POST-BUCKLING ANALYSIS OF SHALLOW SHELLS BY
THE FIELD-BOUNDARY-ELEMENT METHOD

J. D. ZHANG* AND S. N. ATLURI†

Center for the Advancement of Computational Mechanics, School of Civil Engineering, Georgia Institute of Technology,
Atlanta, Ga, 30332, U.S.A.

SUMMARY

The non-linear field-boundary-element technique is applied to the analysis of snap-through phenomena in thin shallow shells. The equilibrium path is traced by using the arc-length method and the solution strategy is discussed in detail. The results show that, as compared to the approaches based on the popular symmetric-variational Galerkin finite element formulation, the current approach based on an unsymmetric variational Petrov-Galerkin field-boundary-element formulation gives a faster convergence while using fewer degrees of freedom. The illustrative numerical examples deal with post-buckling responses of several shallow shells with different geometries.

1. INTRODUCTION

The boundary-element method is increasingly considered as an efficient technique in solving the boundary-value/initial-value problems of mechanics. However, it has not been fully developed in dealing with the analysis of the shell problems, especially the non-linear static and dynamic problems such as of snap-through and post-buckling. As is well known, due to the curvature of the shell, the in-plane displacements and the transverse displacement in a shell are inherently coupled in the kinematics of deformation as well as in the momentum balance relations. Since it is difficult to establish a fundamental solution in infinite space for the entire differential operator in the shell equations, the simple boundary integral representations for displacements, with the mathematical forms as in the case of linear isotropic elasticity, cannot be obtained even for the linear theory of shells. Thus, even in the linear case, the integral representations for displacements involve field (domain) integrals involving displacements, their derivatives and accelerations. In the non-linear case, in general, all the non-linear terms also enter the field integrals. The non-linear case of post-buckling or snap-through analysis is an important challenge to the various techniques of computational mechanics. To analyse the response beyond the limit points in the equilibrium path, the arc-length method proposed by Riks, Crisfield is well known as an efficient technique. Some interesting results have been obtained based on the popular finite element formulation which requires usually large number of degrees of freedom to maintain the accuracy and the rate of convergence.

By considering the fundamental solutions, in the infinite space, to the highest-order linear differential operators of the shell equations as test functions, an unsymmetric variational

*Ph.D. Candidate
†Regent’s Professor and Director

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formulation can be established which gives the integral representations for displacements. In contrast to the so-called boundary-element method based on the boundary-integral equation, however, such integral representations inevitably include the field integrals of the trial solutions and their derivatives. A Petrov-Galerkin discretization of such integral equations with test function spaces different from those of the trial functions would lead to the presently developed field-boundary-element method. It will be shown later in this paper that, because the test functions are infinitely differentiable, the trial functions in the present Petrov-Galerkin approach need not even be continuous across the field-element boundaries. In contrast, in the symmetric-variational Galerkin finite element method, both the test as well as trial functions need to be $C^1$ continuous in the Kirchhoff-Love theory of shells while selective/reduced integration/hour-glass control methods may be needed for $C^0$ elements in the Reissner-Mindlin theories.

In the following, we present the results of snap-through and post-buckling analyses of shallow shells, based on the field-boundary-element formulation in conjunction with the arc-length method. The degrees of freedom in the field-boundary-element formulation can be reduced without losing accuracy, so that the full Newton-Raphson technique may be employed to solve the non-linear equations.

In Sections 2 and 3, the general theory of shallow shells and the general formulation of the field-boundary-element are presented; Section 4 discusses the arc-length method and its implementation based on the field-boundary-element formulation; Section 5 gives several numerical examples; and Section 6 some concluding remarks.

2. NON-LINEAR FIELD-BOUNDARY-ELEMENT FORMULATION

Consider a shallow shell of an isotropic elastic material with the mid-surface described by $z = z(x_i, x_j)$. The von Karman equations of large deformation for the shell may be written for the in-plane equilibrium (ignoring inertia effects):

$$N_{\alpha\beta} + h_3 = 0 \quad (\alpha, \beta = 1, 2) \quad \text{(1a)}$$

and for the out-of-plane equilibrium

$$Dy^2w + \frac{N_{\alpha\beta}}{R_{\alpha\beta}} - h_3 = f_3 + (N_{\gamma\gamma}w_{,\gamma})_{,\gamma} \quad \text{(1b)}$$

where $N_{\alpha\beta}$ are membrane forces; $h_3 = \partial^2 (w_{,\alpha\beta})_{,\gamma} / \partial x_{\gamma}$; $w$ is the transverse deflection of the mid-surface of the shell; $h_i$ ($i = 1, 2, 3$) are body forces; $f_3$ is the load normal to the shell mid-surface; and $D = Et^2/(12(1-v^2))$ where $t$ is the thickness and $E$ and $v$ are the elastic constants; $\nabla^2$ is the biharmonic operator in the variables $x_i$; and

$$R_{\alpha\beta} = 1/z_{\alpha\beta} \quad \text{(2)}$$

are the radii of curvature of the undeformed shell. Along the boundary $\Gamma$, the boundary conditions for the in-plane variables are

$$u_i = \bar{u}_i \text{ at } \Gamma, \quad N_{\alpha\beta}n_{,\beta} = \bar{F}_\alpha \text{ at } \Gamma, \quad \Gamma = \Gamma_{\alpha} \cup \Gamma_{\sigma} \quad \text{(3)}$$

where $n_\alpha$ are the direction cosines of the unit outward normal to $\Gamma$ in the base plane of the shell. The out-of-plane boundary conditions are

$$w = \bar{w} \quad \text{or} \quad \bar{F}_n = \Psi_n \quad \text{or} \quad M_n = \bar{M}_n \quad \text{(4)}$$

where $\Psi_n = \partial w / \partial n$ is the rotation around the tangent to $\Gamma$.
\[ M_a = M_{ij} \nu_i \nu_j \] the normal bending moments;

\[ V_a = -D \frac{\partial}{\partial \nu} (V^2 w) + \frac{\partial}{\partial \nu} M_a + N_a \frac{\partial}{\partial \nu} + N_{a,} \frac{\partial}{\partial \nu} \] the reduced Kirchhoff shear force.

