Numerical studies in dynamic fracture mechanics

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(Received September 1, 1984; in revised form November 8, 1984)

Abstract

This paper provides a summary of recent studies concerning numerical modeling of dynamic crack-propagation. Both "stationary mesh" as well as "moving mesh" finite-element procedures are examined. Simple procedures, using a moving mesh of conventional isoparametric elements in conjunction with certain path-independent integrals for the evaluation of stress-intensity factors for a dynamically propagating crack are presented.

1. Introduction

Dynamic fracture mechanics concerns (i) the onset of crack growth under dynamic (impact) loadings and (ii) dynamic (unstable) crack propagation in stressed solids. The conventional linear elasto-static or quasi-static fracture mechanics applies only up to the end of stable growth under quasistatic loadings and assumes that the onset of unstable crack propagation renders the structure useless. Yet, the prevention of crack-growth initiation itself may be too conservative or too costly an objective for some structural designs. On the other hand, the catastrophic failures caused by unstable crack growth are also obviously intolerable. In these cases, the assurance of crack arrest, as a second line of defense, is essential. This concept has been investigated for design methodologies assuring the integrity of nuclear pressure vessels under thermal shock conditions, LNG ship hulls, and gas transmission pipelines. These studies were reviewed in [1].

The fundamental framework of the subject of dynamic fracture mechanics relies primarily on solutions for dynamic behavior of solids containing cracks. These solutions are characterized by (i) stress wave interactions, (ii) the need to account for kinetic energy in the global energy balance of the fracturing body, and (iii) inertia effects of the material. In the past two decades or so, a number of analytical solutions, which provide a useful understanding of dynamic crack behavior, have been obtained. These analytical solutions are, however, limited to cases of simple loadings and unbounded plane bodies. Moreover, the stress wave interactions, which play an important role in dynamic fracture mechanics, usually render the analytical solutions intractable. Therefore, the use of numerical methods is often indispensable for the analyses of cracks in finite solids.

However, the ability to perform an elastodynamic analysis alone is not enough to assess the situation of dynamic fracture in a solid. Values of the material's resistance to crack propagation, i.e., the dynamic fracture toughness, must also be available. Unfortunately, for materials such as steel, a direct experimental measurement of dynamic fracture toughness, as a function of crack velocity, is often very difficult. To overcome these
difficulties, hybrid experimental-numerical methods of analysis of laboratory tests are often preferred [2,3]. This is another reason why the advancement of dynamic fracture mechanics relies heavily on advances in computational methods for the analysis of propagating cracks.

Numerical methods in dynamic fracture mechanics were critically appraised in 1978 [4]. At that time, a comparison of the finite element and finite difference methods led to the following conclusion. The finite element method was more suitable, for the analysis of stationary cracks under dynamic loadings, due to the fact that the relevant singularities can be modeled in the crack-tip elements. On the other hand, the finite element method was thought to be unsuited for the analysis of dynamic crack propagation, due to the numerical difficulties involved in advancing the crack in a discrete manner.

However, the state of the art of finite element methods in dynamic fracture mechanics has greatly advanced in the intervening years. In large measure, this is the result of the development of finite-element procedures to model propagating singularities and of path-independent integrals which characterize the strength of the fields near a propagating crack-tip. This article is devoted primarily to a review of these developments.

2. Basic concepts of dynamic fracture mechanics

2.1. Crack-tip behavior

A logical classification of dynamic fracture problems may be stated as follows: (i) stationary cracks under dynamic loading, (ii) dynamically propagating cracks under quasi-static loading, and (iii) dynamically propagating cracks under dynamic loading. The elastodynamic fields near the tip of a stationary crack have the same form as those in elasto-statics [5,6]:

\[ \sigma_{ij}^{0} = \sum_{M} \frac{K_{M}}{\sqrt{2\pi r}} f_{ij}^{M}(\theta) \quad (M = I, II, III) \]  \hspace{1cm} (1)

\[ \mu^{0} = \sum_{M} \frac{K_{M}}{2\mu} \sqrt{\frac{r}{2\pi}} g_{M}(\theta) \quad (M = I, II, III), \]  \hspace{1cm} (2)

where \( M = I, II, III \) denotes the “mode” of fracture, \( \mu \) is the shear modulus, and \( (r, \theta) \) are polar coordinates centered at the crack-tip. The superscript \( "0" \) in \( (\quad )^{0} \) denotes the tensor components \( (\quad) \) with respect to the local coordinates \( x_{0} \) attached to the crack-tip (see Fig. 1). The dynamic stress intensity factors \( K_{M} \) for stationary cracks depend on time \( t \), crack length \( a \), loading rate \( \dot{\sigma} \), as well as the boundary conditions.

