An Embedded Elliptical Crack, in an Infinite Solid, Subject to Arbitrary Crack-Face Traction

In this paper, following a critical assessment of earlier work of Green and Sneddon, Segedin, Kassir, and Sih (who obtained solutions for specific cases of normal loading on the crack face and cases of constant and linear shear distribution on the crack face), Shah and Kobayashi (whose work is limited to the case of third-order polynomial distribution of normal loading on the crack face), and Smith and Sih (who work is limited to the case of third-order polynomial variation of shear loading on the crack face), a general solution is presented for the case of arbitrary normal as well as shear loading on the faces of an embedded elliptical crack in an infinite solid. The present solution is based on a generalization of the potential function representation used by Shah and Kobayashi. Expressions for stress-intensity factors near the flaw border, as well as for stresses in the far-field, for the foregoing general loadings, are given.

1 Introduction

The problem of a flat elliptical crack embedded in an infinite solid, of linear elastic material, has attracted much attention in the literature due to its fundamental role in the studies of fracture susceptibility of embedded or surface flaw, three-dimensional, engineering structures. When the solid is subjected to uniform tension at infinity, perpendicular to the plane of the crack (or, equivalently, when the crack face is subjected to uniform pressure, in the complementary problem), Green and Sneddon [1] have solved the problem using the known gravitational potential for a uniform elliptical disk. The case of uniform shear loading along the crack face was treated by Kassir and Sih [2], who obtained an exact solution in terms of two harmonic functions which, as in the tension problem [1], are constant multiples of the aforementioned gravitational potential. Kassir and Sih [2] have also derived expressions for the stress field near the crack border as well as for the stress-intensity factors.

Several investigations reported later were primarily concerned with the generalization of the work in [1] for the cases of the crack surface subjected to various degrees of polynomial pressure distribution normal to the crack surface. The first such generalization was contained in a procedure suggested by Segedin [3] who proposed the use of certain type of ellipsoidal harmonics, and their partial derivatives, which satisfy the Laplace equation. These potential functions were later used by Kassir and Sih [4] in expressing solutions for both the problems of the crack surface (i) under normal load [4, equation 3.24, p. 80] and (ii) under shear loads [4, equation 3.50, p. 86]. However, the contributions of these potentials employed in [4] to each stress component at the crack surface are not linearly independent polynomial functions and hence, one has to make a judicious choice of the potential functions for each degree of polynomial loading individually. By making such judicious choices, without, however, indicating a general procedure for such choices, Kassir and Sih [4] have presented exact solutions for some higher-order homogeneous polynomial loadings normal to the crack face, but limited their analysis to a torsional load in the case of shear loading on the crack face.

Prior to the work of [4] (what appears to us) a more logical choice of the potentials as given in [3] was made by Shah and Kobayashi [5] in representing the solution for the problem of the crack surface under arbitrary normal load. Though not explicitly stated in [5], the choice in [5] can be seen to be such that the individual contributions of the potential functions to the normal stress component on the crack surface are linearly independent, and moreover, form a complete set of polynomials. Shah and Kobayashi [5] have limited their analysis to a third-degree polynomial pressure distribution normal to the crack face, stating that the work involved in deriving the appropriate expressions for the chosen potentials was exorbitant.

In the present paper, because of the previously stated linear independency and completeness of the contributions to stress components, at the crack surface, of the potentials, and due to the analytical convenience they afford, the potentials chosen by Shah and Kobayashi [5] are used to represent solutions for both the problems of crack surface under arbitrary (i) normal and (ii) shear loading. It is
demonstrated to be possible to derive general solutions for arbitrary (normal as well as tangential) crack face tractions. Such general solutions are obtained for stress-intensity factors along the crack periphery for both the cases of (i) arbitrary applied normal stress distribution; and (ii) arbitrary applied shear stress distribution, on the crack face.

For the sake of completeness, the Trefftz’s formulation [8] for a plane surface of discontinuity is briefly sketched. This is followed by the treatment of the foregoing problem of an embedded elliptical crack in an infinite solid. The presentation of the algebraic details of the analysis is kept to a minimum in the interest of clarity as well as reasons of space.

2. Trefftz’s Formulation for a Plane Surface of Discontinuity

Let \( \sigma_{\alpha} (\alpha = 1, 2, 3) \) and \( u_{\alpha} (\alpha = 1, 2, 3) \) denote displacements and stresses, respectively, in a homogeneous, isotropic linear elastic solid. The stress-displacement relations are, by Hook’s law,

\[
\sigma_{\alpha\beta} = G(u_{\alpha\beta} + u_{\alpha} u_{\beta} + \frac{2\nu}{1-2\nu} \delta_{\alpha\beta} u_{\gamma} u_{\gamma})
\]  

(1)

where \( G \) and \( \nu \) are the shear modulus and Poisson’s ratio of the material, respectively. The Navier displacement equations of equilibrium in the absence of body forces are

\[
u_{\alpha,\beta} + (1-2\nu)u_{\alpha,\beta} = 0
\]  

(2)

in rectangular Cartesian coordinates \( x_\alpha (\alpha = 1, 2, 3) \). In the foregoing the notations \( \nu_\alpha = \partial u_\alpha / \partial x_\alpha \) and \( \nu_\alpha = \partial u_\alpha / \partial x_\alpha \) have been employed.

Let \( R \) be a region of discontinuity in the plane \( x_3 = 0 \) such that, after deformation, the material inside \( R \) breaks up with free upper and lower surfaces, and remains continuous outside \( R \). To deal with such a problem it is convenient to consider its complementary problem in which the surface of region of discontinuity is subjected to arbitrary traction \( \sigma_{\alpha\beta} \).

It is well known [7] that the solution for the aforementioned problem can be expressed in terms of four harmonic functions \( \psi \) and \( \phi_\alpha (\alpha = 1, 2, 3) \) in the form

\[
u_\alpha = \phi_\alpha + x_3 \psi_\alpha
\]  

(3)

so that the equations (2) are satisfied if

\[
u_\alpha + (3-4\nu)\psi_\alpha = 0
\]  

(4)


The stress components in terms of \( \phi_\alpha \) and \( \psi \) are

\[
\sigma_{\alpha\beta} = G \left\{ \phi_{\alpha\beta} + \phi_\beta \phi_\alpha + \delta_{\alpha\beta} \psi \phi_\alpha + \delta_{\alpha\beta} \psi \right\}
\]  

(5)

The boundary conditions along the surface of the region of discontinuity are given by

\[
\sigma_{\alpha\beta} = \left. \frac{2G}{1-2\nu} \right| \nu_\alpha \phi_{\alpha\beta} + (1-\nu) (\phi_\beta \nu_\alpha + \psi \phi_{\alpha\beta})
\]  

(6a)

\[
\sigma_{\alpha\beta} = G \left\{ \phi_{\alpha\beta} + (\phi_\beta \phi_\alpha + \psi \phi_{\alpha\beta}) \right\}
\]  

(6b)

inside the region \( R \) in the plane \( x_3 = 0 \), wherein, the notation \( \sigma_{\alpha\beta} \) is used for a prescribed quantity.

