STATIC AND DYNAMIC ANALYSIS OF SPACE FRAMES WITH NON-LINEAR FLEXIBLE CONNECTIONS

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SUMMARY
This paper deals with the effect of non-linearly flexible hysteretic joints on the static and dynamic response of space frames. It is shown that a complementary energy approach based on a weak form of the compatibility condition as a whole of a frame member, and of the joint equilibrium conditions for the frame, is best suited for the analysis of flexibly jointed frames. The present methodology represents an extension of the authors’ earlier work on rigidly connected frames. In the present case also, an explicit expression for the tangent stiffness matrix is given when (i) each frame member, along with the flexible connections at its ends, is represented by a single finite element, (ii) each member can undergo arbitrarily large rigid rotations and only moderate relative rotations and (iii) the non-linear bending-stretching coupling is accounted for in each member. Several examples, with both quasi-static and dynamic loading, are included, to illustrate the accuracy and efficiency of the developed methodology.

INTRODUCTION
In the analysis and design of frame-type space structures, the traditional methods are based on the simple assumptions that the joints are completely either pinned or rigid. However, the experimental investigations of actual joint behaviour have clearly demonstrated that the so-called ‘pinned’ connections do possess a certain amount of rotational rigidity, while the ‘rigid’ connections exhibit some degree of flexibility. Thus, in practice, all types of connections of frame structures are semi-rigid or flexible. The experimental study has also shown that the flexible connections behave non-linearly because of the local distortions, yielding and buckling, etc. in the connections. The flexible connections affect significantly the performance of the structures, such as deformations, stress distributions and dynamic responses. Also, joint flexibility and hysteresis is considered to be a significant source of passive damping of vibration in low-mass structures to be deployed in outer space.

In order to account for the effects of flexible connections on the behaviour of the structures, the modelling of the flexible connections is an essential step. The behaviour of flexible connections is usually described by the moment–rotation curves of the connection in which the slope of the curve corresponds to the rotational rigidity of the connections. A lot of experimental data for various types of connections has been obtained in the past five decades and many models have been proposed to represent the behaviour of the connections. For simplicity, the linear semi-rigid model has been widely used, with concepts such as effective length, rigidity factors and linear rotational springs. However, the approximation of a linear semi-rigid connection is good only when the force at the connection is quite small. When the moment acting at the connection is not small, the rigidity of the connection may change dramatically compared with the initial rigidity.

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and the structure becomes more flexible. Especially under dynamic loading, the non-linear flexible connections will lead to hysteresis loops in the moment–rotation curve, and some energy will be dissipated in the connections. So for both static and dynamic problems, the non-linearity of the connections should be considered.

The non-linear behaviour of the flexible connections can be approximated by bilinear or trilinear functions, or expressed by some types of functions, for example, polynomial, exponential and Ramberg–Osgood functions. The bilinear and trilinear models are simple and present no problems in determining the rigidity of the connections, but they are not accurate enough. Also, the non-smoothness of the curve may cause more numerical difficulties. For the models expressed by the non-linear functions, determining the instantaneous rigidity is very important for the efficiency and accuracy of the analysis. The Ramberg–Osgood function for the moment–rotation relations used by Ang and Morris is a good function to describe the non-linear behaviour of the flexible connections. A scheme for determining the instantaneous rigidity of the connection described by the Ramberg–Osgood function is presented here, which is convenient and accurate for both static and dynamic problems.

Recently, Shi et al. and Chen and Lui used non-linear rotational springs to model the non-linear flexible connections for static problems. In the former paper, the rotational spring was modelled by a connecting element, and treated as an independent element. The ‘connecting element’ approach could easily handle the non-linear flexible connection, but results in an increase in the number of degrees of freedom. In the latter paper, the rotational spring had been treated as an element first, then the element was condensed statically, leading to a more complicated formulation. One purpose of the present paper is to present a very simple and natural approach to account for the behaviour of non-linear flexible connections, without an increase in the number of degrees of freedom. Furthermore, it is demonstrated in this paper that the complementary energy approach, involving the weak form of the compatibility of member deformations, is the simplest way to account for the additional deformation caused by flexible joints.

