A Novel Displacement Gradient Boundary Element Method for Elastic Stress Analysis With High Accuracy

1 Introduction

The boundary element method (BEM) is based upon classical integral equation formulations of boundary value/initial value problems. Although such formulations were originally thought to be primarily of theoretical interest, the engineering applications of this method in linear elastostatics and potential problems can be traced to the works of Jaswon and Symm (1963), Massonnet (1963), and Hess (1967). Later, this method was applied to a wide range of time dependent and vibration problems, and those involving fluid flow and material nonlinearities. Recently, this method has been extended to solve material as well as geometrically nonlinear problems in solid mechanics (see, for example, Zhang and Atluri (1968), O’Donoghue and Atluri (1987), and Im and Atluri (1987)).

Numerical solutions to two-dimensional problems in linear elasticity were obtained by Rizzo (1967) and later for three-dimensional problems by Cruse (1969). The boundary element formulation (as also noted by Bredaia (1978)) can be shown to be based on a weak form of the momentum balance laws in solid mechanics. Here the trial and test functions are quite different from each other. The test functions correspond to fundamental solutions (in this case Kelvin’s singular solution), in finite space, of the differential operator of the problem, and hence are usually infinitely differentiable except possibly at singular points. On the other hand, the trial functions in BEM are required to be differentiable and continuous to a lesser degree (as compared to the finite element method).

The currently popular BEM is based on displacement boundary integral equations in linear elastic solid mechanics. The stress field in this standard BEM formulation is computed by differentiating the displacement field at the source point, to obtain the strain field and then applying Hooke’s law to obtain the stress field. Unfortunately, this method works well in the interior of the body, but at the boundary it gives rise to hypersingular integrals which are intractable from a numerical standpoint. Hence, to circumvent this predicament at the boundary, numerical differentiation of the displacements (Cruse (1969), and Lachat and Watson (1975)) and other ad hoc techniques are used to obtain the displacement derivatives at the boundary. Now, making use of the boundary tractions and the numerically obtained tangential displacement derivatives, the stress field at the boundary is determined.

The hypersingular kernels in the equation for displacement gradients, in the traditional boundary element methods, continue to be of concern to researchers and practitioners in this discipline. In Ghosh et al. (1986), an attempt was made to reduce the order of singularity in the traction kernel in the equation for displacement gradients (see equation (11) to follow later in the present paper), by transferring the derivatives of the traction kernel (i.e., of \( \partial \tau / \partial t \)) to the displacement \( u \), thereby introducing the tangential derivatives of \( u \). It will be shown later, that the present formulation for displacement gradients is fundamentally different from that of Ghosh et al. (1986). Furthermore, the present formulation does not suffer from the following limitations of the approach.
of Ghosh et al. (1986): (i) when the displacements are prescribed at the boundary, the evaluation of tangential derivatives involve numerical differentiation and, hence, leads to inaccuracies, especially for arbitrary boundaries with discontinuous tangents, (ii) their formulation is limited to simply connected domains. For multiply connected regions, the quantities \( u_i \); \( \partial u_i / \partial s \), and \( t^i \) appear as variables and, hence, it is necessary to impose the equations of constraint between \( u_i \) and \( \partial u_i / \partial s \), and (iii) in the case of three-dimensional problems, their formulation is not conducive to numerical implementation.

The directly derived integral equations for the displacement gradients in the presently reported formulation, obtained from a novel weak form of the momentum balance equations, has lower order singularities in the relevant kernels, than in the standard formulation. This makes the singularities to be tractable from a numerical point of view. A detailed numerical treatment of these singularities is given in this paper. The present formulation is more accurate than the standard two-tier approach of evaluating the stress field. This is illustrated through several numerical examples at the end of this paper.