As a typical example, we consider a semi-infinite stationary crack subjected to a (normal) step wave of stress \( \sigma_{M} \) \( (\sigma_{1} = \sigma_{2}, \sigma_{II} = \sigma_{0}^{0}, \sigma_{III} = \sigma_{0}^{0}) \) [5,7,8]. The stress intensity factors vary with time as:

\[ K_{M} = D_{M}(\rho, \mu, \nu) \sigma_{M}\sqrt{t} \quad (M = I, II, III), \]  \hspace{1cm} (3)

where \( D_{M} \) depend only on material properties, with \( \rho \) being the mass density and \( \nu \) Poisson’s ratio. The time \( t \) is taken to be zero at the instant when the stress waves reach the crack-tip.

For dynamically propagating cracks under dynamic or quasi-static external loading, the crack velocity \( C = \dot{a} \) establishes the stress and displacement fields as follows [9–12]:

\[ \sigma_{ij}^{0} = \sum_{M} \frac{K_{M}(C)}{\sqrt{2\pi r}} f_{ij}^{M}(\theta, C) \]  \hspace{1cm} (4)
\[ u_i^0 = \sum_M \frac{K_M(C)}{2\mu} \sqrt{\frac{r}{2\pi}} \, g_M(\theta, C). \]  \hspace{1cm} (5)

The general solutions for the stress and displacement fields near the dynamically propagating crack-tip were obtained in [12] in a unified fashion for all three fracture modes. The stress intensity factors \( K_M \) in (4, 5) may be expressed as products of velocity factors \( k_M(C) \) and "static" factors \( K_M^\ast \), as noted by several researchers [13,14,15]:

\[ K_M(t, C) = k_M(C) \, K_M^\ast (t) \quad (M = 1, \, II, \, III). \]  \hspace{1cm} (6)

The "static" factors \( K_M^\ast \) depend on the current length of the crack, the applied load, the history of crack extension, but not on the instantaneous crack velocity. As discussed in [14] and [15], \( K_M^\ast \) are, in general, not equal to the static stress intensity factors, \( K_{M\infty} \), for a stationary crack of the same length as the moving crack.

For a crack propagating in a self-similar fashion at an angle \( \theta_0 \) measured from the global \( X_1 \) axis as shown in Fig. 1, the energy release rate \( G \) can be expressed as [16]:

\[ G = \left( \frac{C_k}{C} \right) G_k = G_1 \cos \theta_0 + G_2 \sin \theta_0 \]  \hspace{1cm} (7)

and

\[ G_k = \lim_{\epsilon \to 0} \int_{\Gamma} \left[ (W + K)n_k - T_i u_{i, k} \right] dS, \]  \hspace{1cm} (8)

where \( C_k \) are the components of crack velocity in the \( X_k \) system, and \( G_k \) is the energy-release rate due to a unit crack translation along the global coordinate \( X_k \); \( \Gamma_i \) is a contour arbitrarily close to the propagating crack-tip; \( W \) and \( K \) are the strain and kinetic energy densities, respectively; \( n_k \) the outward normal direction cosines; \( T_i \) the traction; and \( (\ )_k \) denotes \( \partial (\ )/\partial X_k \).

We now consider the change of components of \( G \) under a coordinate transformation. Suppose that the angle between the \( X_1 \) and \( x_1^0 \) axes is \( \theta_0 \) (see Fig. 1). The local Cartesian coordinates \( x_i^0 \) can be expressed by

\[ x_i^0 = a_{ij}(\theta_0)X_j; \quad a_{ij}(\theta_0) = \begin{bmatrix} \cos \theta_0 & \sin \theta_0 \\ -\sin \theta_0 & \cos \theta_0 \end{bmatrix}. \]  \hspace{1cm} (9)

\[ \{X_1, X_2, X_3\}: \text{ Global Coordinates} \]

\[ \{x_1^0, x_2^0, x_3^0\}: \text{ Crack-Tip Coordinates} \]

Figure 1. Nomenclature for mixed mode loading.
The same transformation holds for the components of energy release rate \( G_k \). Thus, we have

\[
G_k^0 = \alpha_{ik} G_i \quad \text{or} \quad G_k = \alpha_{ik} G_i^0.
\]  

(10)

Substituting the singular stress and corresponding displacement fields given by (4) and (5) into (8) with \( \theta_0 \) being zero, the relations between the energy release rate and the instantaneous stress intensity factors can be expressed as [12]:

\[
G_1^0 = \frac{1}{2\mu} \left\{ K_{II}^2 A_{I}(C) + K_{III}^2 A_{II}(C) + K_{I}^2 A_{III}(C) \right\}
\]  

(11a)

\[
G_0^0 = -\frac{K_{I}}{\mu} A_{IV}(C).
\]  

(11b)

The crack velocity functions, \( A_{I}(C) \cdots A_{IV}(C) \) are given in [12].

