28}$$

can be established. Through this transformation, strains $h$ can be eliminated from equation (21) and, further if equation (22) and (26), are satisfied a priori, $\dot{u}_g$ can be eliminated from equation (21), to obtain a complementary energy principle,

$$\pi_{C}(t; \dot{u}) = \int_{V_0} \left[ \frac{1}{2} \rho \ddot{u}_g \cdot \dddot{u}_g - \int_{S_{00}} \dot{\tau} \cdot \dddot{u}_g \right] \, dv \tag{29}$$

whose Euler equations and natural b.c. are equations (23), (25), and (27).

For purposes of applications to meaningful boundary-value problems in finite elasticity through the method of finite elements, our primary interest is in the incremental counterparts of the foregoing principles and these are taken up in the following, for simplicity as well as reasons of space, we restrict our attention to the stationary Lagrangean version of these incremental principles.

**Incremental (Total Lagrangean) Formulation**

Here, the loading process is considered in a finite number of increments, and we consider that the state variables in the solid are represented by the sets $C_N$ and $C_{N+1}$ (prior to, and after the addition of the $(N + 1)$th load increment, respectively). The metric of $C_0$ is used to refer to all the state variables in each subsequent state. Let state $C_N$ be defined by the variables, $\{ \sum_{i=1}^{n} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}, \sum_{i=1}^{m} T_{ij} \}$, and a similar set of variables in $C_{N+1}$, with the superscript ($N + 1$). Let the incremental variables in passing from $C_N$ to $C_{N+1}$ be $\{ \Delta T^j, \Delta T^j, \Delta T^j, \Delta T^j, \Delta T^j, \Delta T^j, \Delta T^j, \Delta T^j, \Delta T^j, \Delta T^j \}$. This incremental state is symbolized by the set $C^N$; thus formally, $C^N = C_N + \Delta C$. Let the value of a functional $v$, giving a governing variational principle, be $\pi (C^N)$ and $\pi (C_N^N)$ at states $C_N$ and $C_{N+1}$, respectively. Through the substitution of the respective state variables, one can obtain, in general,

$$\pi(C^N+1) - \pi(C^N) = \pi(C^N + \Delta C) - \pi(C) = \text{const} + \pi(C) \left[ \text{first-order terms of } \Delta C \right] + \pi(C) \left[ \text{second-order terms of } \Delta C \right] + \pi(C) \left[ \text{3rd and higher-order terms of } \Delta C \right] \tag{30}$$

It can easily be shown that, in general, $\dot{\pi}^2 = 0$ if state $C_N$ truly satisfies the relevant field equations. It can also be shown that $\dot{\pi}^2 + \dot{\pi} = 0$ leads to the fully nonlinear incremental field equations, relevant to the given variational principle. However, if the increments are sufficiently small, the incremental field equations can be linearized, and these, for a given variational principle, can be shown to follow from $\dot{\pi}^2 = 0$. Thus, in subsequent discussions, $\dot{\pi}^2$ is ignored; thus paving the way for piecewise linear incremental solutions. As discussed in [4], in order to prevent the piecewise-linear incremental solution path from staying, as little as possible from the true solution, it is generally advisable to retain the term $\dot{\pi}^2$ to generate iterative "correction procedures." Thus the solution procedure can be summarized as: using $\dot{\pi}^2$ to generate solution for $\Delta C$; and using $\dot{\pi}^2$ to check if $\Delta C$ truly satisfies the relevant field equations and boundary conditions.

Under the foregoing considerations, the incremental form $\dot{\pi}^2$ of the functional corresponding to equation (10) can be shown to be

$$\pi_{W}^N (\Delta t; \Delta t; \Delta t) = \int_{V_0} \left[ \Delta W - \rho_o \Delta \dot{u}_g \cdot \Delta \ddot{u}_g - \Delta \ddot{u}_g \cdot \left( \Delta \ddot{u}_g - \sum \Delta g \right) ight] \, dv - \int_{S_{00}} \Delta \ddot{u}_g \cdot \Delta \dddot{u}_g \, ds \tag{31}$$

where $\Delta W = \Delta \ddot{u}_g + \Delta \dddot{u}_g$; and

$$\Delta \ddot{u}_g = \frac{1}{2} S_{mN} \Delta \ddot{u}_g \text{ on } S_{00} \tag{32}$$

$$\Delta \ddot{u}_g = \frac{1}{2} \left( \rho W / \Delta \ddot{u}_g \right) \text{ on } S_{00} \tag{33}$$