2 The BEM Formulation

Let \( \sigma_{ij} \) be the Cartesian components of the Cauchy stress tensor and let \( e_{ij} \) be the corresponding strain components. The equations of linear and angular momentum balance are

\[
\sigma_{ij} + f_j = 0; \quad \sigma_0 = \sigma_0
\]

where \( (\ ) \) denotes differentiation with respect to material coordinates \( x_i \) and \( f_i \) are the body forces per unit volume. The stress-strain relations for the linear elastic body take the following form

\[
\sigma_{ij} = E \epsilon_{ij} \epsilon_{mn}
\]

where, for an isotropic solid, \( E \) is defined as

\[
E = \frac{\lambda + 2 \mu}{\epsilon_{ij} \epsilon_{mn}}
\]

We restrict our attention here to the case of isotropy. The strain-displacement relations take the following form

\[
\epsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji})
\]

Here, \( u_i \) refer to the Cartesian components of displacements. The tractions \( t^i \) on the boundary can be expressed as

\[
t^i = n_i \sigma_{ij}
\]

Here, \( n_i \) are the components of the unit normal to the boundary.

The weak form of the linear momentum balance equation, when a displacement field \( (u_i) \) is taken as the test function, takes the following form:

\[
\int_\Omega (\sigma_{ij} + f_j) u_i \, d\Omega = 0.
\]

(6)

Here, \( \Omega \) and \( \partial \Omega \) refer to the domain and the boundary, respectively, of the problem under consideration. We assume \( u_i \) to be the solutions, in infinite space, of the equation

\[
(E)_{ij} u_{ij} + (\partial \sigma_{ij} / \partial x_i - n_i \epsilon_{ij}) d\Omega = 0
\]

where \( \partial \epsilon_{ij} \) is the component of a unit load along the \( x_i \) coordinate. Thus, \( \tilde{u}_i \) are the fundamental solutions of the Navier equations of linear isotropic elasticity. Specifically, \( u_{ij} \) is the displacement along the \( x_i \)-axis at \( x_m \), due to a unit load along the \( x_i \)-axis at \( x_m \), and \( u_{ij} \) is the displacement along the \( x_i \)-axis at \( x_m \), due to a unit load along the \( x_i \)-axis at \( x_m \). In an infinite solid, of a linear isotropic elastic solid. Thus, the test functions \( u_i \) may be represented as

\[
u_i = u_i \epsilon_{ip}
\]

and correspondingly,

\[
\tilde{u}_i = \frac{n \epsilon_{ij} u_{ij}}{E \epsilon_{ij} \epsilon_{mn}}
\]

(7)

Integrating equation (6) by parts, applying the divergence theorem twice and making use of equations (2), (3), (4), (5), (7), (8), and (9), the standard BEM equations (in the absence of body forces) can be obtained.

\[
C_{ij} u_j (\xi_m) = \int_\partial \Omega \left[ u^*_{ij} (x, \xi_m) \right] t^i (x, \xi_m) \, d\partial \Omega
\]

(10a)

Here, \( C_{ij} = \delta_{ij} \) in the interior and \( 1/2 \delta_{ij} \) at a smooth boundary point.

The above equation, when discretized and applied at the boundary, results in the standard boundary element system of equations:

\[
H u = G T
\]

(10b)

where \( u \) and \( T \) are displacements and tractions, respectively, at the boundary; and \( H \) and \( G \) are the associated matrices.

The standard BEM formulation makes use of equation (10), and obtains the strain field by differentiating it with respect to the load point \( \xi_i \).

Thus,

\[
C_{ij} u_j (\xi_m) = \int_\partial \Omega \left[ \frac{\partial u^*_{ij} (x, \xi_m)}{\partial \xi_i} \right] t^i (x, \xi_m) \, d\partial \Omega
\]

(11)

This method gives rise to \( t^*_{ij} \) type kernels, which are hypersingular, when \( \xi_m \) tends to a boundary point. Hence, this predicate gives rise to a two-tier system of evaluation for the stress field.

With a view towards developing the present "displacement gradient integral equation," directly from the weak form, we take the test function to be \( u_{ij} \) instead of \( u_i \) as in equation (6).