The non-linear in-plane strain–displacement relations are

\[ \varepsilon_{gb} = \frac{1}{2} \left[ u_{,g} + u_{,b} + \frac{2w}{R_{gb}} + w_{,g} + w_{,b} \right] \] (5)

where \( u \) are the in-plane displacements at the shell mid-surface. The in-plane stress–resultant strain relations are

\[ N_{11} = C(\varepsilon_{11} + \nu \varepsilon_{11}) \quad N_{22} = C(\varepsilon_{22} + \nu \varepsilon_{11}) \]
\[ N_{12} = N_{21} = C(1 - \nu \varepsilon_{12}) \] (6)

where \( C = E t / (1 - \nu^2) \). The moment–curvature relations are

\[ M_{11} = -D(w_{,11} + \nu w_{,22}) \quad M_{22} = -D(w_{,22} + \nu w_{,11}) \]
\[ M_{12} = M_{21} = -D(1 - \nu)w_{,12} \] (7)

In order to derive the integral representations for displacements, we shall consider a general weighted-residual formulation. Let \( u \) and \( w \) be the assumed trial solutions, and \( u^* \) and \( w^* \) be the corresponding test functions. The combined weak forms of the equilibrium equations and boundary conditions for the in-plane (equations (1a) and (3)) and out-of-plane (equations (1b) and (4)) deformations, respectively, may be written as

\[ \int_{\Omega} \left( N_{gb,} + b_{gb} \right) u^*_b d\Omega = \int_{\Gamma_r} (P - \mathcal{P}) u^*_b d\Gamma + \int_{\Gamma_n} (\mathcal{u} - u^*_b) P^*_b (u^*_b) d\Gamma \] (8)

and

\[ \int_{\Omega} \left[ D \nabla^4 w + \frac{N_{gb}}{R_{gb}} - b_3 - (N_{gb} w_{gb})_a \right] w^* d\Omega = \int_{\Gamma_r} (\mathcal{P} - V_\nu) w^* d\Gamma + \int_{\Gamma_n} (\mathcal{M}_n - M_n) \psi^* d\Gamma + \int_{\Gamma_n} (\mathcal{\psi} - \psi^*_n) M^*_n d\Gamma \\ + \int_{\Gamma_n} (\mathcal{w} - \mathcal{w}) P^*_n d\Gamma \] (9)

To make a specific choice for the test functions that results in convenient integral representations for the shell-displacements \( u^*_b \) and \( w^* \), we rewrite the in-plane equilibrium equations in a slightly different form, as follows. From the relations between \( (N_{gb}) \) and \( (u_{gb}) \) as given in equations (5) and (6), we may write

\[ N_{gb} = N_{gb}^* + C \kappa_{gb} w + N_{gb}^{(1)} \] (10)

where

\[ N_{11} = C(u_{11} + \nu u_{22}) \quad N_{22} = C(u_{22} + \nu u_{11}) \]
\[ N_{12} = 1/2 C(1 - \nu)(u_{12} + u_{21}) \]

or

\[ N_{gb} = C_{gb} u_{gb} \quad \kappa_{11} = \frac{1}{R_{11}} + \frac{\nu}{R_{22}} \quad \kappa_{22} = \frac{1}{R_{22}} + \frac{\nu}{R_{11}} \quad \kappa_{12} = \frac{1 - \nu}{R_{12}} \] (11)
and the non-linear parts

\[ N_{11}^{(o)} = \frac{C}{2} (w_{1})^2 + v(w_{2})^2; \quad N_{22}^{(o)} = \frac{C}{2} (w_{1})^2 + v(w_{2})^2 \]

\[ N_{12}^{(o)} = \frac{C}{2} (1 - v)w_{1} w_{2} \]

(13)

Use of (10) in (8) results in

\[ \int_{\Omega} \left[ N_{s,p}u_{p}^{*} + C(\kappa_{s,p}w_{p} + N_{s,p}^{(o)} + b_{p} - \rho i_{p})u_{p}^{*} \right] \, d\Omega \]

\[ = \int_{\Gamma_{u}} (P_{s} - \tilde{P}_{s})u_{p}^{*} \, d\Gamma + \int_{\Gamma_{u}} (\tilde{u}_{s} - u_{s}) P_{s}^{*} (u_{p}^{*}) \, d\Gamma \]  

(14a)

Use of the divergence theorem in equation (14a) results in

\[ \int_{\Gamma_{u}} N_{s,p}u_{p}^{*} \, d\Gamma - \int_{\Omega} N_{s,p}u_{p}^{*} \, d\Omega + \int_{\Omega} C(\kappa_{s,p}w_{p} + N_{s,p}^{(o)} + b_{p} - \rho i_{p})u_{p}^{*} \, d\Omega + \int_{\Omega} N_{s,p}^{(o)}u_{p}^{*} \, d\Omega + \int_{\Omega} (b_{p} - \rho i_{p})u_{p}^{*} \, d\Omega \]

\[ = \int_{\Gamma_{u}} (P_{s} - \tilde{P}_{s})u_{p}^{*} \, d\Gamma + \int_{\Gamma_{u}} (\tilde{u}_{s} - u_{s}) P_{s}^{*} (u_{p}^{*}) \, d\Gamma \]  

(14b)

Since the material is linear elastic and isotropic, we have

\[ N_{s,p}u_{p}^{*} = C_{s,p}u_{s}^{*} + \kappa_{s,p}w_{p} = N_{p,s}^{(o)}(u_{p}^{*})u_{s}^{*} \]

(15)

where the definition of $N_{p,s}^{(o)}$ is apparent. Now note that

\[ P_{s} = N_{s,p}u_{p}^{*} \]

or

\[ P_{s} = N_{s,p}^{(o)} + C\kappa_{s,p}w_{p} + N_{s,p}^{(o)} \]

(16a)

(16b)

Using (15, 16b) in equation (14b) and applying the divergence theorem, it is easy to obtain

\[ \int_{\Omega} [N_{s,p}^{(o)}(u_{p}^{*})]_{x} u_{s} \, d\Omega - \int_{\Gamma} (b_{s} - \rho i_{s})u_{s}^{*} \, d\Gamma + \int_{\Gamma} \tilde{P}_{s} u_{s}^{*} \, d\Gamma - \int_{\Gamma} P_{s} u_{s}^{*} \, d\Gamma \]

\[ - \int_{\Omega} C\kappa_{s,p}w_{p} u_{s}^{*} \, d\Omega - \int_{\Omega} N_{s,p}^{(o)} u_{s}^{*} \, d\Omega = 0 \]