Where $W$ is the strain-energy density per unit initial volume, considered to be expressed symmetrically in terms of $g$; and the superscript $N$ is used to denote the state $C_N$. Also, in equation (33), $\Delta g$ is a linear function of $\Delta t$. It can easily be verified that,

$$\Delta t = S_{mN} \Delta \ddot{u}_g + \Delta \dddot{u}_g + \Delta S_{mN} F_{mN} \tag{34}$$

It can also be seen that, to first-order approximation, the incremental rotational equilibrium as derivable from equation (10), can be written as

$$\Delta F = F_N + \Delta S \cdot F_N \tag{35}$$

Substituting for $\Delta t$ in terms of $\Delta t$ from equation (34), it can be seen that the incremental rotational equilibrium is identically satisfied. Thus the incremental rotational equilibrium is inherently built in the principle through the special structure for $\Delta (W)$ (that it is a symmetric function of $\Delta g$). The other variational equations corresponding to $\dot{\pi}^2 = 0$ are, as can be shown easily

$$\nabla \cdot \Delta t = \rho \Delta \ddot{u}_g \tag{36}$$

$$\Delta \ddot{u}_g = \nabla \ddot{u}_g \tag{37}$$

$$\Delta \ddot{u}_g = \Delta \dddot{u}_g \tag{38}$$

$$\Delta \ddot{u}_g = \Delta \dddot{u}_g \tag{39}$$

We now examine the possibility of achieving the contact transformation, $\Delta W = \Delta (W^*) \Delta t = - \Delta (\Delta t); \Delta t$; and if this is possible, and in addition, constraining $\Delta t$ to satisfy equations (38) and (39), a priori, one can eliminate $\Delta \ddot{u}_g$ and $\Delta \dddot{u}_g$ from equation (31)—in which case, we arrive at an incremental truly complementary energy principle involving $\Delta t$ alone. However, we demand that the incremental rotational equilibrium equation (35) (with $\Delta F$ expressed in terms of $\Delta t$) be re-located in the special structure, if any, for the complementary energy density function $\Delta T$. To these ends, we rewrite equation (34) as

$$\Delta \ddot{u}_g = S_{mN} \Delta \ddot{u}_g + E_{mN} F_{mN} F_{mN} \Delta \dddot{u}_g$$

$$\Delta \ddot{u}_g = \delta \Delta \ddot{u}_g + E_{mN} F_{mN} F_{mN} \Delta \dddot{u}_g$$

$$\Delta \ddot{u}_g = \Delta \dddot{u}_g \tag{40}$$

In the foregoing, we used the notation that
\[ E_{\text{num}}^N = (\nabla^2 W / \partial \phi_{\text{num}} \partial \phi_{\text{obj}}) \] at \( C^N \) (41)

which, by definition, has the symmetry properties,

\[ E_{\text{num}}^N = E_{\text{num}}^N = E_{\text{num}}^N = E_{\text{num}}^N \] (42)

Also, we used the notation that

\[ E_{ijkl}^N = S_{ijkl}^N + E_{ijkl}^N F_{jkl}^N F_{kln}^N \]

which, however, has the only symmetry property, that

\[ *E_{ijkl}^N = E_{ijkl}^N \] (44)

Thus, if equation (40) is written in matrix notation,

\[ |\Delta e| = |E| \cdot |\Delta e| \]

The foregoing relation can be inverted\(^3\) to write

\[ \Delta e_{ij} = E_{ijkl}^{-1} \Delta s_{kl} \] or \[ |\Delta e| = |E^{-1}| |\Delta s| \]

(46)

where in general, \( *E_{ijkl}^{-1} = *E_{ijkl}^{-1} \). Thus, using equation (46), the contact transformation can in fact be established to find \( \Delta T \) such that

\[ (\Delta T) / (\Delta s_{ij}) = \Delta e_{ij} = E_{ijkl}^{-1} \Delta s_{kl} \] (47)

If the incremental rotational equilibrium conditions are inherently built into the structure of \( \Delta T \), then the condition

\[ \Delta e_{ijkl}^N + F_{ijkl} N \Delta s_{ijkl} = \text{Symmetric} \] (48)

must be satisfied when \( \Delta e_{ij} \) is expressed in terms of \( \Delta s_{ij} \). Doing so we find the rotational equilibrium to be expressed by the necessary condition that

\[ E_{ijkl}^{-1} \Delta s_{kl} + F_{ijkl} N \Delta s_{ijkl} = \text{Symmetric} \]

(49)

It is easy seen that neither of the two terms in the previous expression is by itself symmetric. The other possible ways in which the aforementioned sum of two terms can be symmetric are: (a) first, one term is a transpose of the other; however, it is easy to see that this is not the case, and (b) second, the first term can be expressed as the sum of a symmetric tensor and the transpose of the second term. However, \( E_{ijkl}^{-1} \) (with the only symmetry property \( *E_{ijkl} = E_{ijkl} \)) cannot be analytically derived. Thus even though \( *E_{ijkl}^{-1} \) is the sum of two terms, it is in general not possible to decompose \( *E_{ijkl}^{-1} \) to suit our purposes. Thus it appears impossible, at present, to prove the assertion (b).