The weak form can now be represented in vector form, rather than in a scalar form as in equation (6), as:

\[
\int_\Omega (\sigma_{ij} + f_j) u_{ij} \, d\Omega = 0.
\]

(12)

Let the test functions satisfy the same field equations as those in equation (8) and (9). Integrating equation (12) by parts, applying the divergence theorem three times and making use of equations (2), (4), (5), (7), (8), and (9), we obtain the following direct integral equation (in the absence of body forces) for displacement gradients

\[
C_{ij} u_{ij} (\xi_m) = \int_\partial \Omega \left[ \frac{\delta_{ij} (x, \xi_m) E \epsilon_{ij} \epsilon_{mn}}{\epsilon_{ij} \epsilon_{mn}} \right] u_{ij} (x, \xi_m) \, d\partial \Omega
\]

(13a)

Here, \( C_{ij} = \delta_{ij} \) in the interior and \( 1/2 \delta_{ij} \) at a smooth boundary point. The kernels \( u_{ij} \) and \( u_{ij}^* \) have the same order of singularity and in general are tractable from a numerical point of view. Note also that as \( \xi_m \) tends to \( \partial \Omega \), the singularities in the new representation, equation (13) are of lower order as compared to those in equation (11).

Equation (13a), when discretized and applied at the boundary, results in the following system of equations:

\[
H u = G T
\]

(13b)
where \( u_a \) and \( T \) are the displacement gradients, and tractions, respectively, at the boundary; and \( H_a \) and \( G_a \) are the associated matrices. It is interesting to note that the order of singularity in \( H_a \) and \( G_a \) terms is the same as that of the terms in \( H \) in equation (10b).

A detailed analysis of the numerical implementation of the singular integrals in equation (13) will follow in the next section. Equations (10) and (13) are valid for simple as well as multiply connected domains \( \Omega \). The surface \( \partial \Omega \) must, of course, include outer as well as inner boundaries of \( \Omega \) in the larger case. Implicit in the derivation of equation (13) is the assumption that the material is homogeneous, i.e., \( E_{inw} \) are independent of material coordinates \( x_m \). Further details of the derivation of (13), along with similar results for displacement gradient integral equations, in the cases of (i) small strain elasto-plasticity; (ii) finite strain elasto-plasticity; and (iii) large deformations of semilinear elastic material, are given in Okada, Rajiyah, and Atluri (1987).

The unknown boundary displacements and tractions could be obtained from equation (10b) as in the standard BEM approach. Once the complete displacement and traction field is completely ascertained on the boundary, equation (13b) could be used to solve for the boundary displacement gradients evaluated on the boundary. Finally, the displacement gradients could be evaluated pointwise, in the interior, (using equation (13a)), once the displacement gradient on the boundary are known. Hence, the stress field could be determined by Hooke’s law in a uniform manner.

3 Numerical Implementation of Singular Integrals in Equation (13)

As in Fig. 1, BOC is the boundary segment where the source point \( O \) lies and let \( P \) be the field point. The straight line EOD is a tangent to the curve BOC at \( O \) and let EC and BD be perpendicular to the line EOD. The points B and C are the ends of the segment under consideration. Let \( r \) be the distance between the source point \( O \) and field point \( P \). For illustrative purposes, let us consider the integral

\[
I_1 = \int_{\partial \Omega} t_j(x_m) \ u_{a,b}^r(x_m, \xi_m) \ d\partial \Omega
\]

(14)

When the traction components \( t_j \) are expressed as nodal tractions multiplied by the associated shape functions, the resulting singular integral to come out of \( I_1 \), would take the following generic form

\[
I_1 = \int_{\partial \Omega} N_{\alpha} t_j(x_m) \ u_{a,b}^r(x_m, \xi_m) \ d\partial \Omega
\]

(15)

Here, \( N_{\alpha} \) is the shape function associated with point \( O \) and \( t_j(x_m) \) corresponds to the nodal traction at the source point, i.e., point \( O \). The singular integral \( I_1 \) can also be written in the following form

\[
I_1 = t_j(x_m) \int_{\partial \Omega} \left( N_{\alpha} - 1 \right) \ u_{a,b}^r(x_m, \xi_m) \ d\partial \Omega + t_j(x_m) \int_{\partial \Omega} \ u_{a,b}^r(x_m, \xi_m) \ d\partial \Omega
\]