(17a)

where

\[ \tilde{P}_{s} = \tilde{P}_{s} \text{ at } \Gamma_{u}, \quad \tilde{P}_{s} = \tilde{P}_{s} \text{ at } \Gamma_{s} \]

(17b)

and

\[ \tilde{u}_{s} = u_{s} \text{ at } \Gamma_{u}, \quad \tilde{u}_{s} = u_{s} \text{ at } \Gamma_{s} \]

(17c)

Now we choose $u_{p}^{*}$ to be the fundamental solution in infinite space of the equation

\[ [N_{s,p}^{(o)}(u_{p}^{*})]_{x} + \delta(x_{s} - \xi_{s})\delta_{s,p} = 0 \]

(18)

where $\delta(x_{s} - \xi_{s})$ is the Dirac delta function at $x_{s} = \xi_{s}$; $\delta_{s,p}$ is the Kronecker delta; and $c_{s}$ denotes that the direction of the application of the point load is along the $x_{s}$ direction. The fundamental solution of (18) will be denoted as $u_{p,0}^{*}$, where $u_{p,0}^{*}$ is the displacement along the $x_{s}$ direction in a plane infinite body at any point $x_{s}$ due to a unit load along the $x_{s}$ direction, applied at the location $x_{s} = \xi_{s}$. Likewise, $P_{s,0}^{*}(x_{s}, \xi_{s})$ will be considered to be the traction along the $x_{s}$ direction on an oriented surface at $x_{s}$, with a unit normal $n_{s}$, due to a unit load along $x_{s}$ at the location $\xi_{s}$. These
solutions are well known and may be written as

\[ u_{\theta \theta \theta}(x_\mu, \xi_\mu) = \frac{1}{8\pi G} \left[ (\nu - 3) \ln \rho + (1 + \nu) \frac{\partial \rho}{\partial \xi_\theta} \frac{\partial \rho}{\partial \xi_\theta} + (1 - \nu) \frac{\partial \rho}{\partial x_\xi} \frac{\partial \rho}{\partial x_\xi} \right] \]  

(19a)

and

\[ P_{\theta \theta \theta}^*(x_\mu, \xi_\mu) = -\frac{t}{4\pi \rho} \left[ \frac{\partial \rho}{\partial x_\xi} \left( (1 - \nu) \delta_{\xi_\theta} + 2(1 + \nu) \frac{\partial \rho}{\partial x_\xi} \right) \right. \]

\[ \left. - (1 - \nu) \left( \frac{\partial \rho}{\partial x_\xi} - \frac{\partial \rho}{\partial x_\xi} \right) \right] \]  

(19b)

where \( \rho = |x_\mu - \xi_\mu| \) is the radius vector \( x_\mu \) to \( \xi_\mu \); and

\[ G = E\ell/[2(1 + \nu)] \]

Due to the property of integrals involving Dirac functions, we have

\[ \int_\Omega \left( N_{\theta \theta \theta}^* \right) u_{\theta \theta \theta} d\Omega = -\int_\Gamma \delta(x_\mu - \xi_\mu) \frac{\partial \rho}{\partial \xi_\theta} u_\theta \left( x_\mu \right) d\Gamma = -u_\theta(\xi_\mu) \]  

(20)

Using (19) and (20) in equation (17a), we have

\[ \gamma u_\theta(\xi_\mu) = \int_\Omega \left( \sigma_{\theta \theta}(x_\mu) - \mu \sigma_{\theta \theta}(x_\mu) \right) u_{\theta \theta \theta}(x_\mu, \xi_\mu) d\Omega + \int_\Gamma \tilde{P}_{\theta}(x_\mu) \psi_{\theta \theta \theta}(x_\mu, \xi_\mu) d\Gamma \]

\[ - \int_\Gamma \tilde{M}_{\theta}(x_\mu) \psi_{\theta \theta \theta}(x_\mu, \xi_\mu) d\Gamma - \int_\Omega C_{\theta \theta \theta} w(x_\mu) u_{\theta \theta \theta}(x_\mu, \xi_\mu) d\Omega \]

\[ - \int_\Omega N_{\theta \theta \theta}(x_\mu) u_{\theta \theta \theta}(x_\mu, \xi_\mu) d\Omega \]  

(21)

It can be shown that, while the coefficient \( \gamma \) in the left-hand side of (21) is unity when \( \xi_\mu \) is in the interior of \( \Omega \), the value of \( \gamma \) is (0.5) when \( \xi_\mu \) falls on the "smooth" boundary \( \Gamma \). Equation (21) is the sought-after integral equation for \( u_\theta \) in a shallow shell.

We now choose the test function \( w^*(x_\mu) \) to be the "fundamental solution" in an infinite plate corresponding to a unit point load at the location \( \xi_\mu \) in the linear theory of Kirchhoff plates. Thus, \( w \) corresponds to the solution of the linear equation

\[ D \nabla^2 w^* = \delta(x_\mu - \xi_\mu) \]  

(22)

in an infinite domain in the base-plane of the shallow shell. It is well known that the solution for \( w^* \) is given by

\[ w^*(x_\mu, \xi_\mu) = \frac{1}{8\pi \rho^2} \ln \rho \]  

(23)

where \( \rho = |x_\mu - \xi_\mu| \).

Using equations (23) and (10) in equation (9) and employing repeated integrations by parts in the resulting equation, one easily obtains the integral equation:

\[ \gamma u_\theta D w(\xi_\mu) = \int_\Gamma \tilde{F}_{\theta}(x_\mu) w^*(x_\mu, \xi_\mu) d\Gamma - \int_\Gamma \tilde{M}_{\theta}(x_\mu) \psi_{\theta \theta \theta}(x_\mu, \xi_\mu) d\Gamma \]

\[ + \int_\Gamma \tilde{P}_{\theta}(x_\mu) \psi_{\theta \theta \theta}(x_\mu, \xi_\mu) d\Gamma - \int_\Gamma \tilde{w}(x_\mu) \psi_{\theta \theta \theta}(x_\mu, \xi_\mu) d\Gamma \]
\[
\begin{align*}
&- \int_\Omega \left[ \frac{N_{\gamma\beta}}{R_{\gamma\beta}} + C_{\gamma,\beta} w \right] (x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Omega \\
&- \int_\Omega \left[ \frac{N_{\gamma\beta}^{(n)}}{R_{\gamma\beta}} - b_3 + \rho \bar{w} - f_3 - (N_{\gamma\beta} w_{\beta} \cdot n_{\beta}) (x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Omega \\
&+ \sum_j \left[ \langle M_j \rangle w^* - \langle M_j^* \rangle w \right]
\end{align*}
\]

where

\[
\begin{align*}
V_n &= - D \frac{\partial}{\partial n} (\nabla \cdot w) + \frac{\partial}{\partial s} M_s + N_{\gamma, \beta} \frac{\partial w}{\partial n} + N_{\gamma, \beta} \frac{\partial w}{\partial s} \\
M_s &= M_{11} n_1^2 + 2M_{12} n_1 n_2 + M_{22} n_2^2 \\
M_{ij} &= (M_{22} - M_{11}) n_1 n_2 + M_{12} (n_1^2 - n_2^2)
\end{align*}
\]