Even though the symmetry (or lack of it) of the term in equation (49) can be decided computationally, for a specific problem, it appears that, in general, we cannot expect the symmetry of the said term. Thus it appears that even though the incremental contact transformation can be achieved to find \( \Delta T \) in terms of \( \Delta e \), since the rotational equilibrium conditions cannot be proved to be built into the structure of \( \Delta T \), the attendant complementary energy principle has little significance.

Thus we are led to believe that, at present, the most consistent and useful development of an incremental complementary energy principle for finite elasticity is that based on the concept wherein the rotational equilibrium conditions follow as a \textit{a posteriori} conditions of the variational principle. To this end, we derive the incremental functional \( \tau^e \) corresponding to equation (21) as \( (\tau^e \) is given in Appendix 1)

\[ \tau^e_{\text{num}} = \tau^e_{\text{num}} = \tau^e_{\text{num}} = \tau^e_{\text{num}} \] (50)

\[ \tau^e_{\text{num}}^N = \tau^e_{\text{num}}^N = \tau^e_{\text{num}}^N = \tau^e_{\text{num}}^N \] (51)

It is noted that the term \( \tau^N \Delta d_{ij} \) is retained in the expression for \( \tau^e \) in the foregoing. This is due to the fact that \( \Delta d_{ij} \) is a rotation, subject to the condition of orthogonality, i.e., \( \tau_{ij} = \Delta d_{ij} \). If one considers a variation of this term, one obtains the constraint conditions, \( \tau_{ij} = \Delta d_{ij} \) is skew-symmetric. Considering the incremental process, one has, on the other hand, the more exact requirement \( (\tau^N + \Delta \tau^T)(\Delta d_{ij} + \Delta d_{ij}) = I \) or, in a variational sense, \( (\tau^N + \Delta \tau^T) \delta d_{ij} = 0 \). In practical application, however, it is convenient to satisfy the orthogonality condition, to a first order, i.e., require \( \tau_{ij} \Delta d_{ij} + \Delta d_{ij} \tau_{ij} = \Delta d_{ij} \tau_{ij} = \text{skew-symmetric} \). In the foregoing

\[ \Delta W = \frac{1}{2} \frac{\partial^2 W}{\partial \phi_{ij} \partial \phi_{mn}} N \Delta h_{ij} \Delta h_{mn} \] (51)

Equation (52) is a piecewise linear relation between \( \Delta R \) and \( \Delta d_{ij} \). Inverting this, we construct the contact transformation,

\[ \Delta W - \Delta R = \Delta d_{ij} = - \Delta R / \partial \Delta d_{ij} = \Delta d_{ij} \] (53)

By requiring the stress \( \Delta d_{ij} \) to satisfy, \( a \text{ priori} \), the constraint conditions

\[ \tau_{ij} + \rho \delta d_{ij} = 0 \] (50)

where \( \delta d_{ij} \) is one can eliminate \( \delta d_{ij} \) from \( \tau^e_{\text{num}} \) in equation (50); further \( \Delta d_{ij} \) is eliminated using the transformation in equation (53). When this is done, we obtain the incremental complementary energy functional,

\[ \tau^e_{\text{num}}(\Delta R; \Delta d_{ij}) = \int_{\Omega} \left[ \Delta R - \Delta d_{ij} \times (\Delta R; (h_{ij} + I)] \right) \] (54)

\[ + \tau^N \Delta d_{ij} \] (55)

With the exact definition,

\[ \tau^e_{\text{num}} = \int_{\Omega} \left[ \Delta R \times \Delta R \right] \] (55)

and the exact orthogonality condition,

\[ \tau_{ij} \Delta d_{ij} = \text{skew-symmetric} \] (56)

noting the stated constraints on \( \tau_{ij} \) (viz., \( \tau_{ij} \Delta d_{ij} + \rho \delta d_{ij} = 0 \) and \( \Delta d_{ij} = \Delta d_{ij} \) on \( S_0 \)), the variational equation \( \delta \tau^e_{\text{num}} = 0 \) can easily be shown to lead to the a \textit{a posteriori} constraints:

\[ \nabla \Delta q_{\text{num}} = \Delta q_{\text{num}} (h_{ij} + I) + \Delta q_{\text{num}} + \Delta q_{\text{num}} \Delta q_{\text{num}} \] (57)

\[ (h_{ij} + I) \times (\Delta h_{ij} + \Delta q_{\text{num}} + \Delta q_{\text{num}} \Delta q_{\text{num}}) = \text{symmetric} \] (58)

\[ \Delta u = \Delta q_{\text{num}} \] on \( S_0 \) (59)