(16)

For simplicity, \( \int_{\partial \Omega} \ u_{a,b}^r(x_m, \xi_m) \ d\partial \Omega \) can be taken to be linear. The integral \( I_2 \) in the above equation is regular and, hence, tractable from a numerical point of view. From here onwards, we will consider only the integral \( I_2 \) which appears to be singular. As per Fig. 1, the integral \( I_2 \) would take the following form

\[
I_2 = \int_{BOC} \ u_{a,b}^r \ dS
\]

(17)

Due to the inherent property of the \( u_{a,b}^r \) kernel, it could be expressed (for two-dimensional problems) in the following form

\[
u_{a,b}^r = \frac{f(\theta)}{r}, \quad f(\theta) = -f(\theta + \pi).
\]

(18)

Here, \( r \) is the distance between the source point and field point. \( \theta \) is the angle made by the line connecting the source point and field point with the \( x \)-axis. Substituting the expression for \( u_{a,b}^r \) in equation (18) into equation (17) we have

\[
I_2 = \int_{BOC} \frac{f(\theta)}{r} \ dS
\]

(19)

Note here, that the angles \( \alpha \) and \( \beta \) are constants and \( \theta \) varies as the field point moves along the arc (Fig. 1). The integrals along BO and OC in equation (19) are regular and, hence, numerically tractable. Let us define the sum of the integrals over the straight lines DO and OE as \( I_2 \). Thus

\[
I_2 = \int_{DO} \frac{f(\alpha)}{r} \ dS + \int_{OE} \frac{f(\beta)}{r} \ dS
\]

(20)

Evaluating \( I_2 \) in the sense of Cauchy Principal Value, we have

\[
I_2 = \lim_{\epsilon \to 0} \left[ -\ln \left| -\ln \left| f(\alpha) \right| \right| + \ln \left| f(\beta) \right| \right] + C.P.V.
\]

(21)

Here, \( r_E \) and \( r_D \) are distances between the source point and points \( E \) and \( D \), respectively. C.P.V. denotes Cauchy Principal Value terms which arise out of the integral \( I_1 \). It so happens that, when equation (13) is considered as a whole, all Cauchy Principal Value terms arising out of the singular kernels \( u_{a,b}^r \) and \( e_{a,b}^r \) cancel out and hence, in effect, there are no free terms arising out of the integral equation (13). Therefore, it is sufficient to consider equation (21) without the C.P.V. terms.

Since EOD is a straight line (Fig. 1)
Table 1  Summary of numerical quadratures employed

<table>
<thead>
<tr>
<th>Case</th>
<th>Type of Integration</th>
<th>Order of Integration Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R &gt; 1.2d$</td>
<td>Gauss Numerical Quadrature</td>
<td>3</td>
</tr>
<tr>
<td>$R &lt; 1.2d$</td>
<td>Gauss Numerical Quadrature Element Sub-divisions: up to 4</td>
<td>$6 \sim 10$</td>
</tr>
<tr>
<td>Log-Singularities</td>
<td>Gauss Numerical Quadrature Element Subdivisions: up to 4 (Quarter Point Coordinate Transformation)</td>
<td>10</td>
</tr>
<tr>
<td>$1/r$-Singularities (Cauchy Principal Value Integral)</td>
<td>1) The Use of Rigid Body Motion</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>2) Dividing the Integral into Two Parts</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>i) $1/r$ Singular Integral on the Straight Line</td>
<td>Analytically</td>
</tr>
<tr>
<td></td>
<td>ii) Non-Singular Integral on the Curved Line</td>
<td>10</td>
</tr>
</tbody>
</table>

$(R$: distance between the source point and an element); $(d$: element size)

Fig. 2  Use of partially discontinuous boundary elements at a corner point and/or where tractions are discontinuous

\[ \beta = \alpha + \pi. \]  \( (22) \)