When the space frame (with non-linearly flexible joints) is subjected to a dynamic load, the non-linearity of the connections will lead to hysteresis loops in the moment–rotation curves and some energy will be dissipated in the connections. Popov and co-workers demonstrated by experiments that the hysteresis loops under repeated and reversed loading are very stable, so, the moment–rotation curves obtained by static experiments can be extended to the dynamic analysis of flexibly connected frames. Up to now, except for a few experimental works, the influence of non-linear flexible connection on the dynamic behaviour of structures was studied analytically in few papers. In the paper by Kawashima and Fujimoto about vibration analysis, they just calculated the frequencies of the structures with linear semi-rigid connections. The effect of hysteretic damping resulting from the non-linear flexible connection on the dynamic response of the frame will be studied in this paper.

MODELLING OF NON-LINEAR FLEXIBLE CONNECTIONS

In general, the behaviour of flexible connections is expressed by its moment–rotation curves. Because of the complexity, the moment–rotation relation is usually determined by experiments. By curve fitting the experimental data, many types of functions were generated to approximate the non-linear behaviour of the flexible connections. Because the behaviour of non-linear flexible connections is actually represented, in a computational model, by the instantaneous rotational stiffness of the connection, i.e. the slope of the moment–rotation curve, the property of the function used for the moment–rotation curve is very important for the numerical analysis of
flexibly jointed frame structures. The polynomial function has a considerable accuracy for the moment–rotation relation, but it may give undesirable negative ‘connection-stiffness’ since the nature of a polynomial is to peak and trough within a certain region. The exponential function used by Chen and Lui\(^9\) can always give a positive derivative, but its stiffness expression is not convenient to be implemented for programming, especially for dynamic problems. The standardized Ramberg–Osgood function for the moment–rotation relation proposed by Ang and Morris\(^9\) has many advantages. It not only can describe the non-linear behaviour very well just by three parameters, and guarantee a positive derivative, but also can take into account of the additional moment at the connection due to the \(P-\Delta\) effect\(^9\) and give a convenient expression for the

\[
\phi = \frac{K_{sM}}{\phi_0} \left[ \frac{1}{\phi_0} \left( \frac{K_{sM}}{\phi_0} \right)^{n-1} \right]
\]

Figure 1(a). The Ramberg–Osgood function

![Graph showing the Ramberg–Osgood function with variables K_sM, \(\phi_0\), and \(\phi\).]

\[
S = \frac{dM}{d\phi}
\]

Figure 1(b). Behaviour of non-linear flexible connection and the rotational rigidity of the connection
instantaneous rotational rigidity of a non-linear flexible connection, as will be demonstrated later on in this paper.

The standardized moment-rotation behaviour is expressed by the Ramberg-Osgood function as

$$\frac{\phi}{\phi_0} = \frac{K \cdot M}{(KM)_0} \left[ 1 + \left( \frac{KM}{(KM)_0} \right)^{n-1} \right]$$

(1)

where $\phi$ is the relative rotation at the connection, $M$ is its corresponding bending moment, $\phi_0$, $(KM)_0$ and $n$ are parameters in which $n$ is a positive real number. As illustrated in Figure 1(a), the three parameters control the shape of the moment-rotation curve. The parameters can be easily determined for a given type of connection. The standardized Ramberg-Osgood functions of moment-rotation relations for five typical connections were given in Ang and Morris' paper.\(^8\)

Differentiating equation (1) with respect to $M$ gives

$$\frac{1}{\phi_0} \frac{d\phi}{dM} = \frac{K}{(KM)_0} \left[ 1 + n \left( \frac{KM}{(KM)_0} \right)^{n-1} \right]$$

(2)

Then the instantaneous rotational rigidity of the non-linear flexible connection, i.e. the slope of the moment-rotation curve, denoted by $S'$, is given by (see Figure 1(b))

$$S' = \frac{dM}{d\phi} = \frac{(KM)_0}{\phi_0 K \left[ 1 + n \left( \frac{KM}{(KM)_0} \right)^{n-1} \right]} \quad M > 0$$

(3)

and the initial rigidity $S^0$ is

$$S^0 = \left. \frac{dM}{d\phi} \right|_{M=0} = \frac{(KM)_0}{\phi_0 K}$$

(4)

Considering that the moment $M$ at the connection can be positive or negative, the general form of equation (3) corresponding to loading or reloading, i.e. the case when $M \, dM > 0$, can be written as

$$S' = \frac{S^0}{1 + n \left( \frac{K|M|}{(KM)_0} \right)^{n-1}}$$

(5)

and the initial rigidity $S^0$ is used for the case of unloading in which $M \, dM < 0$.