Equations (57) and (58) are the exact forms of incremental compatibility and total rotational equilibrium in \( C_{N^e} \), respectively. However noting that in the first-order approximation solution, that (a) the terms like \( \Delta \tau_{ij} \) are omitted in the definition for \( \tau^e \) in equation (55) and (b) the orthogonality condition is approximated as \( \tau_{ij} \Delta d_{ij} + \Delta d_{ij} \tau_{ij} = 0 \) such that \( \tau_{ij} \Delta d_{ij} + \Delta d_{ij} \tau_{ij} = \text{skew-symmetric} \), one obtains the linearized incremental equations, from \( \delta \tau^e_{\text{num}} = 0 \), instead of equations (57) and (58) as

\[ \nabla \Delta q_{\text{num}} = \Delta \tau_{ij} + \Delta h_{ij} \] (60)

\[ \Delta h_{ij} \times \Delta q_{\text{num}} (h_{ij} + I) + (h_{ij} + I) \times (\Delta h_{ij} + \Delta q_{\text{num}} + \Delta q_{\text{num}} \Delta q_{\text{num}}) = \text{symmetric} \] (61)

We note that equation (61) is an approximated form of rotational equilibrium in \( C_{N^e} \). However, since the statement of rotational
equilibrium in the immediately preceding state, i.e., \( C_N \), is built into equation (61), an inherent check for rotational equilibrium for each successive stage may be considered as being built into equation (61).

In applying the complementary energy principle as stated through the functional in equation (54), one has to assume a stress field \( \Delta \tau \) in each element such that not only the equation \( \nabla \cdot \Delta \tau = 0 \) is satisfied in each element, but also that such the principle of action and reaction (or traction reciprocity) is satisfied at the interfaces of adjoining elements. With the notation that: \( V_{om} \) is the volume of the \( m \)th element; \( \partial V_{om} \), its boundary; \( S_{om} \) and \( S_{on} \) are portions of \( \partial V_{om} \) where tractions and displacements, respectively, are prescribed, \( \rho_{tom} \) is that portion of \( \partial V_{om} \) which is common to that of an adjacent element (so-called interelement boundary); superscripts \((+)\) and \((-)\) to denote, arbitrarily, the left and right sides of an interelement boundary in such a way as a boundary is approached; we note the interelement traction reciprocal condition to be of the form,

\[
(\mathbf{n} \cdot \Delta \tau)^+ + (\mathbf{n} \cdot \Delta \tau)^- = 0 \quad \text{at} \quad \rho_{tom} \tag{62}
\]

If the assumed stress field \( \Delta \tau \) in each element doesn't satisfy the foregoing constraint \( e \) priori, then the constraint condition (62) can be introduced into the functional of equation (54) using Lagrange multipliers \( \lambda_{om} \) as follows:

\[
-\pi_{RS}^*(\Delta \tau; \lambda_{om}; \lambda_{om}) = \sum_m \int_{V_{om}} [\Delta \tau \cdot \mathbf{N}^T; \lambda_{om}] + \int_{NT} \left[ \lambda_{om} \cdot (\mathbf{n} \cdot \Delta \mathbf{u}) \right] + \sum_m \int_{\partial V_{om}} \mathbf{n} \cdot \Delta \mathbf{u} \cdot ds \tag{63}
\]

In the foregoing, \( \lambda_{om} \) can be identified as interelement boundary displacements, and are required to be inherently "compatible" (i.e., they must be unique) at the interelement boundary. The variational equations corresponding to equation (63), can be shown to be, in addition to those already stated in equations (57)–(59), the necessary interelement traction and displacement conditions, at \( \rho_{tom} \):

\[
\mathbf{u}^+ = \mathbf{u}^- = \mathbf{u}_{om} \tag{64a}
\]

\[
(\mathbf{n} \cdot \Delta \tau)^+ + (\mathbf{n} \cdot \Delta \tau)^- = 0 \tag{64b}
\]

Consistent with the general philosophy of the finite-element methods based on relaxed requirements of interelement continuity conditions for admissible field variables [4, 11], the finite-element model, as stated through equation (63), is labeled here as an "Assumed Stress Hybrid Finite-Element Model."