Hence, from equations (18) and (21) it is clear that

\[ f(\alpha) + f(3) = 0. \]  \( (23) \)

There, equation (20) takes the following form

\[ I_{\beta} = \ln \left( \frac{\rho}{\rho_{E}} \right) \times f(\alpha) \]  \( (24) \)

The above-mentioned procedure of regularizing the singular integral is explained in the context of a two-dimensional problem for the $\kappa_{s,j}$ kernels. It should be noted here that the kernels $n_{r}E_{r,j}u_{r,j}$ and $\tau_{i}^{p} = n_{x}E_{x,j}u_{x,j}$ appearing in equation (13) will have a similar procedure of regularizing the singular integrals, since the components of the normal $n_{x}$ (or $n_{y}$) are constant over the straight line EOD (Fig. 1). Without any loss of generality, a similar procedure could be adopted in the context of three-dimensional problems as well.

4  Numerical Results

4.1 Preliminaries. Three-noded (quadratic) boundary elements are employed in the present study. When the given tractions are discontinuous at a boundary point, we use double nodes at the point in question (see Fig. 2). Since the presently derived integral representation for displacement gradients is applicable for points at a smooth boundary, we shift the location of a node such as B (in Fig. 2) from a corner, or from a point where tractions are discontinuous, to a location C (as shown in Fig. 2) where the boundary is smooth and/or where the tractions are continuous.

All the numerical integrations are performed by using Gauss quadrature. The order of numerical quadrature is determined by the distance between the source point and nearest nodal point of the element, as shown in Table 1. When this distance is less than roughly the size of the element, an element subdivision technique is employed in the process of numerical quadrature. When logarithmic type singular integrands are involved, a “quarter-point element” type geometric transformations are employed, (leading to Jacobian singularities), and Gauss quadrature is employed in the parent element. When $(1/r)$ type singularities are involved in line integrals, integrations are performed as indicated in Table 1.

In the present paper, the designation of “conventional method” implies that all the displacement gradients and stresses at a boundary point are computed from the algebraic equations at the boundary point in question involving the components of traction, and the gradients, along the tangential direction, of the displacements. The gradients of displacements along the tangential direction are evaluated by
differring the shape functions for displacements in the boundary elements. The stresses at the boundary, computed in this "conventional method" are, in general, discontinuous between the neighboring boundary elements. In such a case, we simply take the average of the two sets of strain and stress components at each interelement nodal point.

4.2 Example Problems.

4.2.1 Radial Expansion of a Thick Cylinder. The geometric and material data for the problem, and the boundary conditions, are shown in Fig. 3. In the boundary element model, a 20-deg segment of the cylinder is considered and displacements are prescribed at the inner surface of the cylinder. Three progressively refined boundary element meshes, as shown in Fig. 4, are considered.

The solid points along the line \( \theta = 0 \) deg in the Fig. 4 are locations where the computed results for stresses \( \sigma_{rr} \) and \( \sigma_{\theta\theta} \) by the present method are the conventional method, respectively, are given in Tables 2 and 3. Tables 2 and 3 show that the stresses at these interior points from both the present method as well as the conventional method, converge to the exact solution, with a progressive refinement of the boundary element mesh.

The solid points along the radius \( r = 7.1764 \) mm, shown in Fig. 4, are locations where the stresses \( \sigma_{rr} \) computed by the present method and the conventional method, for each of the
the three boundary element meshes are shown, respectively, in Figs. 5, 6, and 7. In each of these figures, the locations \( r = 7.1764 \, \text{mm} \) and \( \theta = \pm 10 \, \text{deg} \) are boundary points (see Fig. 4). Figures 5, 6, and 7 show clearly that, in spite of the mesh refinement, the stress \( \sigma_{rr} \) computed by the conventional method, is significantly (one order of magnitude) more erroneous at the boundary point than at the interior points. On the other hand the stresses \( \sigma_{rr} \) computed by the present method, in all the 3 mesh cases, is of equal accuracy at both the boundary points as well as the interior points. It is thus seen that the “boundary layer” type behavior of error in the computed stress that is present in the conventional method is nullified in the present approach.