The problem is further simplified by expressing \( \psi \) and \( \phi_\alpha \) in the form

\[
\psi = \nabla \cdot f = f_{\alpha\alpha}
\]  

(7)

\[
\phi_1 = (1-2\nu)(f_{1,2} + f_{2,3}) - (3-4\nu)f_{1,3}
\]  

(8a)

\[
\phi_2 = (1-2\nu)(f_{2,3} + f_{3,1}) - (3-4\nu)f_{2,3}
\]  

(8b)

\[
\phi_3 = -(1-2\nu)(f_{1,3} + f_{2,3}) - (1-\nu)f_{3,3}
\]  

(8c)

Then, the governing equations, namely,

\[
\psi_{,\alpha} = 0, \quad \phi_{\alpha,\alpha} = 0, \quad (3-4\nu)\psi_{,\alpha} = 0
\]

for \( \psi \) and \( \phi_{\alpha} \) are satisfied in the three functions \( f_{\alpha} (\alpha = 1, 2, 3) \) are harmonic. The stress components \( \sigma_{\alpha\beta} \) in terms of \( f_{\alpha} (\alpha = 1, 2, 3) \) are given by

\[
\sigma_{11} = 2G(\psi_{1,1} + 2\nu(\psi_{2,2} - 2f_{1,3} - 2f_{2,3} + x_3(\nabla \cdot f))_{,1})
\]  

(9a)

\[
\sigma_{22} = 2G(\psi_{2,2} + 2\nu(\psi_{3,3} + 2f_{2,3} - 2f_{1,3} + x_3(\nabla \cdot f))_{,2})
\]  

(9b)

\[
\sigma_{12} = 2G[(1-2\nu)f_{1,3} + (1-\nu)f_{1,2} + f_{2,3} + x_3(\nabla \cdot f)_{,12}]
\]  

(9c)

\[
\sigma_{33} = 2G[\psi_{3,3} - f_{1,3} x_3(\nabla \cdot f)]
\]  

(9d)

\[
\sigma_{13} = 2G[f_{1,3} + f_{2,3} + x_3\nabla \cdot f]
\]  

(9e)

\[
\sigma_{23} = 2G[\psi_{2,3} + f_{1,3} + x_3(\nabla \cdot f)]
\]  

(9f)

The boundary conditions (6) to be satisfied inside the region \( R \) in the plane \( x_3 = 0 \) take the much simpler forms

\[
\sigma_{11}^{(0)} = -2G(\psi_{1,1})
\]  

(10a)

\[
\sigma_{22}^{(0)} = -2G(\psi_{2,2} - f_{1,3} x_3)
\]  

(10b)

in which the boundary condition for \( f_{2} \) is uncoupled from \( f_{1} \) and \( f_{3} \).

It is to be noted that only the symmetric components of \( f_{\alpha} \) with respect to the plane \( x_3 = 0 \) need to be considered for satisfying the boundary conditions (10). If the solid is of infinite extent in all directions and the stress components decay to zero as one moves toward infinity, then the solutions \( f_{\alpha} (\alpha = 1, 2, 3) \) are harmonic functions symmetric in \( x_3 \). In such a case, the problem solving \( f_{2} \) is independent of the problem governing \( f_{1} \) and \( f_{3} \). Kassir and Sih [4] have denoted the former as a symmetric problem and the latter one as a skew-symmetric problem.

3. Embedded Elliptical Crack in an Infinite Solid

Let the region of discontinuity be bounded by an ellipse

\[
\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1, \quad a_1 > a_2
\]  

(11)

in the plane \( x_3 = 0 \). The foregoing geometry of the crack surface is more conveniently described in an elliptoidal coordinate system. The necessary elliptical coordinates \( \xi_\alpha (\alpha = 1, 2, 3) \) are the roots of the cubic equation

\[
\omega (\xi) = 0
\]  

(12)

where

\[
\omega (\xi) = 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} - \frac{x_3^2}{a_3^2}
\]  

(13)

They are connected to the Cartesian coordinates \( x_\alpha \) by the relations

\[
a_1^2(x_1^2 - a_2^2x_2^2) = (a_1^2 - a_2^2)x_1^2 + 2a_2^2x_1x_2 + a_2^2x_2^2
\]  

(14a)

\[
a_2^2(x_2^2 - a_1^2x_1^2) = (a_2^2 - a_1^2)x_2^2 + 2a_1^2x_1x_2 + a_1^2x_1^2
\]  

(14b)

\[
a_3^2(x_3^2 - a_1^2x_1^2 - a_2^2x_2^2) = a_1^2a_2^2
\]  

(14c)

where

\[
-a_1^2 \leq \xi_1 \leq a_1^2, \quad -a_2^2 \leq \xi_2 \leq a_2^2
\]  

(14d)

The expression for \( \omega (\xi) \) in equation (13) may be written in the alternate form

\[
\omega (\xi) = P(\xi)/Q(\xi)
\]  

(15)

where

\[
P(\xi) = (\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)
\]  

(16a)

\[
Q(\xi) = (\xi + a_1^2)(\xi + a_2^2)
\]  

(16b)
The partial derivatives of $\xi_a$ with respect to $x_2$ required later are given by

$$\frac{\partial \xi_a}{\partial x_2} = -\frac{2x_2q(\xi_a)}{(x_2^2 + \xi_a)^2} \frac{P}{\varphi}$$

in which $\beta = 3$, $\alpha_2$ is zero. In the foregoing, the following notation is used: $\partial \xi_a / \partial x_2$ indicates the $a$th partial derivative wrt $x_2$ and $P$ indicates the derivative of $P$ wrt $\xi_a$.

The elliptic boundary (11) in the plane $x_3 = 0$ corresponds to the curve $\xi_1 = 0, \xi_2 = 0$. The crack surface itself namely, the region inside the ellipse (11) in the plane $x_3 = 0$ is given in a simple manner by the surface $\xi_1 = 0$.

The boundary conditions (10) may also be expressed in ellipsoidal coordinates. From practical considerations, however, it is useful to describe the distribution of the applied loads $q_{\alpha}^n$ in Cartesian coordinates. Moreover, the potentials are required to be symmetric with respect to the plane $x_3 = 0$. It is obviously difficult to meet this requirement in ellipsoidal coordinate system.

From previous considerations, it is convenient to carry out the analysis by a judicious use of both Cartesian and ellipsoidal coordinate systems.