**Development of Finite-Element Algebraic Equations**

For clarity and conciseness, only the essential steps pertaining to the finite-element formulation for a three-dimensional case followed by the specific assumptions for a plane-stress case are indicated; further details can be found in [16]. First, the stress tensor \( \Delta \tau \) that satisfies equation (36) is derived from a first-order stress function tensor \( \psi \) such that

\[
\Delta \tau = \sum \mathbf{N} \cdot \mathbf{N}^T + \mathbf{D} \psi \psi^T
\]

where \( \mathbf{D} \psi \psi^T \) is any particular solution. In component form,

\[
\Delta \tau_{ij} = e_{map} \psi_{p,j} + \Delta \tau_{ij}^p
\]

where \( e_{map} \) is the usual alternating tensor. A satisfactory treatment of the boundary condition \( \mathbf{n} \cdot \Delta \tau = \Delta \mathbf{u} \) on \( S_{om} \) for general boundaries requires, in general, the introduction of these as constraint conditions into the functional in equation (63). Thus, by selecting \( \lambda_{om} \) such that \( \lambda_{om} \) on \( S_{om} \) the functional in equation (63) is slightly modified as

\[
-\pi_{RS}^*(\Delta \tau; \lambda_{om}; \lambda_{om}) = \sum_m \int_{V_{om}} [\Delta \tau \cdot \mathbf{N}^T; \lambda_{om}] + \int_{NT} \left[ \lambda_{om} \cdot (\mathbf{n} \cdot \Delta \mathbf{u}) \right] + \sum_m \int_{\partial V_{om}} \mathbf{n} \cdot \Delta \mathbf{u} \cdot ds
\]

For convenience, we now introduce the familiar matrix notation and write

\[
[\Delta \mathbf{u}^m] = [c_{mpn}] \mathbf{v}_{p,j} + [\Delta \mathbf{u}_n^m]^p \quad [\Delta \mathbf{u}] = [A] [\Delta \mathbf{u}] + [\Delta \mathbf{f}] \quad [\Delta \mathbf{r}] \tag{66}
\]

where \( \Delta \mathbf{r} \) are "a" undetermined parameters in the first-order stress functions \( \mathbf{v}_{p,j} \). From these, the tractions at \( \partial V_{om} \) are derived according to the apparent definition, as

\[
[\Delta \mathbf{r}] = [\lambda_{om} \mathbf{u}_n m] = [A^u] [\Delta \mathbf{u}] + [\Delta \mathbf{r}^u] \tag{67}
\]

Now, for a 3-dimensional case, the nine rotation components \( \alpha_j \) are subject to 6 orthogonality relations, thus leaving 3 independent rotation parameters. With this in mind, the assumption for \( \alpha \) can be written as

\[
[\Delta \alpha_j] = [B^u] [\Delta \mathbf{u}] = [B^u] [C^u] [\Delta \mathbf{u}] = [B] [\Delta \mathbf{u}] \tag{68}
\]

where \( \Delta \mathbf{u} \) are the 3 rotation parameters; and \( \Delta \mathbf{u} \) are "b" undetermined parameters in the assumed functions for \( \Delta \mathbf{u} \). The third field variable \( \Delta \mathbf{u}_n \) at \( \partial V_{om} \) is assumed as

\[
[\Delta \mathbf{u}_n] = [L^u] [\Delta \mathbf{u}] \tag{69}
\]

where \( L \) are interpolates that uniquely interpolate for displacements along any boundary segment in terms of their respective values at nodes along that segment.

At this point it is noted that \( t^N, q^N, \) and \( h^N \) are known quantities at \( C_N \) and thus there are no undesired parameters left in them. Thus, from the relation

\[
\Delta \tau = \frac{1}{2} \mathbf{D} \mathbf{N}^T \cdot \Delta \mathbf{u} + \Delta \mathbf{u} \cdot h^N + \Delta \mathbf{u}^T \cdot q^N + \mathbf{N} \cdot \Delta \mathbf{u} \tag{70}
\]

one can construct,

\[
[\Delta \mathbf{r}] = [D_1] [\Delta \mathbf{u}] + [D_2] [\Delta \mathbf{u}] + [\Delta \mathbf{r}^p] \tag{71}
\]

with apparent definitions \( [D_1] \) and \( [D_2] \) in terms of \( \mathbf{A} \) and \( [B] \), and the known \( t^N \) and \( q^N \), as shown through equation (71). Now we consider the matrix representation4 for \( \Delta \mathbf{u} \) as

\[
[\Delta \mathbf{u}] = (h^N + \mathbf{N}) \quad [\Delta \mathbf{u}] = [L^u] [\Delta \mathbf{u}] \tag{72}
\]