4.2.2 Problem of a Tension Plate with a Circular Hole. The geometric, material, and load data are shown in Fig. 8. The progressively refined boundary element meshes are shown in Figs. 9(a), (b), and (c), respectively.

Attention is focused on the solution for the direct stress \( \sigma_{rr} \) along the periphery of the hole and at the boundary segments near the hole, as marked by the lines AB – BC – CD in Fig. 8(b). Along the segment BC, this direct stress \( \sigma_{rr} \) is the usual \( \sigma_{\theta\theta} \). The results for \( \sigma_{rr} \) computed by the present method as well as the conventional method, by using each of the three
 meshes shown in Figs. 9(a), (b), and (c), are shown in Figs. 10, 11, and 12, respectively. The solid lines in Figs. 10, 11, and correspond to the analytical solution for an infinite domain subject to remote tension. It is seen that the direct stress along BC converges, in both the present and conventional methods, to the exact solution, with progressive mesh refinement. Along the segments AB and CD the results for the direct stress $\sigma_T$ from the present method converge to the exact solution with progressive mesh refinement, with the results even for the coarser meshes being quite acceptable. On the other hand, it is seen from Figs. 10, 11, and 12 that the results for $\sigma_T$ along AB and CD, as computed from the conventional method fail to converge to the exact solution even with mesh refinement, with the results for coarser meshes being unacceptable.

4.2.3 Problem of a Tension Plate With a Square Plate. The geometric, material, and load data are shown in Fig. 13, with progressively refined boundary element meshes being given in Figs. 14(a), (b), and (c), respectively.

Attention is focused here on the direct stress $\sigma_T$ along the segment QPO of the periphery of the square hole (see Fig. 13). The results for $\sigma_T$ along QP, computed by the present method are shown in Fig. 15, while those computed by the conventional method are shown in Fig. 16. From Fig. 15, it is seen that the present method, with mesh refinement, leads to a very smooth data for $\sigma_T$ and can capture the singularity in $\sigma_T$ at the corner point P of the square hole. On the other hand, it is seen for Fig. 16 that in the conventional method, it is difficult to obtain a smooth interpolant function of $\sigma_T$ for any of the three meshes employed, and convergence to the correct singular solution is unacceptably slow. The results for $\sigma_T$ along OP, for the present and conventional methods shown in
Figs. 17 and 18 respectively, show similar trends as in Figs. 15 and 16, respectively.

These results clearly demonstrated that the present algorithm yields much better results for stresses and strains at boundary points, and can capture stress-strain singularities at such points much better than the conventional method.

5 Closure

The present method, based on a direct integral representation of the displacement gradients, which are of the nonhyper singular type at the boundary, leads to much better numerical results for boundary stresses than the conventional method wherein the integral representation for displacements is differentiated to obtain strains and stresses. This improvement in numerical accuracy of the present method over the conventional method is especially significant for the components of stress and strain along the coordinate tangential to the boundary.

The numerical implementation of the present novel theoretical formulation does appear to be inefficient. However, the present formulation and its numerical implementation, do produce continuous, and the most accurate stresses. The evaluation of boundary stresses accurately, is also mandatory in problems, such as large strain plasticity, as recently demonstrated by Okada, Rajiyah, and Atluri (1988), who use concepts similar to those in the present paper, to directly develop non-hyper singular integral equations for velocity gradients in large strain problems.

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References


Fig. 15 Stress $\sigma_{xy}$ along the segment QP (see Fig. 12) of square hole. Results by present method.

Fig. 16 Stress $\sigma_{yy}$ along the segment QP (see Fig. 12) of square hole. Results by conventional method.

Fig. 17 Stress $\sigma_{xx}$ along the segment OP (see Fig. 12) of square hole. Results by present method.

Fig. 18 Stress $\sigma_{xx}$ along the segment OP (see Fig. 12) of the square hole. Results by the conventional method.