**Basic Potentials $V_n$ ($n = 1, 2, \ldots$).** Basic potentials useful for the analysis were suggested by Segedin [3]. They are of the form

$$V_n = \int_{E_3} \omega(s)^n \frac{ds}{\sqrt{(Q(s))}} \quad n = 1, 2, \ldots$$

(In fact, the function $V_n$ is known to be harmonic for all real values of $n \geq 0$ [9]. It is symmetric in each $x_2$ since $\omega(s)$ and $\xi_2$ are symmetric in $x_2$.)

Examine the same properties of the functions in equation (18) for the analysis of crack surface under arbitrary load; we first consider the contribution of $V_n$ to the stress components along the crack surface.

By direct differentiation of the expression for $V_n$ with respect to $\omega_\alpha$ we have

$$\frac{\partial V_n}{\partial \omega_\alpha} = \int_{E_3} \omega_\alpha^2 \frac{ds}{\sqrt{(Q(s))}}$$

in which

$$\rho_\alpha = -2/(2\alpha^2 + 1)$$

in which

$$\omega(s) = \frac{\partial \omega_\alpha}{\partial \omega_\alpha}$$

in the plane $x_3 = 0$, one obtains from equations (22), for $\alpha, \beta = 1, 2$, in view of equation (17), that

$$[\partial \xi_a / \partial x_2]_{|s_0} = n(n - 1) \int_{E_3} \frac{\rho_\alpha \rho_\beta \omega_\beta \omega_\alpha}{\sqrt{(Q(s))}} \frac{ds}{\sqrt{(Q(s))}}$$

$$+ n \delta_{\alpha\beta} \int_{E_3} \rho_\alpha \omega_\beta \frac{ds}{\sqrt{(Q(s))}}$$

In the case of the derivative $\partial \xi_1 / \partial x_1$, the expressions (22) contain singular terms. In the limit $\xi_1 \to 0$, however, it can be shown that

$$\lim_{\xi_1 \to 0} [\partial \xi_1 / \partial x_1] = -[\partial \xi_3 / \partial x_1]$$

as it should be since $V_n$ satisfies the Laplace equation.

It can be seen that the expressions in equations (23) and (24) are polynomials in $x_1$ and $x_2$. However, since the functions

$$[\omega_n]^2 = \frac{\delta_{\alpha\beta}}{\xi_1^2}$$

are polynomials in $x_1$ and $x_2$, the aforementioned polynomials in equations (23) and (24) do not form a complete set to represent an arbitrary function of the variables $x_1$ and $x_2$. Hence, if the requirements for $f_{\alpha\beta}$ in the problem are represented as linear combinations of $V_n (n = 1, 2, \ldots, \omega)$, one cannot obtain arbitrary distributions of $\sigma_{\alpha\beta}$ ($\alpha = 1, 2, 3$) along the crack surface. That is, the functions $V_n (n = 1, 2, \ldots, \omega)$ do not form a complete set to represent solutions $f_{\alpha\beta}$ for an arbitrary loading along the crack surface.

**Complete Set of Potentials $F_{KL}$ ($K, L = 0, 1, 2, \ldots$).** Let each component of the applied load $q_{\alpha}^n$ be a polynomial of degree $M$ in $x_1$ and $x_2$. Then the number of linearly independent terms in each component is $\frac{1}{2}(M + 1)(M + 2)$. Hence, to represent the solution for each $f_{\alpha\beta}$, one has to find the same number of linearly independent harmonic functions of the type $V_n$ such that a linear combination of their polynomial contributions to the tractions along the crack surface match with the given polynomial distributions of applied loads exactly. For this purpose, we consider the functions of the type

$$V_n = \frac{\delta_{\alpha\beta}}{\xi_1^2}$$

first suggested by Segedin [3] and later used by Shah and Kobayashi [5] and Kassir and Sih [4], and Smith and Sorensen [6]. It can be easily shown [8] that the aforementioned partial derivatives of $V_n$ are harmonic for $k + i < n$ with polynomial contributions of degree $2n - k - l - 2$ to the tractions along the crack surface. These functions would be suitable for the analysis if the integers $k, l, n$ are restricted by the relations

$$2(k + l + 1) \leq 2n \leq M + k + l + 2$$

However, the number of functions corresponding to the integers $k, l, n$ satisfying the relations (27) are more than $\frac{1}{2}(M + 1)(M + 2)$ for $M \geq 2$. As such, the aforementioned polynomial contributions of these functions are not linearly independent for $M \geq 2$.

In the symmetric problem, wherein only $f_3$ is nonzero, Shah and Kobayashi [5] have chosen the required set of functions for representing $f_3$ by taking $n = k + l + 1 (\leq M + 1)$ and varying $k$ from $0$ to $M$. They have, however, limited their work to $M = 3$ from practical considerations such as exhorbitant work involved in obtaining and using partial derivatives (26) in the analysis. Kassir and Sih [4] have considered some specific (incomplete) homogeneous polynomial loadings up to the degree $M = 6$; thus the results in [4] are inadequate for the solution of a problem of crack face pressure of an arbitrary polynomial variation even of degree 6. In retrospect it appears that for each loading Kassir and Sih [4] have chosen a suitable combination.

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1. Note that

$$\frac{\partial \omega_\alpha}{\partial \beta} = -\frac{2x_2q(\xi_a)}{(x_2^2 + \xi_a)^2} \frac{P}{\varphi}$$

2. One of the reviewers has brought to our attention the efforts of Broekhoven in extending the results of [5] in the case of $M = 4$ in the symmetric problem. Upon further literature search the authors found reference [10], wherein Broekhoven mentions such effort, but no detailed results are given.
of \( h, l, n \) satisfying conditions (27) but no general procedure for the choice was indicated in [4].

In the case of shear loading, Kassir and Sih [2] have considered the problem of crack surface under constant shear for which the solutions for \( f_1 \) and \( f_2 \), which only are nonzero, are constant multiples of \( V_1 \). Later [4], they have also obtained solutions for the problem of crack surface under torsional load.

Smith and Sorensen [6] have considered a complete cubic-polynomial load tangential to the crack face leading to a \((20 \times 20)\) matrix equation. They have not, however, presented in [6] the expressions for the matrix elements. The problem of torsional load, as a special case of their analysis [6], was treated in detail and variation of the stress-intensity factors along the crack border were presented in graphical form in [6]. From the previous observations, it can be seen that, for high-order polynomial shear loadings, no solutions were reported until now in the literature, even though the solutions for \( f_1 \) and \( f_2 \) can be obtained in terms of a suitable set of functions (26).