Noting that, in the foregoing, the only undetermined parameters are \( \Delta \mathbf{u} \), one can clearly write that

\[
[\Delta \mathbf{u}] = (h^N + \mathbf{N}) \quad [\Delta \mathbf{u}] = [L^u] [\Delta \mathbf{u}] \tag{73}
\]

Likewise, since \( N \) is known, we write

\[
[\Delta \mathbf{u}] = [L^u] [\Delta \mathbf{u}] \tag{74}
\]

Finally, we write the matrix representation for the relation between \( \Delta \mathbf{h}_j \) and \( \Delta \mathbf{r}_j \) as

\[
\Delta \mathbf{h}_j = \delta \mathbf{A} \Delta \mathbf{r}_j \quad [\Delta \mathbf{h}] = [E] [\Delta \mathbf{r}] \tag{75}
\]

Using equations (72) and (73) we write

\[
]\int_{V_{om}} \mathbf{N} \cdot [\Delta \mathbf{r}] = \frac{1}{2} \sum_{V_{om}} [\Delta \mathbf{r}] [E] [\Delta \mathbf{r}] \mathbf{d}v = \frac{1}{2} \sum_{V_{om}} \left[ \int [\Delta \mathbf{r}] [D_1]^T [E] [D_1] [\Delta \mathbf{u}] + [\Delta \mathbf{u}] [D_2]^T [E] [D_2] [\Delta \mathbf{u}] \right] \mathbf{d}v
\]

\[
+ 2 \mathbf{L} [\Delta \mathbf{r}] [D_1]^T [E] [D_2] [\Delta \mathbf{u}] + 2 \mathbf{L} [\Delta \mathbf{r}] [D_1]^T [E] [\Delta \mathbf{r}] \mathbf{d}v \tag{76}
\]

4 Due care is exercised in writing the elements of this column vector in such a way that the vector dot product of this with \( [\Delta \mathbf{r}] \) agrees with the tensor trace notation used in equation (66).
with apparent definitions for $H_{11}, H_{22}, H_{12}, \Delta Q_1$, and $\Delta Q_2$ which are, of course, evaluated through integration over $V_{\text{om}}$ by numerical quadrature. Likewise, one can write

$$\int_{V_{\text{om}}} \Delta \frac{T}{T} \left[ \Delta \mathbf{Q} \cdot (h^N + f) \right] dV = \int_{V_{\text{om}}} \left[ \Delta \mathbf{Q} \left[ A \right] + \left[ \Delta \mathbf{Q} \right] \left[ H \right] \right] \left[ \Delta \mathbf{Q} \right] dV$$

Also with apparent definitions for $P$ and $\Delta Q_0$. Similarly,

$$\int_{V_{\text{om}}} \Delta \frac{T}{T} \left[ \Delta \mathbf{Q} \cdot (h^N + f) \right] dV$$

Likewise,

$$\int_{\partial V_{\text{om}}} \Delta \frac{T}{T} \left[ \Delta \mathbf{Q} \cdot \left[ H_{\mu} \right] \right] dS = \int_{\partial V_{\text{om}}} \left[ \Delta \mathbf{Q} \right] \left[ \mathbf{Q} \right] \left[ \Delta \mathbf{Q} \right]$$

Finally,

$$\int_{S_{\text{min}}} \Delta \frac{T}{T} \left[ \Delta \mathbf{Q} \cdot \left[ H_{\mu} \right] \right] dS = \int_{S_{\text{min}}} \left[ \Delta \mathbf{Q} \right] \left[ \mathbf{Q} \right] \left[ \Delta \mathbf{Q} \right]$$

Using equations (77)-(81) one can rewrite equation (66) as

$$- \pi_{HS} = \sum_{\alpha = 1}^{n} \left[ \left[ \Delta \mathbf{Q} \right] \left[ \mathbf{Q} \right] \left[ \Delta \mathbf{Q} \right] \right]$$