2.2. Dynamic fracture criteria

The criteria governing the motion of the crack (growth initiation, propagation, and arrest) can be expressed, when linear elastic conditions prevail, in terms of the stress intensity factor \( K \) and the experimentally determined critical values that are taken to be material specific.

In the conventional, static LEFM, the condition for the onset of crack growth is often expressed as

\[
K = K_c,
\]  

(12)

where \( K_c \) is called the static fracture toughness.

In elastodynamic fracture, one has to consider two counterparts of fracture toughness of materials [1]. First, for the onset of growth of a rapidly loaded stationary crack, we have

\[
K = K_d(\dot{a}) \quad (\text{or} \quad K = K_d(\dot{K}))
\]  

(13)

where \( K_d \) is the dynamic "initiation" toughness, \( K_d \) may also depend on the prevailing temperature. For the limiting case of \( \dot{a} = 0 \), \( K_d \) is identical with \( K_c \). Second, for dynamically propagating cracks under quasi-static as well as dynamic loading conditions, it has been suggested that

\[
K = K_D(\dot{a}),
\]  

(14)

where \( K_D \) is the dynamic propagation toughness for propagating cracks. There is some controversy as to whether the property \( K_D(\dot{a}) \) is truly geometry-independent. On the other hand, for viscoelastic materials \( K_D \) may also depend on the crack acceleration \( \dot{a} \) and/or other higher-order time derivatives of crack length. According to the criterion given by (14), crack arrest occurs when the stress intensity factor becomes smaller than or equal to a critical value. This can be expressed as

\[
K \leq K_D(0) \equiv K^{\text{dyn}},
\]  

(15)

where \( K^{\text{dyn}} \) denotes the dynamic crack arrest toughness. In the above, the superscript "dy" is used to distinguish a material property from the so-called arrest toughness \( K_{1a}(= K^{\text{ac}}) \), which is utilized in the current ASME Boiler and Pressure Vessel Code, Section XI.

3. Finite element modeling of dynamic crack propagation

In the finite element method, the solid continuum is represented by a mesh of a finite number of elements. To simulate the crack propagation, two different concepts of
computational modeling may be considered, namely, stationary (fixed) mesh procedures and moving (distorting) mesh procedures. For each procedure, several alternate schemes have been presented in literature.

3.1. Stationary mesh procedure

We review first the feature of the stationary mesh procedure. In early applications of the finite-element method to dynamic crack propagation [17,18], the crack-tip motion was modeled by discontinuous jumps. This was done by changing the location of the crack-tip from one node to the next ("node-shifting" procedure) along the crack axis, in time $\Delta t$ employed in the time integration scheme. To obtain accurate finite element solutions for wave propagation problems, one must use a small time increment $\Delta t$ – usually the time for dilatational stress waves to travel the distance between the two closest finite element nodes. Since the velocity of crack propagation is usually significantly lower than the wave velocity, the crack-tip will move during this time increment $\Delta t$ to a position somewhere in between the two successive nodal points. Also the sudden increase of crack length, and the release of constraint on the displacement, induce spurious high-frequency oscillations in the finite element solutions. To overcome these difficulties, several attempts have been made at releasing the nodes gradually, over a period of time. These are discussed below.

Keegstra et al. [19,20] have proposed a model, as shown in Fig. 2a, which allows for the presence of non-zero "holding-back forces" at more than one node behind the advancing crack-tip. Yagawa et al. [21] us a Lagrange multiplier approach to enforce the displacement boundary condition on the "uncracked" part of the surface of the crack-tip element [marked as segment (CD) in Fig. 2b] in situations when, in time $\Delta t$, the crack-tip actually propagates to a location in between two adjacent nodes. Note that the traction $T_2$, as shown in Fig. 2b, is used as a Lagrange multiplier on segment CD. Other "gradual" nodal release mechanisms were presented in the literature, with different rates of release of nodal