\(\gamma_w = 1\) for \(\xi_{\mu} \in \Omega\); \(\gamma_w = \frac{1}{2}\) for \(\xi_{\mu} \in \Gamma\) (smooth)

In equation (24) the terms with the superposed symbol 'A' should be taken to imply the respective prescribed values, if any, at \(\Gamma\); otherwise, they are to be treated as the unknown solution variables. Also, the symbol \(\langle \cdot \rangle \) denotes the jump in the quantity \((\cdot)\) at a corner at \(\Gamma\), in the direction of the increasing arc length along \(\Gamma\); and the summation (1 to \(J\)) extends to all the \(J\) such corners.

Using equation (11) and the divergence theorem, it is easy to see that

\[
\int_\Omega \frac{N_{\gamma\beta}}{R_{\gamma\beta}} (x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Omega = - \int_\Gamma C_{\gamma,\beta} n_{\beta} u_{\beta}(x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Gamma
\]

\[
- \int_\Omega C_{\gamma,\beta} u_{\beta}(x_{\mu}) [w^*(x_{\mu}, \xi_{\mu})]_\beta d\Omega
\]

(25)

Use of (25) in (24) results in the final integral equation for \(w\) as follows:

\[
\begin{align*}
\gamma_w D w(x_{\mu}) &= \int_\Gamma \hat{V}_n(x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Gamma - \int_\Gamma M_{ij}(x_{\mu}) \psi^*_i(x_{\mu}, \xi_{\mu}) d\Gamma \\
&+ \int_\Gamma \hat{\psi}_n(x_{\mu}) M_{ij}^*(x_{\mu}, \xi_{\mu}) d\Gamma - \int_\Gamma \hat{w}(x_{\mu}) V_{ij}^*(x_{\mu}, \xi_{\mu}) d\Gamma \\
&- \int_\Omega C_{\gamma,\beta} n_{\beta} u_{\beta}(x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Omega + \int_\Omega C \left[ \langle K_{\gamma,\beta} w^*(x_{\mu}, \xi_{\beta}) \rangle \right]_\beta u_{\beta}(x_{\mu}) d\Omega \\
&- \int_\Omega C_{\gamma,\beta} w(x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Omega - \int_\Omega \frac{N_{\gamma\beta}^{(n)}}{R_{\gamma\beta}} (x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Omega \\
&+ \int_\Gamma [b_3 - \rho \bar{w} + f_3 + (N_{\gamma\beta} w_{\beta} \cdot n_{\beta})] (x_{\mu}) w^*(x_{\mu}, \xi_{\mu}) d\Omega \\
&+ \sum_j \left[ \langle M_j \rangle w^* - \langle M_j^* \rangle w \right]
\end{align*}
\]

(26)

Since \((\partial w / \partial n)\) is also an independent variable at \(\Gamma\), an integral relation for \((\partial w / \partial n)\) should be derived. Towards this purpose, consider a second fundamental solution,

\[
w^* = \frac{1}{2\pi \rho \ln \rho \cos \phi}
\]
where $\phi$ is the angle between the outward normal to $\Gamma$ and the radius $\rho$ (see Stern\textsuperscript{13}). The resulting integral equation is

$$
\gamma D \frac{\partial w}{\partial n} = \int_{\Gamma} \bar{\psi}_a(x_\sigma) w_a^* (x_\sigma, \xi_\sigma) d\Gamma - \int_{\Gamma} \bar{M}_a(x_\sigma) \psi_a(x_\sigma, \xi_\sigma) d\Gamma
$$

$$
+ \int_{\Gamma} \{ \psi_a(x_\sigma) M_a^*(x_\sigma, \xi_\sigma) d\Gamma - \int_{\Gamma} [\tilde{\omega}(x_\sigma) - \tilde{\psi}(\xi_\sigma)] V_a^*(x_\sigma, \xi_\sigma) d\Gamma \}
$$

$$
- \int_{\Omega} \{ C_{s \rho} u_\rho \tilde{u}_a(x_\sigma) w_a^* (x_\sigma, \xi_\sigma) d\Omega + \int_{\Omega} C_{s \rho} w_a^* (x_\sigma, \xi_\sigma) \tilde{u}_\rho d\Omega \}
$$

$$
- \int_{\Omega} \frac{C_{s \rho}}{R_{s \rho}} w_a^* (x_\sigma, \xi_\sigma) d\Omega - \int_{\Omega} \{ N_{s \rho} (x_\sigma) w_a^* (x_\sigma, \xi_\sigma) - \frac{\rho \tilde{\omega} + f_3 + (N_{s \rho} w_{s \rho})_{z_a}}{2} \}
$$

$$
+ \sum_{\tilde{\gamma}} [ \langle M_a \rangle w_a^* - \langle M_{a \theta} \rangle \tilde{w}] \}
$$

(27)

Remarks

In summary, equations (21), (26) and (27) represent the complete set of integral equations for $u$, $w$ and $\tilde{w}/\tilde{n}$. An examination of these equations reveals the following features.

(i) For given body forces $\bar{b}_a$, the integral relations for $u_a$ (equation 21) involves the trial functions $u_a$ only at the boundary $\Gamma$. On the other hand, due to the curvature induced coupling of the trial functions (u and w) in the shallow-shell problem, the integral relations for $u_a$ contain a domain-integral (over $\Omega$) involving the trial function for w. If the in-plane inertia forces $(\rho \tilde{\omega})$ appear in the problem, then the integral relations for $u_a$ involve a domain-integral (over $\Omega$) of $\tilde{u}_a$ as well.