As shown in [17] for the linear elastic case, the matrix $H_{11}$ cannot by itself be inverted, due to the fact that certain combinations of linear terms in $X_1, X_2, X_3$ in the assumed functions for stress-functions $\psi_j$ produce zero stress-energy; thus there exists a nonzero vector $\Delta \mathbf{Q}$ for which the stress-energy is zero; however, the entire matrix consisting of $H_{11}, H_{22}, H_{12}$ in equation (82) can be inverted. Also in equation (82), $\Delta \mathbf{Q}$ and $\Delta \mathbf{Q}$ are independent for each element, where as $\Delta \mathbf{Q}$ are common to a set of adjoining elements. Thus, varying $\pi_{HS}$ with respect to $\Delta \mathbf{Q}$ and $\alpha$, we obtain the equations at the element level as

$$H_{11} \frac{T}{T} (h^N + f) \left[ \Delta \mathbf{Q} \right] = \left[ \mathbf{Q} \right] \left[ \Delta \mathbf{Q} \right]$$

where $H$ is defined in the apparent way. In the foregoing $\Delta \mathbf{Q}$ can be interpreted as residual forces to check the rotational equilibrium in $C_N$. Solving for $\Delta \mathbf{Q}$ from equation (83) in terms of $\Delta \mathbf{Q}$ and substituting in equation (82), we obtain

$$- \pi_{HS} = \sum_{\alpha = 1}^{n} \left[ \left[ \Delta \mathbf{Q} \right] \right] \left[ \mathbf{Q} \right] \left[ \Delta \mathbf{Q} \right]$$

where $\mathbf{Q}$ is a stress parameter in the present example is 10.

The special case when $\Delta \mathbf{Q}_1 = - \Delta \mathbf{Q}_2$ and other $\Delta \mathbf{Q}$'s are zero corresponds to a zero stress-energy state and this is responsible for the need to invert the matrix $H$ as a whole in equation (91). The two-dimensional rotational field tensor $\mathbf{Q}$ is assumed as

$$\left[ \alpha_1 \alpha_2 \alpha_3 \right] = \left[ \cos \alpha \sin \alpha \right]$$

It is easy to see that the foregoing satisfies the required orthogonality
condition, \( a_{ij}a_{kj} = b_{kl}(i, j, k = 1, 2). \) The incremental rotation field \( \Delta \theta = \theta^{N+1} - \theta^N \) is then assumed as:

\[
\Delta \alpha_{11} \Delta \alpha_{12} \Delta \alpha_{21} \Delta \alpha_{22} = [-\sin \theta^N \cos \theta^N - \cos \theta^N - \sin \theta^N] \Delta \theta \quad (33)
\]

The incremental rotation field can be seen to satisfy the required linearized orthogonality condition, viz.,

\[
\theta^N \cdot \Delta \theta^T + \Delta \theta \cdot \theta^NT = 0
\]

or

\[
a_{ij}^N \Delta \alpha_{kj} + \lambda a_{ij} a_{kj}^N = 0 \quad (i, k = 1, 2) \quad (94)
\]

It is also noted that if \( \Delta \theta \) is sufficiently small, the incremental rotation field can also be considered to satisfy the exact orthogonality condition,

\[
a_{ij}^N \Delta \alpha_{kj} + \lambda a_{ij} a_{kj}^N + \Delta a_{ij} \Delta \alpha_{kj} = 0 \quad (i, k = 1, 2) \quad (95)
\]

to an accuracy of order \((\Delta \theta)^2\); The variation of \( \Delta \theta \) over the element can be assumed as a general polynomial in \( x_1 \) and \( x_2 \) with parameters \( \Delta a_{ij} \). However, in the present calculations, \( \Delta \theta \) is assumed to be constant over each element. Finally, the geometry of the finite-element considered is that of a 4-noded rectangular element in the undeformed configuration, \( C_0 \). The third field variable, \( u_y \) at \( \Delta V_{\text{def}} \), is simply assumed to be a linear function on each segment of the boundary of the rectangle. The specific constants \( \mu, f, \) and \( \alpha \) that are chosen for the present numerical example are as in Fig. 1. A 6 X 6 nonuniform mesh for a quarter of the sheet (only which needs to be analyzed due to symmetry) is considered in \( C_0 \) configuration, as also shown in Fig. 1 (each element has 8 displacement degrees of freedom). The considered total average stretch of 100 percent of the sheet is imposed in 20 increments of 5.0 percent when Newton-Raphson-type iterations (due to corrective terms arising from the functional \( \kappa \)) as in equation (30)) are used; whereas no iterations are used (i.e., \( \kappa \) is excluded from the formulation), the total stretch is imposed in 40 increments of 2.5 percent each. An average of 3 iterations in each increment we found necessary to achieve convergence.