The instantaneous rigidity of the non-linear connection, $S'$, in equation (5) is a function of the moment at the connection. Such an expression for $S'$ has some advantages. Because the moment acting on the connection is the same as that at the connecting end of the structural member, the connection moment can be very easily evaluated. Therefore equation (5) is very suitable for a computational model.

The non-linear equation of moment-rotation relation can be solved by its incremental form. Supposing $M_i$ and $\phi_i$ were known, then

$$M_{i+1} = M_i + \Delta M$$

(6)

Corresponding to the increment $\Delta M$, the increment of $\phi$ can be approximated as

$$\Delta \phi = \Delta M / S'$$

(7)
and

$$\phi_{i+1} = \Phi_{i} + \Delta \phi$$

(8)

The difference between $\Phi = \phi(M_{i+1})$ defined in equation (1) and $\Phi_{i+1}$ approximated in equation (8) can be corrected by some iterative scheme, and equations (6) and (8) become

$$M_{i+1} = M_{i} + \sum_{k} \Delta M^{(k)}$$

(6a)

$$\phi_{i+1} = \phi_{i} + \sum_{k} \Delta M^{(k)}/S_{k}$$

(8a)

in which the superscript $k$ indicates the iteration.

**TANGENT STIFFNESS OF A FLEXIBLY CONNECTED BEAM COLUMN: AN EXPLICIT EXPRESSION**

The behaviour of a non-linear flexible connection can be represented by a non-linear rotational spring in which the rotational rigidity of the spring corresponds to the slope of the moment-rotation curve. The flexibly jointed frame is modelled as a collection of beam-columns with rotational springs at the ends of each beam-column. Unlike the model used in the authors’ previous paper,\(^{11}\) the beam together with the springs is considered as a single element in the present study.

A procedure for the elasto-plastic large deformation analysis of rigidly jointed space frames was presented by Shi and Atluri,\(^{14}\) which was based on the weak forms of governing equations, assumed stress fields and the incremental forms for the non-linear equations. When the beam-column is connected by rotational springs at its ends, the compatibility conditions for the beam as a whole will be different from those in Reference 14. We examine the new compatibility conditions here.

A typical beam element with rotational springs at its ends is illustrated in Figure 2 in which $X$ is the global reference system, $X'$ and $X''$ are the co-ordinates in undeformed and deformed configurations respectively. It was assumed that the element has a constant cross-section with two perpendicular principal axes. Therefore, the governing equations of an element can be established in the principal axes of the element.

Because of the non-linearity of the connection, the compatibility conditions will be written in the incremental form.

As illustrated in Figures 2 and 3, in the presence of rotational springs at the ends of the beam-column, the incremental nodal rotations measured in the 'co-rotational' local basis $\hat{e}$, denoted by $\Delta \theta^{*}_{i}$ ($i = 2, 3$, $x = 1, 2$), are composed of two parts: $\Delta \theta_{i}^{**}$ and $\Delta \theta_{\phi_{i}}$, where $\Delta \theta_{i}^{**}$ is the part resulting from the elastic beam and $\Delta \theta_{\phi_{i}}$ is the contribution of the rotational spring. $S_{i}$ is the instantaneous rigidity of the spring about axis $x_{i}$ and at node $x$.

Using the nomenclature of deformations as defined in Figure 2, the incremental curvatures in the local co-ordinates of the deformed element are given by

$$\Delta k_{1} = \Delta \theta_{1,1}^{*}$$

$$\Delta k_{2} = -\Delta \theta_{2,1}^{*}$$

$$\Delta k_{3} = \Delta \theta_{3,1}^{*}$$

(9)

Let $\Delta N$, $\Delta M_{i}$ ($i = 1, 2, 3$) be the increments of the Cauchy stress resultant and stress couples in the
\[ \Delta \phi_i = \Delta \phi_{i+1} + \Delta \Theta_i \quad (i = 2, 3) \]
\[ \Delta \phi_i = \Delta \phi_{i+1} + \Delta \Theta_i \quad (i = 2, 3) \]