(ii) Again, due to the curvature-induced coupling of the in-plane and out-of-plane displacements in the presently considered shallow shell, the integral equations for $w$ and $\tilde{w}/\tilde{n}$, equations (26) and (27), respectively, contain domain-integrals (over $\Omega$) involving trial functions for both $w$ and $u_a$.

(iii) In the non-linear problem, the non-linear terms $N_{s \rho}$ and $(N_{s \rho} w_{s \rho})_{z_a}$ involving trial functions for both $w$ and $u_a$ inevitably bring the domain-integrals (over $\Omega$) into the equations.

(iv) For reasons (i) to (iii) above, unlike the classical homogeneous isotropic elasto-statics\textsuperscript{1} wherein a discretization of the relevant integral equations requires the use of basis functions for the displacements at the boundary alone, the present non-linear shallow-shell formulation requires the assumption of basis functions for the trial solutions $u_a$ and $w$ at the boundary $\Gamma$ as well as in the interior $\Omega$. Thus, the present solution methodology may, strictly speaking, be classified as a hybrid boundary-element/field-(finite)-element method based on a direct discretization of integral equations. We name this method the field-boundary-element method.

(v) Suppose now that in equations (21), (26) and (27) we let $\xi_\sigma$ tend to a point on the boundary, i.e. $\xi_\sigma \in \Gamma$. Thus, we obtain three integral relations for the boundary values of $u_a$, $w$ and $\tilde{w}/\tilde{n}$. An examination of these relations reveals that, in order to discretize these equations, one needs to assume only very simple trial functions $u_a$, $w$, $\tilde{w}/\tilde{n}$ not only at the boundary but also in the interior of $\Omega$. For instance, $\Omega$ may be discretized into a number of finite elements and $\Gamma$ into a number of boundary elements. As $\xi_\sigma$ tends to $\Gamma$, the integral relations (21), (26) and (27) clearly show that $w$ and $u_a$ need only be piecewise differentiable and need not even be $C^0$ continuous at the element boundaries. In contrast, it is recalled that in the Galerkin finite element method, $u_a$ need be $C^0$
continuous and \( w \) be \( C^1 \) continuous in each element. The difficulties with such a finite element approach are too well documented in the literature to warrant further comment here.

(vii) At each point on the boundary, two of the in-plane variables \( u_x (z = 1, 2) \), \( P_z \) (\( z = 1, 2 \)) are specified; and the other two are unknown. Likewise, two of the out-of-plane variables, \( V_n \), \( M_x \), \( \psi_n \), and \( w \) are specified; and the other two are unknown. At each point in \( \Omega \), as seen from equations (21), (26) and (27), the three displacements \( u_x \) and \( w \) are unknown. Thus, if equations (2), (26) and (27) are discretized through finite as well as boundary elements and the nodal values of the variables are appropriately understood to be either specified or unknown, one would easily obtain exactly as many equations as the number of unknowns, so that the problem may be considered as well posed.

Note that due to the appearance of the non-linear terms in the obtained integral equations (21), (26) and (27), the discretized interior/boundary-element equations may be solved through an incremental approach. This will be discussed in detail in Section 3.

3. INCREMENTAL APPROACH AND SOLUTION STRATEGY

In the incremental approach, the load and the prescribed boundary conditions are applied in small but finite increments.

Consider that the shell is at the end of the \( k \)th load increment. All quantities which have been known during the previous \( k \) steps of analysis are denoted by a superscript \( 'k' \). The displacement increments are denoted as

\[
\Delta u = u^{k+1} - u^k; \quad \Delta w = w^{k+1} - w^k
\]

With these notations, the integral equations in incremental form may be written as

\[
\gamma(u^k + \Delta u) = \int_{\Omega} \left[ [H^k + \Delta b - \rho(\bar{u}^k + \Delta \bar{u})] u^*_{\theta \theta} \right] d\Omega
\]

\[
+ \int_{\Gamma} (\bar{P} + \Delta \bar{P}) u^*_{\theta \theta} d\Gamma - \int_{\Gamma} (\bar{u}^k + \Delta \bar{u}^k) P_{\theta \theta} d\Gamma - \int_{\Omega} C_{\nu,\beta}(w^k + \Delta w) U_{\theta \theta}^* d\Omega
\]

\[
- \int_{\Omega} (N_{\nu,\beta}^{\nu,\beta} + \Delta N_{\nu,\beta}^{\nu,\beta} + \text{higher-order terms}) u^*_{\theta \theta, \beta} d\Omega
\]

\[
\gamma \cdot D(w^k + \Delta w) = \int_{\Gamma} (\bar{\nu}^k + \Delta \bar{\nu}) \nu^* d\Gamma - \int_{\Gamma} (\bar{M}^k + \Delta \bar{M}) \nu^* d\Gamma
\]

\[
+ \int_{\Omega} \left[ \left( \frac{\Delta N_{\nu,\beta}^{\nu,\beta}}{R_{\nu,\beta}} + \frac{\Delta N_{\nu,\beta}^{\nu,\beta}}{R_{\nu,\beta}} \right) C_{\nu,\beta}(w^k + \Delta w) \right] w^* d\Omega
\]

\[
- \int_{\Omega} \left[ \left( \frac{\Delta N_{\nu,\beta}^{\nu,\beta}}{R_{\nu,\beta}} + \frac{\Delta N_{\nu,\beta}^{\nu,\beta}}{R_{\nu,\beta}} \right) + \text{higher-order terms} \right] w^* d\Omega
\]

\[
+ \int_{\Omega} \left[ H^k + \Delta b - \rho(\bar{\nu}^k + \Delta \bar{\nu}) + f_\nu^k + \Delta f_\nu \right] w^* d\Omega
\]

\[
+ \int_{\Omega} \left[ \left( N_{\nu,\beta}^{\nu,\beta} + \Delta N_{\nu,\beta}^{\nu,\beta} \right) (w^k + \Delta w) \right] w^* d\Omega
\]

\[
+ \sum_{\Gamma} \left[ \left( M_{\nu,\beta}^k + \Delta M_{\nu,\beta} \right) w^* - \left( M_{\nu,\beta}^k \right) (w^k + \Delta w) \right]
\]

\[(29a)\]

\[(29b)\]
where the increments of the non-linear parts of in-plane forces are

\[
\Delta N_{11}^{\alpha \beta} = C [w_{\alpha}^{\beta} \Delta w_{\beta} + v w_{\alpha}^{\beta} \Delta w_{\beta}]
\]