![Figure 2. "Gradual-nodal-release" techniques.](image-url)
forces. Suppose that the actual crack-tip is located at point C in between the finite element nodes B and D as shown in Fig. 2c, and that "b" and "d" are the lengths of segments (BC) and (BD), respectively. The "holding-back" force \( F \) at node B is gradually reduced to zero as the crack-tip reaches the node D. The various schemes for accomplishing this are summarized below:

\[
\text{Malluck and King [22]} \quad \frac{F}{F_0} = \left( 1 - \frac{b}{d} \right)^{1/2} \quad (16a)
\]

\[
\text{Rydholm, Fredriksson and Nilsson [23]} \quad \frac{F}{F_0} = \left( 1 - \frac{b}{d} \right)^{3/2} \quad (16b)
\]

\[
\text{Kobayashi, Mall, Urabe and Emery [24]} \quad \frac{F}{F_0} = \left( 1 - \frac{b}{d} \right) \quad (16c)
\]

where \( F_0 \) is the original reaction force when the crack-tip is located at the node B. It is important to recognize that the above procedure of moving the crack-tip from B to D involves several time-steps \( \Delta t \). In the use of eight-noded isoparametric elements, it was demonstrated in [25] that both the corner and midside nodes on the crack element should be released simultaneously. Studies using different nodal release mechanisms, as in (16a-c), were discussed in [26,27]. From the authors' experience, the use of "linear relaxation" (as in (16c), while employing eight-noded elements, yields smoother results than the other "nonlinear" relaxation techniques.

3.2. Moving mesh procedure

Next we present a synopsis of the moving (shifting) mesh procedures in which the mesh moves with the crack-tip.

Bazant, Glazik, and Achenbach [28] developed a non-singular finite element method for moving the entire mesh with the crack-tip. However, this procedure has two obvious limitations: (i) it is restricted to infinite bodies whose surfaces and/or bimaterial interfaces are parallel to the direction of crack propagation; and (ii) more importantly, it cannot be applied to bodies having finite dimensions in the direction of crack propagation. An early attempt toward a singular element for propagating cracks was made by Aberson, Anderson and King [29,30], who used the eigenfunctions appropriate for a stationary crack [31] as basis functions in the propagating singular element. The crack-tip moves within the singular element, between the nodes A and B as shown in Fig. 3(a). When the crack-tip reaches the node B, the mesh pattern is changed suddenly, as illustrated in the figure. However, it has been reported [32] that this procedure did not yield meaningful results. Another attempt at using Williams' eigenfunctions [31] was made by Patterson and Oldale [33,34]. The singular element [see Fig. 3(c)] has 13 nodes and is topologically equivalent to two assembled eight-noded isoparametric elements. The location of this singular element is suddenly changed by a distance equal to the size of the regular elements ahead of the crack-tip, when the crack-tip propagates a critical distance (about 80% of the length of the element) within the singular element. This model violates the displacement compatibility condition at the interfaces between the singular element and the surrounding regular elements.

In both the singular elements shown in Figs. 3(a) and (c), for stationary or quasi-statically propagating cracks, the stress intensity factor can be determined directly as one of the unknown coefficients of Williams' stationary-crack eigenfunctions. However, since the near-tip fields depend on crack velocity for dynamically propagating cracks, the coefficient of the singular eigenfunction for a stationary-crack does not correspond to the stress intensity factor for a propagating crack. Thus, care must be exercised in inferring the dynamic stress intensity factors. An empirical formula for this purpose was derived by the authors in [35].

Aoki, Kishimoto, and Sakata presented a singular element procedure [36,37] wherein
only the appropriate rigid motion and the eigenfunction corresponding to the singular stress field for a propagating crack were incorporated. The crack propagates in the singular element until it reaches the point B indicated in Fig. 3(b). Then the singular element suddenly changes its location as illustrated in the bottom half of Fig. 3(b). Since the size of the element in [36,37] is in general much larger than the region of validity of the singular solution alone, noticeable errors may occur in the determination of stress intensity factors. Omission of the constant stress field and higher order stress fields also limits the applicability of such an element in studying physical problems of interest such as crack-branching, etc.

In developing a special element, certain simple requirements have to be met in order to ensure the convergence of the solution. The displacement field in the element must: (i) include the appropriate rigid body motions, (ii) be able to represent a constant strain (stress) field, and (iii) be continuous at the interfaces between elements. In addition, to avoid kinematic modes in the special element, i.e. zero energy modes other than rigid modes, certain requirements on the number of parameters in the element basis functions have to be satisfied. Suppose a special element has \( N \) unknown parameters, i.e., \( N \) eigenfunctions including appropriate rigid body motions and \( M \) unconstrained nodal degrees of freedom. It can then be shown [35] that one has to satisfy an inequality, \( N \geq M \). The developments in [29,30,33,34,36] and [37] violate some or all of these criteria.