Figure 2. Increments of element local rotations

Figure 3. A typical beam-column with rotational springs at the ends in \( x_1-x_2 \) plane
convected element co-ordinates, then, as shown by Shi and Atluri,\textsuperscript{14} the trial stress fields obtained from the linear and angular momentum balance relations are

\[
\Delta N = \Delta n \\
\Delta M_1 = \Delta m_1 \\
\Delta M_2 = (1 - \dot{x}_1/l)\Delta^1 m_2 + (\dot{x}_1/l)\Delta^2 m_2 \\
\Delta M_3 = (1 - \dot{x}_1/l)\Delta^1 m_3 + (\dot{x}_1/l)\Delta^3 m_3
\]  

(10)

where \( \Delta m_i (i = 1, 2; i = 2, 3) \) are the moments at the element ends, \( \dot{x} \) refers to the node and \( \dot{x} \) refers to the direction. Corresponding to the incremental stress resultant and stress couples, the increment \( \Delta W_e \) of the complementary energy density will be

\[
\Delta W_e = \frac{1}{2} \left[ \frac{(\Delta N)^2}{EA} + \frac{(\Delta M_1)^2}{EI_1} + \frac{(\Delta M_2)^2}{EI_2} + \frac{(\Delta M_3)^2}{EI_3} \right]
\]  

(11)

where \( EA \) is the tensile stiffness of the element and \( EI_i (i = 1, 2, 3) \) is the flexural rigidity of the element about axis \( \dot{x}_i \).

The stress-strain relations between the conjugate pairs of the mechanical and kinematic variables are of the form

\[
\frac{\partial \Delta W_e}{\partial (\Delta N)} = \Delta h \\
\frac{\partial \Delta W_e}{\partial (\Delta M_1)} = \Delta k_1
\]

\[
\frac{\partial \Delta W_e}{\partial (\Delta M_2)} + \delta(\dot{x}_1)\Delta^1 \phi_2 + \delta(\dot{x}_1 - l)\Delta^2 \phi_2 = -\Delta k_2
\]

\[
\frac{\partial \Delta W_e}{\partial (\Delta M_3)} + \delta(\dot{x}_1)\Delta^1 \phi_3 + \delta(\dot{x}_1 - l)\Delta^3 \phi_3 = \Delta k_3
\]

(12)

in which \( \Delta h \) is the incremental stretch in \( \dot{x}_i \) direction, \( \delta \) is the Dirac delta function and \( \delta(\dot{x}_i - x_0) \Delta \phi \) represents the effects of the rotational springs at the ends of the element. For a small strain problem we have

\[
\Delta h = \Delta u_{\dot{x}_i,1}
\]

(13)

Rather than considering the point-wise compatibility conditions, the weak form of the compatibility conditions is written for the element as a whole. Let \( \Delta v \) be the test function corresponding to \( \Delta N \), \( \Delta \mu_i (i = 1, 2, 3) \) are the test functions corresponding to \( \Delta M_i \), then the weak forms of the compatibility conditions are

\[
\int_0^l \frac{\partial \Delta W_e}{\partial (\Delta N)} \Delta v \, d\dot{x}_1 = \int_0^l \Delta h \Delta v \, d\dot{x}_1 = \Delta v \int_0^l \Delta u_{\dot{x}_i,1} \, d\dot{x}_1 = \Delta v(\Delta u_{\dot{x}_i,1})\bigg|_0^l = \Delta v \Delta H
\]

(14)

\[
\int_0^l \frac{\partial \Delta W_e}{\partial (\Delta M_1)} \Delta \mu_1 \, d\dot{x}_1 = \int_0^l \Delta \theta_{\dot{x}_i,1} \Delta \mu_1 \, d\dot{x}_1 = (\Delta^2 \theta_{\dot{x}_i} - \Delta^3 \theta_{\dot{x}_i}) \Delta \mu_1
\]