\[
\Delta N_{12}^{\alpha \beta} = C [w_{\alpha}^{\beta} \Delta w_{\beta} + v w_{\alpha}^{\beta} \Delta w_{\beta}]
\]

\[ (30) \]

\[
\Delta N_{12}^{\alpha \beta} = C \frac{1 - v}{2} [w_{\alpha}^{\beta} \Delta w_{\beta} + w_{\alpha}^{\beta} \Delta w_{\beta}]
\]

and

\[
\Delta N_{\gamma \beta} = \Delta N_{\gamma \beta} + C \kappa_{\gamma \beta} \Delta w + \Delta N_{\gamma \beta}^{\alpha \beta} + \text{higher-order terms}
\]

where the definition of the increments of the linear part is apparent. Here, the incremental form of equation (27) is similar to that of equation (29b), and its treatment follows the same routine.

In equations (29a, b), the higher-order terms involve the products of the incremental displacements. In solving for these unknown incremental displacements, those higher-order terms are ignored. In the considered incremental equations, the terms with the superscript 'k' should have satisfied the equilibrium conditions at the end of the kth load increment; however, the equilibrium conditions are in fact not exactly satisfied because of the absence of the higher-order terms. Therefore, an 'equilibrium correction' iteration is employed at each step.

Note that in equation (29b), the non-linear term \((N_{\gamma \beta}^{k} + \Delta N_{\gamma \beta}) (w^{k} + \Delta w)\) can be written as

\[ (N_{\gamma \beta}^{k} + \Delta N_{\gamma \beta}) (w^{k} + \Delta w) = N_{\gamma \beta}^{k} w_{\gamma}^{k} + N_{\gamma \beta}^{k} \Delta w_{\beta} + N_{\gamma \beta}^{k} w_{\beta}^{k} + \text{higher-order terms} \]

(31)

Ignoring the higher-order terms and examining (30) and (31), we may see that those non-linear terms are linearized with respect to the displacement increments. Using (30) and (31) in equations (29a, b) and applying the divergence theorem, we may obtain the final integral equations in terms of unknown displacement increments:

\[
\gamma (u_{\alpha}^{k} + \Delta u_{\alpha}) = \int_{\Omega} \left[ (b_{\alpha}^{k} + \Delta b_{\alpha} - \rho \dot{u}_{\alpha}^{k} + \Delta \dot{u}_{\alpha}) \right] u_{\alpha}^{*} \, d\Omega
\]

\[ + \int_{\Gamma} (\mathbf{f}_{\alpha}^{k} + \Delta \mathbf{f}_{\alpha}) \mathbf{u}^{*}_{\alpha} \, d\Gamma - \int_{\Gamma} (\mathbf{t}_{\alpha}^{k} + \Delta \mathbf{t}_{\alpha}) \mathbf{P}^{*}_{\alpha} \, d\Gamma
\]

\[- \int_{\Omega} C_{\kappa_{\gamma \beta}} (w^{k} + \Delta w) u_{\gamma \beta}^{*} \, d\Omega + \int_{\Omega} N_{\gamma \beta}^{k} w_{\gamma \beta}^{k} \, d\Omega + \int_{\Gamma} N_{\gamma \beta}^{k} w_{\gamma \beta} \Delta \mathbf{w} \, d\Gamma
\]

\[ - \int_{\Omega} N_{\gamma \beta}^{k} w_{\gamma \beta} \Delta w \, d\Omega + \Delta w_{\beta}^{k} \]

(32a)

\[
\gamma_{\alpha} D (w^{k} + \Delta w) = \int_{\Gamma} \left[ (\mathbf{F}_{\alpha}^{k} + \Delta \mathbf{F}_{\alpha}) w^{*} \right] \, d\Gamma - \int_{\Gamma} (\mathbf{M}_{\alpha}^{k} + \Delta \mathbf{M}_{\alpha}) \mathbf{u}^{*} \, d\Gamma
\]

\[ + \int_{\Gamma} (\mathbf{\phi}_{\alpha}^{k} + \Delta \mathbf{\phi}_{\alpha}) \mathbf{M}^{*}_{\alpha} \, d\Gamma - \int_{\Gamma} (\mathbf{\psi}_{\alpha}^{k} + \Delta \mathbf{\psi}_{\alpha}) \mathbf{V}^{*}_{\alpha} \, d\Gamma
\]

\[- \int_{\Gamma} \left[ C_{\kappa_{\beta \gamma}} w_{\beta}^{*} + A_{\beta} \right] \Delta u_{\gamma} \, d\Gamma + \int_{\Omega} \left[ C_{\kappa_{\beta \gamma}} w_{\beta}^{*} + B_{\alpha} \right] \Delta u_{\gamma} \, d\Omega
\]

\[- \int_{\Gamma} \left[ C_{\kappa_{\beta \gamma}} w_{\beta}^{k} + N_{\gamma \beta}^{k} w_{\gamma \beta}^{*} \right] \Delta \mathbf{w} \, d\Gamma
\]

\[ + \int_{\Omega} \left[ C_{\kappa_{\beta \gamma}} w_{\beta}^{k} + N_{\gamma \beta}^{k} w_{\gamma \beta}^{*} - C_{\kappa_{\beta \gamma}} R_{\gamma \beta}^{k} w^{*} - B_{\alpha} \right] \Delta w \, d\Omega
\]
\[ + \int_{\Omega} \left( (\mathcal{K}_{w^k} w^k \mathcal{K}_{b^k})_{ij} - \frac{\mathcal{N}_{w^k}}{R_{w^k}} C^{w^k}_{b^k} + b^k + \Delta b^k \right) + \rho(\ddot{w}^k + \Delta \ddot{w}) + f^k + \Delta f^k \right] w^k \, d\Omega \]

\[ + \sum_k \left[ \langle M_{r^k} + \Delta M_{r} \rangle w^k - \langle M_{r^k} \rangle (w^k + \Delta w) \right] \quad (32b) \]

where the constants \( A \) and \( B \) may be given as follows:

\[ A_1 = C \left[ w_1^w w_1^w, n_1 + \frac{1-v}{2} (w_1^w w_2^w + w_2^w w_1^w), n_1 \right] \quad (33a) \]