In the present paper, Shah and Kobayashi's representation [5] for the solution is extended to the general case of arbitrary loading by expressing the solutions components, \( \alpha = 1, 2, 3 \), in the form

\[
f_{kl} = \sum_{k} \sum_{l} C_{i,j,k,l} f_{kl}
\]

where, by definition,

\[
f_{kl} = \phi_{r}^{i} \phi_{q}^{j} V_{k+l+1}
\]

and \( C_{i,j,k,l} \) are unknown constants to be determined from analysis.

**Partial Derivatives of \( F_{kl} \)**

By successive differentiation, it can be shown that, since \( \alpha(\xi_3) = 0 \),

\[
F_{kl} = \int_{\xi_3}^{\infty} \phi_{r}^{i} \phi_{q}^{j} \omega^{k+l+1} \frac{ds}{Q(s)}
\]

and

\[
\phi_{r}^{i} \phi_{q}^{j} \phi_{r}^\prime \phi_{q}^\prime \omega^{k+l+1} \frac{ds}{Q(s)}, \quad \alpha = 1, 2, 3
\]

Differentiating both sides of equation (31) with respect to \( \xi_3 \), we obtain the second-order partial derivatives of \( F_{kl} \) required for the evaluation of stress components along the crack face in the form

\[
\phi_{r}^{i} \phi_{q}^{j} \phi_{r}^\prime \phi_{q}^\prime \omega^{k+l+1} \frac{ds}{Q(s)}
\]


\[
F_{kl} = \int_{\xi_3}^{\infty} \phi_{r}^{i} \phi_{q}^{j} \phi_{r}^\prime \phi_{q}^\prime \omega^{k+l+1} \frac{ds}{Q(s)}
\]

In the foregoing (\( \prime \) denotes the factorial of the respective quantity. The derivation of expression (33) is given in Appendix of the present paper.

**Expressions for Boundary Errors.** To satisfy boundary conditions (10), it is necessary to evaluate only the derivatives

\[
\phi_{r}^{i} \phi_{q}^{j} \phi_{r}^\prime \phi_{q}^\prime \phi_{r}^\prime \phi_{q}^\prime \omega^{k+l+1} \frac{ds}{Q(s)}
\]

along the crack surface \( \xi_3 = 0 \). The former three derivatives along the crack surface are given by

\[
\phi_{r}^{i} \phi_{q}^{j} \phi_{r}^\prime \phi_{q}^\prime \phi_{r}^\prime \phi_{q}^\prime \omega^{k+l+1} \frac{ds}{Q(s)} = 0, \quad \alpha = 1, 2
\]

since

\[
\phi_{r}^{i} \phi_{q}^{j} \phi_{r}^\prime \phi_{q}^\prime \phi_{r}^\prime \phi_{q}^\prime \omega^{k+l+1} \frac{ds}{Q(s)} = 0, \quad \alpha = 1, 2
\]

In the evaluation of the derivative \( \phi_{r}^{i} \phi_{q}^{j} \phi_{r}^\prime \phi_{q}^\prime \phi_{r}^\prime \phi_{q}^\prime \omega^{k+l+1} \frac{ds}{Q(s)} \)
\[ X = \frac{x^{2p-2k-i}}{(2p-2-2q-k)!} \frac{x^{2j-i}}{(2q-l)!} \]

\[ J \]  

(Cont.)

\[ \text{in which} \]

\[ p_0 = 1 + \text{integer value of} \ (k + l + 1)/2 \]

\[ q_0 = \text{integer value of} \ (l + 1)/2 \]

\[ J = \text{finite part of} \ \lim_{|s| \to \infty} \int_{|s|}^{\infty} \frac{2ds}{\sigma(a^2 + s)^{p-1}(a^2 + s)^q \sqrt{\eta(s)}} \]

**Crack Surface Under Normal Load.** Let the normal load \( \sigma_{nn} \) along the crack surface in the symmetric problem be expressed in the form

\[ \sigma_{nn} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{M} \sum_{l=0}^{N} A_{ijkl}^2 (a^2 + s)^{2p-2k-i} (a^2 + s)^{2q-l} \]

(43)

so that the values of \( i, j, k \) specify the symmetries of the load with respect to the axes of the ellipse.

The solution for the function \( f_2 \) in terms of the potentials \( F_{mn} \) is assumed in the form

\[ f_2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{M} \sum_{l=0}^{N} C_{ijkl}^2 F_{3k-2i+2l+1} \]

(44)

In the foregoing, for purposes of clarity, a potential such as \( F_{mn} \) is represented as \( F_{3k-2i+2l+1} \); thus it is to be understood that \( m \) and \( n \) take the values \( (2k + 2l + i) \) and \( (2i + j) \), respectively.

The aforementioned expressions for \( \sigma_{nn} \) and \( f_2 \) are substituted in the boundary condition (100). The polynomial expansion of the second-order derivative of each of the potentials \( F_{ij} \) is obtained from equation (41). The coefficients of like power terms on both sides of equation (100) are equated to each other leading to the following set of linear algebraic equations for the determination of the parameters \( C_{ijkl}^2 \):

\[ (-1)^{m+i+j} (2m - 2n + i)(2n + j)! (-1)^{m} \] \[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{M} A_{ijkl}^2 J^{(i)j} (2m - 2n + i) \]

\[ \frac{1}{(2m)} A_{ijkl}^2 \]

\[ m = 0, 1, 2 ... M \]

\[ n = 0, 1, 2 ... m \]

(45)

in which

\[ I_{ij} = \int_{|s|}^{\infty} \frac{2ds}{\sigma(a^2 + s)^{p-1}(a^2 + s)^q \sqrt{\eta(s)}} \]

(47)

and

\[ J^{(i)j} = \text{finite part of} \ \lim_{|s| \to \infty} \int_{|s|}^{\infty} \frac{2ds}{\sigma(a^2 + s)^{p-1}(a^2 + s)^q \sqrt{\eta(s)}} \]

(48)

The foregoing equations (45) may be solved in successive steps as outlined as follows:

(i) From the \( M + 1 \) equations corresponding to \( m = M, n = 0, 1, 2, ..., M \), solve for the \( M + 1 \) coefficients

\[ C_{ijkl}^2 \]

\[ i = 0, 1, 2, ..., M \]

(ii) Substitute the values of coefficients obtained in step (i), in the \( M \) equations corresponding to \( m = M - 1, n = 0, 1, 2, ..., M - 1 \). Solve these \( M \) equations for the \( M \) coefficients

\[ C_{ijkl}^2 \]

\[ i = 0, 1, 2, ..., M - 1 \]

(iii) Continue the process solving, at the \( M + 1 \) step, the \( r + 1 \) equations corresponding to \( m = r, n = 0, 1, 2, ..., r \) for the \( r + 1 \) coefficients

\[ C_{ijkl}^2 \]

\[ i = 0, 1, 2, ..., r \]

and taking values of \( r = M - 2, M - 3, ..., 1, 0 \) in succession.