(15)
\[
\int_0^L \left[ \frac{\partial W_e}{\partial (\Delta M_2)} + \delta(x_1) \Delta^1 \phi_2 + \delta(x_1 - l) \Delta^2 \phi_2 \right] \Delta \mu_2 \, d\xi_1
\]
\[
= \int_0^L \left[ \frac{\partial W_e}{\partial (\Delta M_3)} \Delta \mu_2 \, d\xi_1 + \Delta^1 \phi_2 \Delta^1 \mu_2 + \Delta^2 \phi_2 \Delta^2 \mu_2 = \Delta^2 \theta^*_2 \Delta^2 \mu_2 - \Delta^1 \theta^*_2 \Delta^1 \mu_2 \right]
\]
\[
(16)
\]
\[
\int_0^L \left[ \frac{\partial W_e}{\partial (\Delta M_3)} \Delta \mu_3 \, d\xi_1 + \Delta^1 \phi_3 \Delta^1 \mu_3 + \Delta^2 \phi_3 \Delta^2 \mu_3 = \Delta^2 \theta^*_3 \Delta^2 \mu_3 - \Delta^1 \theta^*_3 \Delta^1 \mu_3 \right]
\]
\[
(17)
\]
In equations (16) and (17), on the right hand side, the identities \( \int_0^l \theta^*_x \, d\xi_1 = 0 \) and \( \int_0^l \theta^*_y \, d\xi_1 = 0 \) are used. The increments of the spring rotations can be expressed as
\[
\Delta^* \phi_i = (-1)^{\alpha} \frac{\Delta^2 M_i}{S_{ij}} \quad \text{for each } i = 2, 3; \text{ and } \alpha = 1, 2
\]
(18)
where \( S_i \) is the instantaneous rigidity of the spring as defined in equation (5). Substituting equations (10), (11) and (18) into equations (16) and (17), and recalling \( \Delta \mu_i \) is the variation of \( \Delta M_i \), one has
\[
(\Delta \mu_i, \Delta \theta_i) \frac{l}{6EI_l} \begin{bmatrix} 2+a_i & 1 \\ 1 & 2+b_i \end{bmatrix} \Delta M_i = (\Delta \mu_i)^T \Delta \theta_i \quad \text{no sum on } i; \text{ for } i = 2, 3
\]
(19)
in which \( EI_i \) and \( l \) are the flexural rigidity and length of the element respectively, and the following notations are used:
\[
\Delta \mu_i = \{ \Delta^1 \mu_i, \Delta^2 \mu_i \}^T
\]
\[
\Delta \theta_i = \{ -\Delta^1 \theta^*_i, \Delta^2 \theta^*_i \}^T
\]
\[
\Delta M_i = \{ \Delta^1 M_i, \Delta^2 M_i \}^T
\]
\[
a_i = \frac{6EI_i}{S_{ij}}, \quad b_i = \frac{6EI_i}{2S^2_{ij}}
\]
(20)
Equation (19) can also be rewritten as
\[
\Delta M_i = \frac{6EI_i}{l} \frac{1}{(2 + a_i)(2 + b_i) - 1} \begin{bmatrix} 2 + b_i & -1 \\ -1 & 2 + a_i \end{bmatrix} \Delta \theta_i
\]
\[(\hat{A}^{-1}_{oo})^T \Delta \theta_i \quad \text{(no sum on } i, \text{ for } i = 2, 3)\]
(21)
In equation (17), \( a_i \) and \( b_i \) are the modifying terms resulting from the rotational springs at the element ends. It is evident that equation (19) goes back to the rigid connection when \( S \to \infty \), \( 2S \to \infty \), and to the pinned connection when \( S \to 0 \), \( 2S \to 0 \). Therefore, equations (19) and (20) are valid for all kinds of connections.

By letting
\[
\Delta \sigma = \{ \Delta n, \Delta m_1, \Delta^1 m_2, \Delta^2 m_2, \Delta^1 m_3, \Delta^2 m_3 \}^T
\]
\[
\Delta d = \{ \Delta H, (\Delta^2 \theta^*_1 - \Delta^2 \theta^*_2), -\Delta^1 \theta^*_2, \Delta^2 \theta^*_2, -\Delta^1 \theta^*_3, \Delta^2 \theta^*_3 \}^T
\]
(22)
\[
\hat{A}^{-1}_{oo} = \begin{bmatrix} \frac{EI_i}{l} & 0 \\ \hat{A}^{-1}_{oo} & \end{bmatrix}
\]
(23)
(24)
Equations (14)-(17) can be written in a matrix form

\[ \Delta \sigma = \hat{A}_{sd}^{-1} \Delta d \]  

(25)

For the small deformation problem, \( \hat{A}_{sd}^{-1} \) is the element stiffness matrix in the local co-ordinates \((\hat{\epsilon})\). \( \hat{A}_{sd}^{-1} \) can be transformed into the global co-ordinates in the usual manner. It should be pointed out that the element stiffness matrix for the large deformation problem is the combination of the transformations of \( \hat{A}_{sd}^{t} \) with a deformation dependent matrix, as explained in Shi and Atluri’s paper.\(^{14}\) It should be noted that the derivation of the explicit expression for the tangent stiffness matrix for the finitely deformed beam follows the same lines as in Reference 14, except for the modifications in equations (14)-(25) above.