\[ B_1 = C \left[ (w_1^w w_1^w)_{i1} + \nu (w_2^w w_2^w)_{i2} + \frac{1-v}{2} [(w_1^w w_1^w)_{i2} + (w_2^w w_1^w)_{i1}] \right] \quad (33b) \]

and \( A_2, B_2 \) can be similarly written by cycling the subscripts; and the constants \( A_w \) and \( B_w \) may be written as

\[ A_w = C (w_1^w)^2 w_1^w, n_1 + (w_2^w)^2 w_2^w, n_2 + \nu (w_1^w)^2 w_2^w, (w_1^w, w_2^w, n_1) + \frac{1-v}{2} [(w_1^w)^2 w_2^w, n_2 + (w_2^w)^2 w_1^w, n_1] \quad (33c) \]

\[ B_w = C \left[ 2 (w_1^w)^2 w_1^w, w_2^w, w_2^w, n_1 + (w_2^w)^2 w_2^w, w_1^w, w_2^w + w_1^w, w_2^w, (w_1^w, w_2^w, n_1) \right] + \frac{1+v}{2} [(w_1^w)^2 w_2^w, w_2^w, w_2^w, n_1 + (w_2^w)^2 w_1^w, w_2^w, w_1^w, n_2] + \frac{3-v}{2} (w_1^w, w_2^w, w_1^w, w_2^w, n_1) + \frac{1-v}{2} [(w_1^w)^2 w_2^w, w_2^w, w_2^w, n_1 + (w_2^w)^2 w_1^w, w_2^w, w_1^w, n_2] \]

We discretize the domain (\( \Omega \)) as well as the boundary \( \Gamma \) by using some appropriate interpolation functions for the unknown displacement increments. By carrying out the indicated integrations, we may obtain the field-boundary-element equations with the displacement increments as unknowns. For each load increment, these equations have to be solved iteratively because the equilibrium conditions are only approximately satisfied due to the absence of the higher-order terms.

The full Newton–Raphson algorithm is used to obtain the solution. This involves domain integrations for constructing the coefficient matrix, and reduction of the matrix in each iteration. Since, in the present field-boundary-element method, the coefficient matrix is not as large as in the usual finite element approach, the reduction of this matrix is not as critical, and moreover, it is found that the solution converges very rapidly.

In the present numerical implementation, at the beginning of each load increment, the equilibrium is checked first by subtracting the integrals of other terms with the superscript \( k \) from the total load integration; the residual load vector is used to solve the displacement increments; the obtained displacement increments are then used to update the coefficient matrix and check the equilibrium again and so on.
4. THE ARC-LENGTH METHOD FOR THE SOLUTION OF TANGENT STIFFNESS EQUATIONS OF THE FIELD-BOUNDARY-ELEMENT APPROACH

Let the system be at the beginning of the $n$th load increment. The equilibrium equation can be written as

$$K_0^n \Delta p_0 = \lambda_0 q - r_0$$  \hspace{1cm} (34)

where $K_0^n$ is the coefficient matrix obtained by carrying out the integrations associated with the unknown trial solutions in the field-boundary-element equations (32a), (32b) (and a simpler equation for $\partial W/\partial n$) for details, see Zhang and Atluri\textsuperscript{[15]}, here, in this incremental equation, $K_0^n$ is constructed based on the previous solution vector $p_0$ up to the $(n-1)$th load increment. As a matter of fact, $K_0^n$ is equivalent to the tangent stiffness matrix in the finite element method; $\Delta p_0$ is the incremental unknown vector. In the field-boundary-element method, however, the vector $p_0$ includes both boundary forces and displacements as well as the displacements in the interior of $\Omega$; $\lambda_0$ is the current load parameter; $q$ is the fixed load vector; $r_0$ is the internal force vector which should have been equilibrated with the external load up to the $(n-1)$th load increment.

Equation (34) is an approximate one because the higher-order non-linear terms have been ignored. Therefore, the unknown incremental vector should be decided by iterations, i.e. we have

$$K_0^n \Delta p_i = \lambda_i q - r_i$$  \hspace{1cm} (35)

where $r_i$ is obtained by using the previous solution vector $p_i$ up to the $(i-1)$th iteration; and, if the modified Newton-Raphson solution technique is used, $K_0^n$ is the same as $K_0^n$ during the current increment; or it should be updated after each iteration if the full Newton-Raphson technique is used; $\Delta p_i$ is the current displacement increment and we have

$$\Delta p_i = \Delta p_{i-1} + \delta p_{i-1}$$

$$p_i = p_{i-1} + \Delta p_{i-1}$$  \hspace{1cm} (36)

The central concept of the arc-length method is to decide the value of the load parameter $\lambda_i$ in other words, to give the suitable step length as well as the direction of the load increment when tracing the equilibrium path. A general formula to decide $\lambda_i$ can be given as\textsuperscript{[6]}

$$\Delta p_i^T \Delta p_i + b \Delta \lambda_i^2 q^T q = \Delta l^2$$  \hspace{1cm} (37)

where $b$ is a scaling parameter and $\Delta l$ is a prescribed incremental length. Equation (37) is the so-called displacement load control. Some experiences have shown that it is preferable to set $b=0$, that is, to use the displacement control only.\textsuperscript{[6],[7],[12]} From equation (35), we have

$$\delta p_i = \lambda_i (p_i - p_{i-1})$$  \hspace{1cm} (38)

where $p_i = (K_0^n)^{-1} q$ and $p_{i-1} = (K_0^n)^{-1} r_{i-1}$, and from equation (36) one has

$$\Delta p_i = \lambda_i (p_i - p_{i-1})$$  \hspace{1cm} (39)

Substituting (39) into equation (37) and letting $b=0$, we have

$$a \lambda_i^2 + b \lambda_i + c = 0$$  \hspace{1cm} (40)

where

$$a = p_i^T p_i; \hspace{0.5cm} b = 2 p_i^T (\Delta p_{i-1} - p_i);$$

$$c = p_i^T p_i - 2 p_i^T p_{i-1} + [\Delta p_{i-1}^T \Delta p_{i-1} - \Delta l^2]$$