Fig. 1 shows the total axial load necessary to achieve various levels of average stretch. The results with finer increments and no iterations were not noticeably different from those with twice large increments and with iterations, and hence no distinction is made between these in Fig. 1. Preliminary results for a similar problem (1 in. X 1 in.) sheet, with a uniform 6 X 6 mesh, with 40 increments and no iterations were reported in [4]. From these and the present results, judging from a computational time viewpoint it appears to us that, in the present type of formulation, it is simpler to consider smaller increments with no iterations.

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Footnote: \(^{(9)} \) It is possible to retain terms of order \((\Delta \theta)^2\) in the expression for \( \Delta \theta \) in equation (93). In fact, when this was done, the convergence of the present results was observed to be somewhat accelerated. Further details of this are omitted due to space reasons.

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Journal of Applied Mechanics

SEPTEMBER 1978, VOL. 45 / 545
APPENDIX 1

The incremental functional \( \Pi^I \) corresponding to equation (29) is

\[
\Pi^I = \int_{\Omega} \left[ \left( \frac{\partial W}{\partial \Delta u} \Delta u + \frac{\partial W}{\partial \Delta l} \Delta l \right) : \Delta \dot{u} + \dot{\alpha} \cdot \Delta \dot{u} + \dot{\alpha} \cdot \Delta \dot{l} \right] + \frac{1}{2} \left( \dot{\alpha} \cdot \Delta \dot{u} - \dot{\alpha} \cdot \Delta \dot{l} \right) ds
\]

As stated in the text, the functional \( \Pi^I \) is used in generating correct iteration procedures.

APPENDIX 2

The strain-energy density function \( W \) for the Blatz-Ko material is indicated in equation (68) with \( \frac{\partial W}{\partial \Delta u} \) as the principal invariants of the deformation tensor \( \Gamma \) and \( \dot{\alpha} \) are principal invariants of the stretch tensor \( \Gamma + \dot{\alpha} \).

\[
\Delta W = \frac{1}{2} \frac{\partial W}{\partial \Delta u} (\Delta u) : \Delta \dot{u} + \dot{\alpha} \cdot \Delta \dot{u} - \dot{\alpha} \cdot \Delta \dot{\alpha}
\]

Through lengthy, but relatively straightforward algebra, we derive the following:

\[
\Delta W = \frac{\mu}{2} \left[ \left( \frac{1}{2} \frac{\partial W}{\partial \Delta u} (\Delta u) : \Delta \dot{u} + \dot{\alpha} \cdot \Delta \dot{u} \right) - \dot{\alpha} \cdot \Delta \dot{\alpha} \right]
\]

where, the following notation applies:

\[
\Delta h_1 = \Delta h_{11} + \Delta h_{21} + \Delta h_{31}
\]

\[
\Delta h_2 = (2 + h_{23} + h_{12}) \Delta h_{11} - h_{21} \Delta h_{12} - h_{22} \Delta h_{21} + (2 + h_{11} + h_{32}) \Delta h_{22} + (2 + h_{11} + h_{22}) \Delta h_{32}
\]

\[
\Delta h_3 = (1 + h_{12}) \Delta h_{11} + (1 + h_{13}) \Delta h_{12} + (1 + h_{23}) \Delta h_{21} + (1 + h_{23}) \Delta h_{22} + (1 + h_{11}) \Delta h_{32}
\]

\[
\Delta h_{12} = \Delta h_{11} + \Delta h_{12} + \Delta h_{21} + \Delta h_{31} + \Delta h_{32} + \Delta h_{33}
\]

\[
\Delta h_{13} = (1 + h_{13}) \Delta h_{11} + (1 + h_{23}) \Delta h_{12} + (1 + h_{23}) \Delta h_{21} + (1 + h_{11}) \Delta h_{32} + (1 + h_{11}) \Delta h_{33}
\]

\[
\Delta h_{21} = (1 + h_{12}) \Delta h_{11} + (1 + h_{23}) \Delta h_{12} + (1 + h_{23}) \Delta h_{21} + (1 + h_{23}) \Delta h_{32} + (1 + h_{11}) \Delta h_{33}
\]

In the foregoing, \( h_{ij} \) are known quantities and the superscript N has been omitted for convenience. It is now seen that \( \Delta W \) in equation (88) is a quadratic form in \( \Delta h_{ij} \). Thus the incremental constitutive law

\[
\frac{\partial W}{\partial \Delta l} = \Delta h_{ij}
\]
is linear, can be inverted, and hence through the contact transformation, we find such that

$$\Delta h_i = \frac{\partial \Delta R}{\partial \Delta r_i}$$  \hspace{1cm} (102)

$$\Delta R = \Delta r_i \Delta h_i - \Delta W$$  \hspace{1cm} (101)  \hspace{1cm} Further details can be found in [16].