It may be mentioned here that Shah and Kobayashi [5] have confined their work to \( M = 1 \) in each of the symmetric groups \( i, j \) = (0, 0), (0, 1), and (1, 0) and to \( M = 0 \) in the case of doubly antisymmetric loading corresponding to \( i = j = 1, 1 \). Kassir and Sih [4] have considered homogeneous polynomial loadings corresponding to \( M = 1, 2, 3 \) in the doubly symmetric group \( i, j \) = (0, 0) and to \( M = 1 \) in the two groups \( i = j = 0, 1, 1 \). Solutions in [4] were obtained, however, by using combinations of \( F_{ij} \) different from those given in equation (44) except in the case of \( M = 1 \) in the (0, 0) group.

**Crack Surface Under Shear Load.** As in the symmetric problem, the applied shear load components \( e_{ij}^0 \) (\( i = 1, 2 \)) in the skew-symmetric problem may be taken in the form

\[ e_{ij}^0 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{M} A_{ijkl}^0 (a^2 + s)^{2m-2k+i} (a^2 + s)^{2n-l} \]

(49)

and the solutions for \( f_3 \) may be assumed in the form

\[ f_3 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{M} \sum_{l=0}^{N} C_{ijkl}^3 F_{3k-2i+2l+1} \]

(50)

in which the upper value for \( k \) is dependent on \( \alpha, i, j, \) and \( M \).

Due to the skew-symmetric nature, the problem decomposes into the following two problems denoted by \( F_{ij} (\alpha, \beta, M) \) and \( F_{ij} (\beta, \alpha, M) \), \( \alpha \neq \beta \):

(i) Problem \( F_{ij} (\alpha, \beta, M) \): \( e_{ij}^0 \) is symmetric and \( e_{ij}^0 \) is antisymmetric in both \( x_1 \) and \( x_2 \).

(ii) Problem \( F_{ij} (\beta, \alpha, M) \): \( e_{ij}^0 \) is symmetric in \( x_1 \) and antisymmetric in \( x_2 \).

In the latter of the just mentioned two problems, the expressions taken for the load components, the series assumed for the solutions, and the derived linear algebraic equations governing the coefficients \( C_{ijkl}^3 \) are given as follows:

(i) Problem \( F_{ij} (\alpha, \beta, M) \):

\[ \sigma_{ij} = \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{k=0}^{M} \sum_{l=0}^{N} A_{ijkl}^0 (a^2 + s)^{2m-2k+i} (a^2 + s)^{2n-l} \]

(51)

where

\[ \sigma_{ij} = 0 \]

(52)

and

\[ f_{ij} = \sum_{k=0}^{M} \sum_{l=0}^{N} C_{ijkl}^3 F_{3k-2i+2l+1} \]

(53)

where

\[ f_{ij} = 0 \]

(54)

\[ \left(\frac{1}{2}\right)^{M-N} \sum_{k=0}^{M-N} \sum_{l=0}^{N} C_{ijkl}^3 \sum_{m=0}^{M} \sum_{n=0}^{N} A_{ijkl}^0 (a^2 + s)^{2m-2k+i} (a^2 + s)^{2n-l} \]

(55)

and

\[ C_{ijkl}^3 = \sum_{k=0}^{M-N} \sum_{l=0}^{N} C_{ijkl}^3 \]

(56a)

\[ \left(\frac{1}{2}\right)^{M-N} \sum_{k=0}^{M-N} \sum_{l=0}^{N} C_{ijkl}^3 \]

(56b)

in which

\[ \left(\frac{1}{2}\right)^{M-N} \sum_{k=0}^{M-N} \sum_{l=0}^{N} C_{ijkl}^3 \]

(57a)
\[ J_{x2}^{(2)} = \left. \frac{1}{\sqrt{Q(s)}} \frac{ds}{d\alpha} \right|_{\alpha = 1, 2} \]

(56a)

\[ J_{x2}^{(1)} = \int_0^\infty \frac{1}{(\alpha^2 + s)^{k-\frac{1}{2}}} ds \]

(56b)

In the particular case of \( M = 0 \), we have

\[ \sigma_{00} = A_{\alpha\beta}^{(0)0} \quad \sigma_{0\beta} = 0 \]

\[ f_{\alpha}^{(01)} = C_{\alpha\beta}^{(0)0} F_{\alpha\beta} \quad f_{\bar{\beta}}^{(1\beta)} = 0 \quad \alpha \neq \beta = 1, 2 \]

(59)

The equations (56) reduce to

\[ 2G[(1 - \nu)\nu^{(0)0} + \nu \nu^{(0)1} - \nu \nu^{(1)0}] C_{\alpha\beta}^{(0)0} = A_{\alpha\beta}^{(0)0} \]

(60)

so that, in the case of constant shear, the solutions \( f_{\alpha}^{(01)} \) (\( \alpha = 1, 2 \)) are constant multiples of \( F_{\alpha\beta} \).

As in the symmetric problem equations (56) may be solved in successive steps by solving first for the (\( 2M + 1 \)) coefficients

\[ C_{\alpha\beta}^{(0)0}, \quad l = 0, 1, 2 \ldots M \]

and

\[ C_{\alpha\beta}^{(0)1}, \quad l = 0, 1, 2 \ldots M - 1 \]

from the \( (2M + 1) \) equations consisting of \( M + 1 \) equations of (56a) for \( m = M, n = 0, 1, 2 \ldots M \) and \( M \) equations of (56b) for \( m = M - 1, n = 0, 1, 2 \ldots M - 1 \).

\( \text{i) Problem P}_\alpha(\alpha, \beta, M) \):

\[ \sigma_{00} = x_2 \sum_{m=0}^{M} \sum_{n=0}^{m} A_{\alpha\beta}^{(0)0} x_2^{m-2n} \times C_{\alpha\beta}^{(0)0} \]

(61)

\[ \sigma_{0\beta} = x_2 \sum_{m=0}^{M} \sum_{n=0}^{m} A_{\alpha\beta}^{(0)1} x_2^{m-2n} \times C_{\alpha\beta}^{(0)1} \]

(62)

\[ f_{\alpha}^{(01)} = \sum_{k=0}^{M} \sum_{l=0}^{k} C_{\alpha\beta}^{(0)1} F_{\alpha\beta} \]

(63a)

\[ f_{\bar{\beta}}^{(1\beta)} = 0 \]

(63b)

\[ \left( \frac{1}{2G} \right) A_{\alpha\beta}^{(1)0}, \quad n = 0, 1, 2 \ldots M \]

\[ \times \sum_{k=0}^{M} \sum_{l=0}^{k} (-1)^{l+1} \left[ \left( 1 - \nu \right) I_1^{(0)0} I_2^{(0)l} - \nu I_1^{(0)1} I_2^{(1)l} \right] \]

(64a)

\[ \left( \frac{1}{2G} \right) A_{\alpha\beta}^{(1)1}, \quad n = 0, 1, 2 \ldots M \]

\[ \times \sum_{k=0}^{M} \sum_{l=0}^{k} (-1)^{l+1} \left[ \left( 1 - \nu \right) I_1^{(0)0} I_2^{(1)l} - \nu I_1^{(0)1} I_2^{(1)l} \right] \]

(64b)

As in the previous problem, equations (64) may be solved in successive steps by solving first for the \( (2M + 1) \) coefficients

\[ C_{\alpha\beta}^{(0)0}, \quad C_{\alpha\beta}^{(0)1}, \quad l = 0, 1, 2 \ldots M \]

from the \( (2M + 1) \) equations corresponding to \( m = M, n = 0, 1, 2 \ldots M \).