If we assume an exact satisfaction of equation (37) for each increment, the terms in brackets in the formula for $c$ may be ignored.
Taking a proper value of $\Delta l$ and solving equation (40), we may obtain $\lambda_i$ for each iteration which should make the solution point closer and closer to the true equilibrium path. As a matter of fact, equation (40) gives a constraint sphere and usually it cuts the equilibrium path at two points, i.e. equation (40) has two different roots. To choose an appropriate one, we may consider the angle $\theta$ between $\Delta P_i$ and $\Delta P_{i-1}$ where

$$
\cos \theta = \frac{\Delta P_i^T \Delta P_{i-1}}{\Delta l^2} = 1 + \frac{1}{\Delta l^2} (\lambda_i \Delta P_{i-1}^T P_i - \Delta P_{i-1}^T P_i)
$$

(41)

To avoid the so-called ‘double back’, i.e. to avoid reversing along the equilibrium path, one should choose the root that makes the value of $\cos \theta$ positive. If both values of $\cos \theta$ are positive, we may take the root as the one which is closest to the linear solution of equation (40).\textsuperscript{3}

The incremental length $\Delta l$ is usually decided by trial in the first load increment. Using some guessed load parameter $\lambda_i$, $\Delta l$ can be chosen as

$$
\Delta l = \lambda_i / \sqrt{dP_i^T dP_i}
$$

(42)

and it can be kept fixed, unless in some increments the number of iterations becomes more than a desired number (usually 4), when $\Delta l$ may be reduced. In our experience for the shell problem, $\lambda_i$ can be taken such that the largest deflection will be around 10 percent of the thickness, that is, to let deformation remain in the linear scope.

In the current work, based on the field-boundary-element formulation, the full Newton-Raphson technique is used to solve the non-linear system equations. Because of the advantage of having fewer degrees of freedom by the present formulation, the cost of refactorizing the tangent matrix after each iteration is not significant.

The difference between the current formulation and the popular displacement finite element method in applying the arc-length method is that in the field-boundary-element method, the solution vector is a mixture of displacement and boundary force components. To implement the displacement control, equation (37) should be changed to

$$
\Delta P_i^T \Delta P_i = \Delta l^2
$$

(43)

and equation (39) to

$$
\Delta P_i = \lambda_i P_i + \Delta P_{i-1} - P_i
$$

(44)

where the terms with the superposed bar are the vectors which have the non-linearly varying displacement components only.

In the literature, one may find discussions on some techniques such as line search to accelerate the convergence. In the current work by the full Newton-Raphson technique, the rate of convergence is quite satisfactory (3 to 4 iterations on average for each load increment), so that these accelerating techniques were found to be not necessary.

5. NUMERICAL RESULTS

The first problem consists of a shallow spherical shell, with a square base, as shown in the inset of Figure 1. The shell is loaded by a point load at the crown. The relevant geometrical and material-property data are given in Figure 1. In the present field-boundary-element method, a $3 \times 3$ mesh over a quarter of the shell (see the inset in Figure 1) with a total of 64 degrees of freedom is used. As shown in Figure 1, the present solution agrees excellently with:

(i) the analytical solution by Licester,\textsuperscript{11}
(ii) the numerical solutions obtained by Bathe and Lo\(^3\) using triangular finite elements based on the usual Galerkin displacement formulation, with 206 degrees of freedom (results from Bathe and Lo\(^3\) being close to those of Leicester\(^,1\) are not shown in Figure 1 for the sake of clarity);

(iii) the numerical results of Dvorkin and Bathe\(^8\) using continuum-based quadrilateral Galerkin finite elements, with 80 degrees of freedom (not shown in Figure 1, for clarity);

(iv) the results obtained by Bergan \textit{et al.}\(^7\) using triangular finite elements based on a mixed variational Galerkin formulation, with 180 degrees of freedom;

(v) the results obtained by Dhatt\(^8\) using a triangular finite element based on a discrete-Kirchhoff type Galerkin formulation, with 134 degrees of freedom.

From this set of results, the potential of the present field-boundary-element method in shell analysis is evident. Also earlier results\(^1\) concerning free vibration and transient response of shallow shells indicate that the present method can yield a larger number of eigenvalues of vibration with a greater accuracy, while using a fewer number of degrees of freedom as compared to the Galerkin finite element method.

The second example concerns a shallow spherical shell with a circular base (shown schematically in Figure 2). The shell is subject to a concentrated load at the crown. Note that while the problem is axisymmetric, the formulation that is employed is in a Cartesian reference frame for the sake of generality. Thus, one quadrant of the problem is analysed by using different meshes, as shown in Figure 3, to study the convergence properties of the present approach. The three meshes in Figure 3 consist of 26, 51 and 61 degrees of freedom, respectively. The load versus crown-deflection curves, obtained for each of the 3 meshes respectively, are shown in Figure 4. The variation of the
Figure 2. Schematic of a circular-based shallow spherical shell

(a) 26 D.O.F.  
(b) 51 D.O.F.  
(c) 61 D.O.F.

Figure 3. Different field-boundary-element (FBE) meshes for the analysis of the circular-based shallow spherical shell

Figure 4. Convergence of the load-deformation curve for various FBE meshes shown in Figure 3
computed bending moments $M_r$ and $M_\theta$ at cross sections along the radial length $r$ from the apex of the shell, after the fifth load increment, are shown in Figure 5. These results were obtained by using mesh (b) in Figure 3. For each load increment, convergence was achieved with 3 or 5 iterations. These results indicate that the present method is fully capable of accounting for stress concentrations/singularities near concentrated loads. Also using the mesh of Figure 3, several shells with varying radii of principal curvature were analysed, and the corresponding load-deflection diagrams are shown in Figure 6. As seen from Figure 6, as the radii of curvature increase, the load value at snap-through decreases, and the load-deflection response curve becomes much smoother in the post-critical range.

6. CONCLUSIONS

This paper presented an exploratory study concerning the use of the field-boundary-element method in the analysis of large deformations and post-buckling response of shallow shells. In this method, the fundamental solutions in infinite space to the highest-order linear differential operator of the problem are used as test functions. The results were found to be highly encouraging. The reduced-order modelling capability of the present method appears to make it attractive for purposes of implementing ‘deformation-control’ algorithms, wherein the limiting factor is the order of the Riccati equations that must be solved. The present study points to a need for further exploration of the mathematical convergence aspects of the field-boundary-element method.

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Figure 5. Bending moments $M_r$ and $M_\theta$ along the radial direction of a point-loaded shallow spherical shell of Figure 2.
(Results shown are for FBE mesh shown in Figure 3(b))
Figure 6. Load-deformation diagrams for shells of various radii of principal curvature

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