Referring to the complementary energy analysis of finitely deformed, rigidly jointed frames as discussed in Shi and Atluri,\(^{14}\) the only difference in the analysis due to the presently considered non-linear, hysteretic joint flexibility is the following: the incremental compatibility conditions in the presence of flexible joints, i.e. equations (14)-(17) above, contain additional terms as compared to the incremental counterparts of the compatibility conditions, equations (41)-(44) in Reference 14, for rigidly connected frames. Following the details given in equations (41)-(78) as well as the Appendix of Reference 14, it is seen that the matrix \( A_{sd} \) of equations (A21)-(A24) of Reference 14 must now be replaced by the matrix \( \hat{A}_{sd} \) given in equation (24) above. Thus, the explicit expression for the tangent stiffness matrix of a flexibly connected, finitely deformed space frame is given by

\[ K = A_{sd}^{t} \hat{A}_{sd}^{-1} A_{sd} + A_{dd} \]  

(26)

wherein the expressions for \( A_{sd}^{t}, A_{sd}, A_{dd} \) are given in the Appendix of Reference 14. It should be noted that \( \hat{A}_{sd}^{-1} \) is explicitly given in equation (24), and the explicit expressions (without involving element integrations) for \( A_{sd}, A_{dd} \) and \( A_{dd} \) are given in Reference 14. It may be seen that \( \hat{A}_{sd}^{t} \) is the so-called ‘linear’ stiffness matrix in the local co-ordinates \( \hat{\epsilon} \) of the deformed beam-member, \( A_{dd} \) and \( A_{dd} \) are deformation-dependent transformations; and \( A_{dd} \) is the additional deformation-dependent stiffness.

The analysis of the frame with non-linear flexible connections is a kind of materially non-linear analysis. The system non-linear equations is approximated by the incremental tangent stiffness matrix equations, and a modified arc-length method\(^{15}\) is employed to solve the system equations.

**HYSTERETIC DAMPING RESULTING FROM THE NON-LINEAR FLEXIBLE CONNECTION**

Under the dynamic loading, the non-linearity of the flexible connection will result in the hysteretic loop in the moment–rotation curve from the cyclic displacement at the connection. The area enclosed by the loop corresponds to the energy dissipated in a cycle of the oscillation. Such hysteretic damping plays a very important role for the dynamic behaviour of the structure. Because the hysteretic loop of the non-linear flexible connection is very highly reproducible, even in the presence of the pronounced local buckling at the connection,\(^{12,13}\) as illustrated in Figure 4, the moment–rotation curve obtained by static experiments can be extended to dynamic analysis. So, the Ramberg–Osgood function used for the static problem is employed here again. In this case, the hysteretic loop is determined by the Ramberg–Osgood function, the unloading criterion and the initial rigidity of the connection. The unloading criterion is

\[ M \Delta M < 0 \]  

(27)

where \( M \) is the total moment at the connection and \( \Delta M \) is its increment and the unloading
criterion is checked in each time step for all rotational springs. Corresponding to the loading and unloading, the instantaneous rigidity of the spring for the dynamic problem \( S_d \) will take, respectively, the value

\[
S_d = S^1 \quad \text{for loading} \tag{28}
\]

\[
S_d = S^0 \quad \text{for unloading} \tag{29}
\]

Similarly, the element stiffness matrix \( \tilde{A}_{es}^{-1} \) in the local co-ordinates will be

\[
\tilde{A}_{es}^{-1} = \tilde{A}_{es}^{-1}(S^1) \quad \text{for loading} \tag{30}
\]

\[
\tilde{A}_{es}^{-1} = \tilde{A}_{es}^{-1}(S^0) \quad \text{for unloading} \tag{31}
\]

in which \( S^1 \) was defined in equation (5), \( S^0 \) in equation (4) and in equation (24). The tangent stiffness matrix of element in the global co-ordinates can be built up by the local tangent element stiffness matrix \( \tilde{A}_{es}^{-1} \) in the same way as in the static problem, but takes different rigidities for the spring according to loading and unloading to represent the hysteretic loop of the moment–rotation curve. Let \( K^e = \sum_{\text{elements}} K_i^e \) be the system tangent stiffness matrix, \( M \) the system mass matrix, \( C \) the damping coefficient matrix (\( M \) and \( C \) can be established in the usual way), then the equations of motion of a non-linear system can be written in the incremental form