It is to be noted that, in the case of arbitrary polygonal loadings, the integer \( M \) in the summation in equation (48) can assume different values \( M(\alpha, i, j) \) for different combinations of \( \alpha, i, \) and \( j \). In such a case, we define, for \( \alpha \neq \beta = 1, 2 \).

\[ M_{1,2} = \max \{ M(\alpha, 0, 0), M(\beta, 1, 1) + 1 \} \]

and

\[ M_{3,5} = \max \{ M(\alpha, 0, 1), M(\beta, 1, 0) \} \]

(65)

Then the solution of the skew-symmetric problem for arbitrary polygonal loading is given by a linear sum of the solutions of the four problems

\[ P_1(\alpha, \beta, M_{1,2}), P_2(\alpha, \beta, M_{3,5}), \quad \alpha \neq \beta = 1, 2 \]

(66)

**Stress-Intensity Factors** \( k_\infty(\alpha = 1, 2, 3) \). Kassir and Sih [2] have shown that in the vicinity of the periphery of an elliptical crack on the plane \( x_3 = 0 \), the ellipsoidal coordinates become

\[ \xi_1 = -a_1 \sin \theta + a_2 \cos^2 \theta \]

(67a)

\[ \xi_2 = 0 \]

(67b)

\[ \xi_3 = 2a_1 \sigma_\tau_a \sin \theta \cos \theta \eta^{1/2} \]

(67c)

where \( \eta \) is the radial distance normal to the crack border in the plane \( x_3 = 0 \), and \( \theta \) is the angle in the parametric equations of the ellipse

\[ x_1 = a_1 \cos \theta \quad \text{and} \quad x_2 = a_2 \sin \theta \]

(68)

The normal and tangential components \( \sigma_{\alpha 3}, \sigma_{\beta 3} \) near the border of the elliptical crack (where \( n \) and \( \tau \) are directions normal and tangential to the crack border in the plane \( x_3 = 0 \)) in the plane \( x_3 = 0 \) are given by the relations

\[ \sigma_{\alpha 3} = \sigma_{\beta 3} \cos \beta + \alpha \beta \sin \beta \]

(69a)

\[ \sigma_{\beta 3} = -\sigma_{\alpha 3} \sin \beta + \beta \cos \alpha \]

(69b)

where \( \beta \) is the angle between the outward normal of the crack border (in the plane \( x_3 = 0 \)) and the \( x_1 \)-axis, is related to \( \theta \) by the equations

\[ \cos \beta = a_2 \cos \beta / \sqrt{A} \quad \sin \beta = a_1 \sin \theta / \sqrt{A} \]

(70)

\[ A = a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta \]

(71)

The stress-intensity factors \( k_\infty \) are defined as

\[ K_1 = \lim_{\eta \to 0} \left( 2\pi \right)^{1/2} \sigma_{33} \eta^{3/2} \]

(72a)

\[ K_2 = \lim_{\eta \to 0} \left( 2\pi \right)^{1/2} \sigma_{33} \eta^{3/2} \]

(72b)

and

\[ K_3 = \lim_{\eta \to 0} \left( 2\pi \right)^{1/2} \sigma_{33} \eta^{3/2} \]

(72c)

From equations (67), (69)–(72), the expressions for \( K_\infty \) take the following forms:

\[ K_1 = \frac{\pi}{a_1 a_2} \left( 1/4 \right) \text{lim}_{\eta \to 0} \left( \xi_{33}^2 \right)^{1/2} \]

(73a)

\[ K_2 = \frac{\pi}{a_1 a_2} \left( 1/4 \right) \text{lim}_{\eta \to 0} \left( \xi_{33}^2 \right)^{1/2} \]

(73b)

\[ K_3 = \frac{\pi}{a_1 a_2} \left( 1/4 \right) \text{lim}_{\eta \to 0} \left( \xi_{33}^2 \right)^{1/2} \]

(73c)

From equations (9d)–(9f), we have

\[ \text{lim}_{\eta \to 0} \left( \xi_{33}^2 \right) = -2G \text{lim}_{\eta \to 0} \left( \xi_{33}^2 \right) \]

(74)

and

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\[
\lim_{\varepsilon \to 0} \left| \hat{\varepsilon}^{(i)} \right| \left| \Delta \varepsilon = a, b \right. = -2G \lim_{\varepsilon \to 0} \left| \hat{\varepsilon}^{(i)} \right| \left( 1 - \nu \right) \left( 1 - \nu \right) = 0 \quad \alpha \neq \beta = 1, 2 \quad (75)
\]

Since the solutions for \( f_{in} \) are linear combinations of potentials \( F_{kn} \), one needs the following quantities for evaluating the right-hand side expressions in equations (74) and (75).

\[
G_{kn} = \lim_{\varepsilon \to 0} \left| \hat{\varepsilon}^{(i)} \right| \left| \Delta \varepsilon = a, b \right. = \lim_{\varepsilon \to 0} \left| \int \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right| \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right|
\]

\[
= \left( -2 \right)^{1/2} \left( 2 + 1 \right) \left( \frac{\cos \theta \sin \theta}{a^1} \right)^{1/2} \left( \alpha \right) \rangle = 1, 2 \quad (77)
\]

where \( x, x, a, b \) are given by equation (68).

By using equations (74), (75), series solutions obtained for \( f_{in} \), and the equations (76) and (77), the stress-intensity factors \( K_i \) are evaluated from equations (76). The expressions for \( K_i \) thus obtained are given next.