A moving singular element procedure which satisfies all the above criteria for convergence and which employs eigenfunctions [12,40] for a dynamically propagating crack, has recently been extensively developed [38,39]. Thus, in the singular element, it is assumed

\[
u_i^j(x_1^0, x_2^0, 0) = U_j(x_1^0, x_2^0, C) \alpha_j(t) \quad (i = 1, 2; \quad j = 0, 1, 2, \ldots, N - 1),
\]

where \((x_1^0, x_2^0)\) is the moving coordinate * system shown in Fig. 1. The functions \(U_j\) include the zero stresses and rigid body motions \((j = 0)\), the singular stresses and corresponding displacements \((j = 1)\), the constant stresses and linear displacements \((j = 2)\), and the higher-order terms \((j \geq 3)\). Thus, the coefficient \(\alpha_j\) is equivalent to the dynamic stress intensity factor. The displacement functions given by (17) satisfy the requirements (i) and (ii) mentioned above for the convergence of the solution. The compatibility of displacements, velocities, and accelerations at the boundary between the singular element

* The coordinate system \(x_1^0, x_2^0\) is assumed to be always centered at the crack-tip, and to move with the crack-tip velocity \(C\).
and regular elements (Type B elements in Fig. 4) can be satisfied through a least-square approach [38,39].

In the "moving mesh" procedure [38,39], the singular element translates in each time step for which crack growth occurs as illustrated in Fig. 4. Thus the crack-tip always remains at the center of the singular element throughout the analysis. The regular elements (the type B elements in Fig. 4) surrounding the moving singular element are continuously distorted. To simulate a large amount of crack propagation, the mesh pattern around the moving element is periodically readjusted as also shown in Fig. 4. An important feature which distinguishes the moving-mesh procedures from the nodal release techniques is that the size of the increment of crack growth is not affected by nodal spacing but can be made as small as desired. Therefore, the displacement boundary condition ahead of the actual crack-tip can be satisfied exactly in the moving mesh procedures. Another attractive feature of the moving singular element procedure [38,39] is that the dynamic stress intensity factor at each time step can be calculated directly as the coefficient of one of the eigenfunctions, e.g., \( \alpha_i \), for a mode I case.

To simplify the moving singular element procedure, a study was undertaken in [35] and [41] to address the effect of alternatively modeling the moving region A in Fig. 4 with either the "quarter-point" singular elements (model A') or the regular (non-singular) isoparametric elements (model A''). In the moving (non-singular) isoparametric element procedure, one has to calculate the stress intensity factors indirectly. A most accurate

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**Figure 4. Moving element procedures.**
estimate can be obtained by using certain path-independent integrals as discussed later in Section 5. While simplified procedures may lead to acceptable estimates of “overall parameters” like the stress-intensity factors, the use of more refined approaches, such as the moving singular element approach detailed above, is still an indispensable tool for purposes of basic research into fracture phenomena such as crack branching, etc.

4. Finite element simulation of dynamic fracture

Since a computational model can only respond in a manner as prescribed by input data, numerical simulations of dynamic fracture in materials require experimentally determined information. This approach may be labeled as the hybrid experimental-numerical method [2,3]. Computational simulations of dynamic crack propagation and crack arrest, for a specified initial flaw size, specimen geometry, and applied load, can be conducted in either of two different ways [4]. One of these is the so-called “generation phase simulation” in which the variation of stress intensity factor can be determined, using an experimentally measured crack-propagation history (a vs. t and/or C vs. t) as the input data into the computational model. From this calculation, one determines the dynamic fracture toughnesses through (13) and (14). Once the material properties $K_d$ and $K_p$, as defined earlier, are determined numerically or experimentally, they may be used in the second type of computational simulation, a “prediction phase simulation”. In this calculation, the crack propagation history (a vs. t) can be determined by specifying the initial conditions and material fracture toughness data as inputs to the computational model. The prediction phase simulation is also sometimes called “application”, “propagation” or “inverse” simulation.

In the following, we consider the finite element simulation of dynamic crack propagation and arrest in a Wedge-loaded Rectangular Double Cantilever Beam (RDCB) specimen (see Fig. 5). Due to symmetry, only the upper half of the specimen is modeled by finite elements. The moving singular element is shown hatched at the beginning of crack propagation. In the experiment conducted by Kalthoff et al. [42], several test specimens were studied, wherein cracks were initiated from blunt notches, each with an apparent initiation stress intensity factor $K_{IQ}$ larger than the fracture toughness $K_{IC}$. In their

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Figure 5. Finite element mesh at $t = 0$, of DCB specimen.
report [42] the RDCB specimens with $K_{10}$ values of 2.32 and 1.33 Mpa·m$^{1/2}$ were identified, respectively, as specimens with numbers 4 and 17. For convenience, the same identifications are used here. The specimens were made of a photoelastic material with the properties of $E = 3.38$ Gpa, $\nu = 0.33$.