\[
M \Delta \ddot{u} + C \Delta u + K^e \Delta u = \Delta P \tag{32}
\]

where \( \Delta u \), \( \Delta \dot{u} \), \( \Delta \ddot{u} \) and \( \Delta P \) are increments of displacement, velocity, acceleration and external loading vectors. After establishing \( K^e \) by equations (30) and (31), equation (32) can be solved by any time integration scheme.\(^{16}\)

**NUMERICAL EXAMPLES**

The efficiency and accuracy of the present approach are demonstrated by the following numerical examples. In order to compare the present results with others, three examples which were solved by other researchers are analysed here again.
Example 1

The three storey single bay frame shown in Figure 5(a) is taken from Arbabi's paper and the linear semi-rigid connection was used there. However, the Young's modulus was not given there. The lateral displacements corresponding to rigid and semi-rigid connection are shown in Figures 5(b) and (c) respectively. While the two sets of results are close for the case of a rigid connection, present results for the flexible connection are different from those in Reference 7, but close to those of the three storey three bay frame in the same paper. A possible reason for the difference is that the single bay frame was reduced to a cantilevered beam by some assumption in Reference 7.

Example 2

This example concerns a T-shaped frame with a non-linear flexible connection. The geometry and load conditions are shown in Figure 6 and the behaviour of the connection is given by the

![Diagram of a T-shaped frame with load conditions and lateral deflections](image)

Figure 5. Three storey single bay frame

![Diagram of a T-shaped frame with load sequence 1 and 2](image)

Figure 6. Geometry and loading conditions
moment–rotation curve, as illustrated in Figure 7. Because only the slope of the curve is used in
the modelling of the connection, it is better to interpolate the derivative of the curve directly
rather than to fit the curve. The ratios of the rotation/translation at point A for both the rigid and
flexible connections are plotted in Figure 8. The results agree quite well with those given by Lui.17
Some similar results can be found in Chen and Lui’s paper.9

Example 3

The geometric and material data of a simple three-dimensional frame are given in Figure 9. The
non-linear moment–rotation behaviour of the connection that Lui used for a similar plane
toggle17 is illustrated in Figure 10. The deflections for both cases of rigid supports and flexible
supports for the 3-D case of Figure 9 are shown in Figure 11.

Figure 7. The flexible connection and its moment–rotation behaviour

Figure 8. The ratio of rotation/translation at point A
Example 4

Now we study two dynamic problems. Kawashima and Fujimoto gave the analytical and experimental natural frequencies of the L-type frame with a linear semi-rigid connection, as indicated in Figure 12, and material property $EI = 4230$ kg cm², $v = 0.47$. The present results are also shown in the figure. It can be seen that the present results are very good.
Figure 11. The load-deflection curve at point A

Figure 12. Geometry and natural frequencies of L-frame with rigid connection and semi-rigid connection

Example 5

Kawashima and Fujimoto considered only the influence of a linear flexible connection on the frequencies. Here we will consider the effects of a non-linear flexible connection on the dynamic response. A header plate connection with the instantaneous rigidity

$$S' = \frac{1400}{1 + 4.32 \cdot 0.1184M^{3.32}}$$
is used for the non-linear connection. The responses of the structure with a linear or non-linear connection are illustrated in Figure 13. The response corresponding to a non-linear connection, indicated by the curve with crosses, shows that the hysteretic damping resulted from the non-linearity of the connection really affects the response considerably.

CLOSURE

The weak form of the incremental governing equations in the complementary energy approach provides a simple way to handle the non-linear behaviour of flexible connections. By the present approach, there is no difference in the treatment for a flexibly jointed plane frame and a flexible jointed space frame, and there is no difficulty in handling a flexible torsional connection if such data are available. The Ramberg-Osgood function not only can represent the non-linear behaviour well, but also has some advantage for computational modelling. In order to consider the structural response of lattice structures more accurately, non-linear flexible connections should be used for both static and dynamic problems. The numerical examples demonstrate that the approaches presented in this paper are very efficient and accurate for both static and dynamic analysis of flexibly jointed space structures.

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