(i) Symmetric Problem. From equations (44), (73a), (74), and (76), the expression for the stress-intensity factor \( K_1 \) is obtained as

\[
K_1 = -2G \int \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right| \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right|
\]

\[
= 8G \int \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right| \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right|
\]

\[
= 8G \int \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right| \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right|
\]

\[
= 8G \int \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right| \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right|
\]

(ii) Skew-Symmetric Problem. The solutions for \( f_1 \) and \( f_2 \) in the problem are linear sums of the solutions of the problems \( F_1 \) and \( F_2 \) which take the form

\[
f_1 = \sum_{i=0}^{M} f_{i}^{(i)}(x) \sum_{j=0}^{M} f_{j}^{(j)}(x) \sum_{k=0}^{M} f_{k}^{(k)}(x) \sum_{l=0}^{M} f_{l}^{(l)}(x) \sum_{i=0}^{M} F_{i}^{(i)}(\Delta \varepsilon = a, b)
\]

and

\[
f_2 = \sum_{i=0}^{M} f_{i}^{(i)}(x) \sum_{j=0}^{M} f_{j}^{(j)}(x) \sum_{k=0}^{M} f_{k}^{(k)}(x) \sum_{l=0}^{M} f_{l}^{(l)}(x) \sum_{i=0}^{M} F_{i}^{(i)}(\Delta \varepsilon = a, b)
\]

The stress-intensity factors \( K_2 \) and \( K_3 \) corresponding to the foregoing series solutions (79) are given by

\[
K_2 = 8G \int \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right| \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right|
\]

\[
K_3 = 8G \int \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right| \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \left( \hat{\varepsilon}^{(i)} \right)^{(2)} \right|
\]

in which

\[
H_1 = \sum_{i=0}^{M} f_{i}^{(i)}(x) \sum_{j=0}^{M} f_{j}^{(j)}(x) \sum_{k=0}^{M} f_{k}^{(k)}(x) \sum_{l=0}^{M} f_{l}^{(l)}(x) \sum_{i=0}^{M} F_{i}^{(i)}(\Delta \varepsilon = a, b)
\]

\[
H_2 = \sum_{i=0}^{M} f_{i}^{(i)}(x) \sum_{j=0}^{M} f_{j}^{(j)}(x) \sum_{k=0}^{M} f_{k}^{(k)}(x) \sum_{l=0}^{M} f_{l}^{(l)}(x) \sum_{i=0}^{M} F_{i}^{(i)}(\Delta \varepsilon = a, b)
\]

4 Concluding Remarks

In the foregoing, we have presented a general solution for the problem of an infinite linear elastic solid containing a flat elliptical crack, whose faces are subjected to an arbitrary polynomial variation of normal as well as tangential tractions. This represents a generalization of the cited earlier works of other authors [1-6]. The expressions for the three-modes of stress-intensity factors, \( K_1, K_2, \) and \( K_3 \) along the flaw border, for the considered general loading, are given.

The expressions for the stresses in the far-field for the considered problem of arbitrary loading on the crack face, are given in Appendix of the present paper.

One of the most pressing needs in applied fracture mechanics is the accurate and cost-effective evaluation of stress-intensity factors along the border of embedded or surface flaws in complex structural geometries such as aircraft attachment lugs, nuclear reactor pressure-vessel nozzle junctions, etc. The shapes of these flaws are often assumed, to a first approximation, as elliptical or part-elliptical. In solving these complex practical problems, several approaches such as, the Schwartz-Neuman alternating technique [10, 11], the boundary-integral equation technique [12], and singularity-finite element methods [13, 14] have been reported in literature. It is generally recognized [15] that even though the alternating technique may be the simplest and most cost-effective technique, the results obtained so far through this technique are not as accurate as those obtained through the finite element and boundary-integral-equation approaches.

In the alternating technique, as applied to the problem of cracks in finite solids, two solutions are needed, generally. One of these solutions is for stresses in the uncracked finite body at the location of the considered crack, and the other solution is for the problem of an infinite body with a crack whose faces are subject to arbitrary normal as well as shear traction components. For cracks in complex finite bodies, such as described earlier, the first solution previously mentioned, would in general lead to a rather complex stress-field at the location of the considered crack. Because of the limitation of the available analytical results for the second solution discussed in the foregoing [1-6], the stress-fields of the aforementioned first problem were always approximated by polynomials of order \( \leq 3 \) in the use of the alternating technique [10, 11]. Since this limitation has been overcome in the present paper, the results of the present paper may effectively be employed in devising a more accurate and cost-effective alternating-solution technique for analyzing complex, flawed, structural geometries. The results of our efforts in this direction will be reported shortly.

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References


**APPENDIX**

We consider here evaluation of potentials $F_k$ and their partial derivatives required for calculating displacements and stress components at a point away from the crack surface.

Earlier, Shah and Kobayashi [5] have derived, for values of $k + l$ up to 3 expressions for $F_k$ and their partial derivatives up to second-order in terms of complete elliptic integrals and Jacobian elliptic functions. In a subsequent investigation [16], they have also obtained expressions for some third-order partial derivatives of $F_k$ required in the evaluation of stress components $\sigma_{yy}$, $\sigma_{yz}$, and $\sigma_{yz}$ in the symmetric problem. It appears that, in deriving the aforementioned expressions, they have expressed the power term $\omega^{k+l+1}$ as a polynomial in $\chi^2 (\alpha = 1, 2, 3)$ and carried out necessary differentiations. Kassir and Sih [4] have, however, adopted the chain rule of differentiation involving total derivatives with respect to $\chi$ and obtained expressions in a slightly different form for their potentials and partial derivatives up to second-order in the analyses of both symmetric and skew-symmetric problems.

In the present paper, we derive the necessary expressions for a general potential $F_k$ and its partial derivatives by both the aforementioned procedures. For this purpose, it is convenient to consider the required partial derivatives of $F_k$ as a sum $(H_0 + H_1)$ of an integrated component $H_0$ and an integral component $H_1$ of the form

$$H_0 = \frac{1}{\sqrt{Q(t)}} \int_{t_0}^{t} \frac{d\omega}{\omega} \omega^{k+l+1} \frac{ds}{\sqrt{Q(t)}}$$

Then the component $H_0$ and the form of $H_1$ in $F_k$ and its partial derivatives are as listed as follows:

(i) $F_k$: 

$$H_0 = 0$$

and

$$k_1 = k, \quad l_1 = l, \quad m_1 = 0 \quad \text{in} \quad H_1$$

(ii) $\partial^0_k F_k$: 

$$H_0 = 0$$

and

$$k_1 = k + \delta_{10}, \quad l_1 = l + \delta_{20}, \quad m_1 = \delta_{30} \quad \text{in} \quad H_1$$

(iii) $\partial^1_k F_k$: 

$$H_0 = F_k^{(0)}$$

$$= \frac{\Gamma(\omega^{k+1}+1)}{\sqrt{Q(t)}} \left[ \omega^{k+1} Q^{-\alpha/2} \right]$$

$$k_1 = k + \delta_{10} + \delta_{10}, \quad l_1 = l + \delta_{20} + \delta_{20}, \quad m_1 = \delta_{30} + \delta_{30} \quad \text{in} \quad H_1$$