First we present the results of generation phase simulation for No. 4 Specimen[43]. The experimentally measured crack length and crack velocity history curves, i.e., $a = a(t)$ and $C = C(t)$ as given in Fig. 6, were used as the input data for the generation phase simulation. The variation of the computed $K_1$ values agrees very well with the experimentally measured (using an optical method) data.

In the prediction phase simulation, the crack velocity in each time step will be determined from the given fracture toughness versus crack velocity relation, i.e., (14). Before proceeding with the prediction phase simulation from time $t_1$ to $t_2 = t_1 + \Delta t$, one has to know the position of crack-tip at time $t_2$. To this end, the crack velocity at time $t_1 + \Delta t/2$ is predicted by using the following Taylor series expansion:

$$K_{10} \left(t_1 + \frac{\Delta t}{2}, C + \Delta C \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \left( \frac{\Delta t}{2} \right) \frac{\partial}{\partial C} + \Delta C \frac{\partial}{\partial C} \right\}^n K_1(t_1, C_1),$$

(18)

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**Figure 6. Results of generation phase simulation (No. 4).**
Figure 7. Fracture toughness versus crack velocity relation measured experimentally. [42].

Figure 8. Results of prediction phase simulation (No. 4).
where \( K_{ip} \) is the predicted \( K_1 \) at \( t_i + \Delta t/2 \) while \( K_1(t_1, C_1) \) is known. One may now rewrite (18) as:

\[
K_{ip} = K_1(t_1, C_1) + \left( \frac{\Delta t}{2} \right) \dot{K}_1(t_1, C_1) + \frac{(\Delta t)^2}{8} \ddot{K}_1(t_1, C_1) + R, \tag{19}
\]

where \( R \) is the residue; alternatively, it may be considered as a corrector of the prediction. The form of \( R \) depends on the type of variational principle used in the analysis, as discussed in detail in [43] and [44]. In most analyses based on the principle of virtual work, a zero corrector \( (R = 0) \) gives enough accuracy in the numerically calculated \( K_1 \) values.

In the moving singular element procedure [38,39], one has \( K_1(t_i) = \alpha_i(t_i); \dot{K}_1(t_i) = \dot{\alpha}_i(t_i); \) and \( \ddot{K}_1(t_i) = \ddot{\alpha}_i(t_i) \) in which \( \alpha, \dot{\alpha}, \) and \( \ddot{\alpha} \) are directly calculated as the parameters for the singular element. For other non-singular element procedures, \( \dot{K}_1 \) and \( \ddot{K}_1 \) can be determined using simple finite difference schemes, for example, \( \dot{K}_1 \equiv \frac{K_1(t_i) - K_1(t_{i-1})}{\Delta t} \), etc.

The experimentally measured \( K_{ip} \) vs. \( C \) curve for the RDCB specimen [42] is shown in Fig. 7. Using a \( K_{ip} \) value given by (19) in this experimental data, the crack velocity at \( (t_i + \Delta t/2) \) was determined in the prediction phase simulation [43]. The simulation results for No. 4 specimen with \( K_{iq} = 2.32 \text{ Mpa-m}^{1/2} \) are shown in Fig. 8. The variation of stress intensity factors and the computed crack propagation and crack velocity histories are compared with those determined experimentally by Kalthoff et al. [42]. In the prediction phase simulation, the crack was assumed to have arrested when \( K_{ip} \leq K_{ip}^a = K_{IP}(0) \), where \( K_{ip}^a \) is the dynamic arrest fracture toughness of the material.

5. Use of path-independent integrals

It is well known that in mode I elasto-static fracture the \( J \) integral can be used for an efficient and simple computation of the relevant crack-tip parameters. This is so because, under appropriate assumptions of material homogeneity, etc., the strength of the crack tip fields (\( K_1 \), for instance) is governed also by an integral evaluated over a path that is far removed from the crack tip. Since the computed stress and displacement data in regions far removed from the crack tip is, in general, relatively insensitive to the details of the modeling of the crack tip region, this far-field integral can be evaluated with reasonable accuracy using relatively coarse finite element models of the structure. In this section, we discuss the computational details of the analysis of dynamic crack propagation using “path-independent” integrals which, however, also involve domain-integrals, due to the presence of material inertia.