(iv) $\partial^2_k F_k$: 

$$H_0 = \frac{\partial}{\partial x_3} F_k^{(0)} = \frac{\partial^2}{\partial x_3^2} F_k^{(0)}$$

$$\omega^{k+1} = \frac{\partial}{\partial x_3} \left[ \omega^{k+1} \sqrt{Q(t)} \right] \left[ \omega^{k+1} \sqrt{Q(t)} \right]$$

$$k_1 = k + \delta_{10} + \delta_{10}, \quad l_1 = l + \delta_{20} + \delta_{20}, \quad m_1 = \delta_{30} + \delta_{30} \quad \text{in} \quad H_1$$

The integrand in equation (83) may be evaluated in two ways mentioned earlier. In the first procedure, the power term $\omega^{k+l+1}$ is expanded in terms of $x_3^2 (\alpha = 1, 2, 3)$ and term-by-term differentiations are carried out. Then we get

$$\frac{\partial^2}{\partial x_3^2} \left[ \frac{\partial}{\partial x_3} \omega^{k+1} \right] = \frac{(k + l + 1)!}{(k + l + 1 - 1)!} \sum_{p=0}^{k+1} \sum_{q=0}^{l+1} \frac{(-1)^p}{q! (p + q)! (p + q)!} \frac{(2p - 2q)! (2q - 2p)!}{(2p - 2q)! (2q - 2p)!} \frac{\chi^2}{x^2}$$

and

$$m_1 = \delta_{30} + \delta_{30}$$

(85)

(86)

(87)

(88)

The expression in (83) may also be used in (89) and (88) for the evaluation of the integrated parts $H_0$. However, much simpler expressions for these integrated parts are derived from the second procedure described in the following.

In adopting the second procedure involving chain rule of differentiation, we note that

$$\partial^2_k \omega^{k+1} = \sum_{p=0}^{k+1} \sum_{q=0}^{l+1} \frac{(k + l + 1)!}{(k + l + 1 - 1)!} \frac{(-1)^p}{q! (p + q)! (p + q)!} \frac{(2p - 2q)! (2q - 2p)!}{(2p - 2q)! (2q - 2p)!} \frac{\chi^2}{x^2}$$

and

$$m_1 = \delta_{30} + \delta_{30}$$

(90)

The aforementioned integrals can be evaluated in terms of incomplete elliptic integrals of the first and second kinds and Jacobian elliptic functions.

The expression in (90) can also be used in (89) and (88) for the evaluation of the integrated parts $H_0$. However, much simpler expressions for these integrated parts are derived from the second procedure described in the following.

In adopting the second procedure involving chain rule of differentiation, we note that

$$\partial^2_k \omega^{k+1} = \sum_{p=0}^{k+1} \sum_{q=0}^{l+1} \frac{(k + l + 1)!}{(k + l + 1 - 1)!} \frac{(-1)^p}{q! (p + q)! (p + q)!} \frac{(2p - 2q)! (2q - 2p)!}{(2p - 2q)! (2q - 2p)!} \frac{\chi^2}{x^2}$$

and

$$m_1 = \delta_{30} + \delta_{30}$$

(91)

(92)

(93)

(94)
\[ A_{i}^{(1)} = A_{j}^{(1)} \] when \( I = \left[ \frac{i + 1}{2} \right] + 1 \) (94) (Cont.)

Using the form for a partial derivative of the power term, we get
\[
\frac{\partial^{(k+1)j+i+l+1}}{\partial \xi_{p} \partial \xi_{q} \partial \xi_{r}} = (k + l + 1)! \sum_{p=0}^{k+1} \sum_{q=0}^{l+1} \sum_{r=0}^{i+1} A_{p}^{(k)} A_{q}^{(l)} A_{r}^{(i)}
\times A_{m}^{(m)} \rho_{1}^{l} \rho_{2}^{m} \rho_{3}^{p} \rho_{4}^{q} \xi_{5}^{p} \xi_{6}^{q} \xi_{7}^{r} \xi_{8}^{s} Q(s) N^{N}
\]
(95)

where
\[
N = k + l + 1 + p + q + r - k_1 - l_1 - m_1.
\]
(96)

By substituting the foregoing expression (95) into equation (83), one gets an alternate expression for \( H_{1} \) involving integrals of the type
\[
J = \int_{0}^{\infty} \frac{\omega^{N}}{(a^{2} + s)^{p_{1} - p_{2}}(a^{2} + s)^{q_{1} - q_{2}}(a^{2} + s)^{r_{1} - r_{2}}} \sqrt{Q(s)}
\]
(97)

which are relatively complicated in comparison with the integrals in equation (91).

However, in view of the property \( \omega(\xi_{s}) = 0 \), the form (95) is convenient for finding the integrated components \( H_{1} \) in equations (86) and (88). The contributions of terms in (95) to \( H_{1} \) are from terms corresponding to \( N = 0 \). Hence, we get the following simple expressions for the integrated components in the second and third-order partial derivatives of \( F_{ik} \):

\[
F_{ik}^{(3)} = \frac{(k + l + 1)!}{(\xi_{3} - \xi_{1})(\xi_{3} - \xi_{2})} \left[ \rho_{1}^{m_1} \rho_{2}^{p_1} \rho_{3}^{q_1} \sqrt{Q(s)} \right]_{s = \xi_{3}}
\]

(98)
The expression for \( H_{0} \) in (88) is obtained as

\[
H_{0} = \frac{Q}{\partial x_{\gamma}} F_{ik}^{(0)} + \frac{(k + l + 1)!}{\xi(\xi_{3} - \xi_{1})(\xi_{3} - \xi_{2})} \sqrt{Q(s)} \left[ \rho_{1}^{m_1} \rho_{2}^{p_1} \rho_{3}^{q_1} \right]
\]
\[
\times \left[ \frac{(k_1 - 1)k_1}{2p_1 x_{1}^2} + \frac{(l_1 - 1)l_1}{2p_2 x_{2}^2} + \frac{(m_1 - 1)m_1}{2p_3 x_{3}^2} \right]
\]
(99)
in which \( k_1, l_1, \) and \( m_1 \) are given by equations (87).

The partial derivative of \( F_{ik}^{(0)} \) in (99) may be obtained from the expression (98) by treating \( \xi_{s} (\alpha = 1, 2, 3) \) as functions of \( x_{\alpha} (\alpha = 1, 2, 3) \).

Substituting the foregoing second and third-order partial derivatives of \( F_{ik} \) appropriately into equations (90a)–(90f), the expression for each of the six stress components in the far-field can easily be written down. These lengthy expressions are omitted here for the sake of conciseness and clarity.