5.1. Path independent-integrals for elastodynamically propagating cracks

Through an elaborate discussion of conservation laws, a path-independent integral \( J \) for an elastodynamically propagating crack, which has the meaning of rate of change of Lagrangean per unit crack extension, has been derived in [16]. Later, through a simple modification of an integral in [16], a path-independent integral \( J^p \), which is equivalent to the rate of energy release \( G \) in elastodynamic crack propagation, has been derived [12]. The integrals in [12] and [16] were defined along spatially fixed paths. On the other hand, using paths which move rigidly with the propagating crack tip, Bui [45] has derived a path-independent integral which is also equivalent to the energy release rate. Also Kishimoto, Aoki, and Sakata [46,47] have derived a path-independent integral \( J \) for spatially fixed paths, which is equivalent to the energy release rate only for a stationary crack in a solid under dynamic motion.
We assume that either the crack is not kinking or is not abruptly changing the direction and velocity of propagation, i.e., the crack propagation is more or less self-similar. This condition is more precisely stated as follows. Let the vector of crack velocity at time $t_1$ be $C(t_1)$ and at $t_2 = t_1 + dt$ be $C(t_2)$. The vector $C(t_2)$ may be decomposed into components $C_1(t_2)$ and $C_2(t_2)$ along and normal to vector $C(t_1)$, respectively. Thus, we consider situations where $C(t_1) = |C(t_1)|$ and $C_1(t_2) \gg C_2(t_2)$.

In this case, the energy release rate $G$ per unit of crack-extension is given by (7) and (8). It is noted that (8) is expressed in terms of the global cartesian coordinate system $X$, shown in Fig. 9.

The components of the path independent integral which are equivalent to the components of energy release rate were derived in [12] *:

$$J' = \lim_{\epsilon \to 0} \int_{\Gamma_1} \left[ (W + K) n_k - T_i u_{i,k} \right] dS$$

$$= \lim_{\epsilon \to 0} \int_{\Gamma_1} [ (W + K) n_k - T_i u_{i,k} ] dS + \int_{V_1 - V} [ \rho \ddot{u}, u_{i,k} - \rho \ddot{u}, u_{i,k} ] dV. \quad (20)$$

The symbols $\Gamma, \Gamma_1, V_1, V_1'$, and $\Gamma_c$ are explained in Fig. 9. Note that the far-field contour $\Gamma$ is fixed in space. The relations of the various integrals to an energy-release rate are given by

$$G = \frac{C_1}{C} J' = \frac{C_1}{C} \left[ J_k + \int_{\Gamma_{1_{c}}} Kn_{k} dS \right] = \frac{C_1}{C} \left[ J_k + 2 \int_{\Gamma_{1_{c}}} Kn_{k} dS \right]. \quad (21)$$

For a crack that does not propagate under dynamic loading, all the integrals, $J_k, J_k', J_k''$ are directly related, as seen from (21), to the energy-release rate at incipient crack propagation. This is so because, for a non-propagating crack, the kinetic energy is

* These are components in $X_1$ system (see Fig. 1).
non-singular and hence $\int_K K n_d dS \rightarrow 0$ in the limit. However, during crack-propagation, $K$ is singular near the crack-tip, and $\int_K K n_d dS$ remains finite in the limit. Using the asymptotic solutions for elastodynamic crack propagation, given in (4) and (5), the relations between the various path-independent integrals and the instantaneous stress-intensity factors were derived in [12]. The agreement of (20) with that in [48] for the steady-state case has been shown in [49]. *

5.2. Numerical studies of the use of path-independent integrals

A numerical study [50] using the moving singular element procedure indicates that the path-independent integrals $J'$, $J$, and $J$, give distinctly different numerical results, as may be expected from a theoretical point of view (see (21)). Although the moving singular element procedure gives highly accurate solutions, especially for the detailed stress distribution near the propagating crack-tip, this procedure may be difficult to apply by users of general-purpose finite element codes, because of its sophistication. From this point of view, a study was undertaken [51] to use the path-independent integrals along with less-sophisticated finite elements such as isoparametric elements indicated in Fig. 4. The moving-mesh procedure together with regular isoparametric elements (model $A''$) near the crack-tip gave results in close agreement with those obtained by using the singular element (model $A$), while the results using quarter-point singular elements (model $A'$) were considerably different than those from using the singular element for the case of $C = 0.6 C_0$. Thus, the model $A'$ can be applied only to slow speed crack propagation, i.e. $0 \leq C \leq 0.3 C_0$.

In the following an example problem of symmetrical dynamic propagation from the tips of a central crack in a square plate is presented. This problem may be considered to be similar to that treated analytically in [52], except that [52] treated an infinite body with

* For a comprehensive survey of recent work on path-independent integrals in inelastic and dynamic fracture see [55].
Numerical studies in dynamic fracture mechanics

Table 1. Comparison of path independent integrals and their path independency ($C = 0.6C_0$)

<table>
<thead>
<tr>
<th>Time</th>
<th>Integral [N/m]</th>
<th>Isoparametric elements (model $A''$)</th>
<th>Singular element (model $A$)</th>
<th>Difference (model $A''$-model $A$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Path 1</td>
<td>Path 2</td>
<td>Path 3</td>
</tr>
<tr>
<td>$t = 20\Delta t$</td>
<td>$J'_1$</td>
<td>19.35</td>
<td>20.25</td>
<td>20.07</td>
</tr>
<tr>
<td>($a/w = 0.3$)</td>
<td>$\bar{J}_1$</td>
<td>19.35</td>
<td>20.25</td>
<td>20.07</td>
</tr>
<tr>
<td></td>
<td>$J_1$</td>
<td>19.35</td>
<td>20.25</td>
<td>20.07</td>
</tr>
<tr>
<td>$t = 40\Delta t$</td>
<td>$J'_1$</td>
<td>23.21</td>
<td>23.12</td>
<td>22.95</td>
</tr>
<tr>
<td>($a/w = 0.4$)</td>
<td>$\bar{J}_1$</td>
<td>23.21</td>
<td>23.12</td>
<td>22.95</td>
</tr>
<tr>
<td></td>
<td>$J_1$</td>
<td>23.21</td>
<td>23.12</td>
<td>22.95</td>
</tr>
<tr>
<td>$t = 60\Delta t$</td>
<td>$J'_1$</td>
<td>28.05</td>
<td>27.99</td>
<td>27.98</td>
</tr>
<tr>
<td>($a/w = 0.5$)</td>
<td>$\bar{J}_1$</td>
<td>28.05</td>
<td>27.99</td>
<td>27.98</td>
</tr>
<tr>
<td></td>
<td>$J_1$</td>
<td>28.05</td>
<td>27.99</td>
<td>27.98</td>
</tr>
</tbody>
</table>

A crack of zero initial length. Figure 10 shows the finite element breakdown for the initial configuration $t = 0$. A time-independent tensile stress acts at the edge of plate parallel to the crack axis. Three contour paths are considered as shown in Fig. 10.

The results obtained from the models $A''$ and $A$ are compared in Table 1 for different instants of time and different far-field paths. Reference to Table 1 shows that, while the model $A$ (eigenfunction singular-element) gives distinctly different values for the $J'$, $\bar{J}$, and $J$ integrals (see Fig. 7), the model $A''$ (non-singular isoparametric elements) gives exactly the same values for the three different integrals throughout the crack propagation history. This is due to the fact that, since the model $A''$ does not model the singularity in the kinetic energy density $K$, the limit of $\int K_{ii} dS$ in (21) tends to zero. For the model $A$, however, this singularity was correctly incorporated; thus $\int K_{ii} dS$ remains finite. Table 1 also indicates the path independency of the integral values obtained by the moving isoparametric elements. Noting that the relations between each of the three integrals $J'$, $\bar{J}$, and $J$ on the one hand, and the $K$-factors on the other, are distinctly different from one another [12], it may be seen from Table 1 that the use of the $J'$ integral (20) and its relation to $K$-factors as given in (11), in conjunction with the use of ordinary (non-singular) isoparametric elements near the propagating crack tip constitutes a computational procedure of sufficient accuracy to analyze fast elastodynamic crack propagation. This assertion was confirmed also in the case of mixed-mode dynamic crack propagation in [53], wherein the path-independent integrals $J'_i$ were used in conjunction with: (i) a moving regular-isoparametric finite-element mesh and (ii) a mode-decomposition method [53,54] to evaluate the mixed mode $K$-factors.

Acknowledgements

The results presented herein were obtained during the course of investigations supported by the US Office of Naval Research under contract N00014-78-C-0636. The authors gratefully acknowledge this support as well as the encouragement received from Drs Y. Rajapakse and A. Kushnir. It is a pleasure to thank Ms J. Webb for her tireless efforts in preparing this typescript.

References

Résumé

Le mémoire fournit des synthèses des études récentes relatives à la modélisation numérique de la propagation de fissures dynamiques. On examine, à la fois, le maillage "stationnaire" et le maillage "mobile" utilisés dans les procédures d'éléments finis. On présente des procédures simples utilisant un maillage "mobile" d'éléments conventionnels isoparamétriques utilisé avec certaines intégrales indépendantes du parcours, en vue d'évaluer les facteurs d'intensité de contrainte dans le cas d'une fissure en cours de propagation dynamique.