An analysis of an explicit algorithm and the radial return algorithm, and a proposed modification, in finite plasticity

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Abstract. This paper presents an analysis of Nemat-Nasser's (1991) explicit algorithm and a radial return algorithm for isotropic Von Mises materials undergoing large deformations. It is found that the final-radial return algorithm can result in a simple scalar equation (for the constant of proportionality that defines the plastic flow), which can be easily solved. A new modified algorithm is also proposed. All the three algorithms are efficient. No iterative scheme is required. For proportional loading, all the three algorithms work well. But, in cases where the direction of stress cannot well follow the direction of the deformation rate, which occurs when spin effect is significant or when the direction of deformation rate keeps changing, the explicit algorithm has a problem with its convergence; the final-radial return algorithm oscillates for large time steps; while the presently proposed modified algorithm can provide reasonable results even for large time steps, without any convergence problems.

Notation

- \( \dot{\gamma} \) objective rate of (\cdot)
- \( D \) rate of deformation
- \( D^p \) plastic deformation rate
- \( \Omega \) spin tensor
- \( \mu = \sigma \cdot \sigma \cdot \sigma \) current elasticity tensor
- \( \sigma \) Kirchhoff stress
- \( D = \dot{D} + \sqrt{D : D^T} \) rotation tensor
- \( \gamma = \sqrt{1 \cdot D : D^T} \) shear parameter
- \( h \) hardening parameter
- \( n \) hardening exponent
- \( \dot{\gamma} \) deviatoric part of (\cdot)
- \( D^e \) elastic deformation rate
- \( C \) current elasticity tensor
- \( \tau = \sqrt{1 \cdot \sigma \cdot \sigma \cdot \sigma} \) identity tensor

1 Introduction

The integration of elastoplastic constitutive relations has been studied by many authors. The radial return algorithm has been considered as one of the most efficient and accurate methods. In general, the radial return algorithm requires iterative schemes to solve a nonlinear equation or a set of nonlinear equations. Nemat-Nasser (1991) proposed an explicit algorithm based on some physical arguments. It was found that this algorithm works well for certain cases.

In this paper, an analysis of Nemat-Nasser's (1991) algorithm and the radial return algorithm is presented, for isotropic Von Mises materials with isotropic hardening. A new modified algorithm is also proposed in this paper. Also, it is found that the final-radial return algorithm can result in a simple nonlinear equation for isotropic Von Mises materials. This analytical and numerical results show that the performance of Nemat-Nasser's (1991) explicit algorithm is not satisfactory when the direction of stress keeps changing, as for non-proportional loading. In this case, Nemat-Nasser's (1991) explicit algorithm may have problems in convergence; the final-radial return algorithm may oscillate for large time step marching; but the presently proposed modified algorithm has no problems in convergence and can give a reasonable solution even for large time steps.
2 Preliminaries

Assume that the rate of deformation is decomposed as

\[ D = D^e + D^p \quad (1) \]

where \( D \) is the rate of deformation, \( D^e \) is plastic deformation rate, \( D^p \) is plastic deformation rate.

An objective rate of Kirchhoff stress, \( \dot{\sigma} \), is determined by the elastic deformation rate tensor, \( D^e \), through the current elasticity tensor, \( C \), as:

\[ \dot{\sigma} = C : D^e. \quad (2) \]

We will use the following form of an objective rate of stress

\[ \dot{\sigma} = \dot{\sigma} - \Omega \dot{\Omega} + \sigma \Omega \]

where \( \Omega \) is some spin tensor. The objective rate is the Jaumann rate, if we take \( \Omega \) as the skew-symmetric part of the velocity gradient.

Nemat-Nasser (1991) introduced a normalized deviatoric tensor \( \mu \), defined as:

\[ \mu = \frac{\sigma'}{\sqrt{2\tau}} \quad (4) \]

where \( \sigma' \) is the deviatoric part of Kirchhoff stress \( \sigma \); and \( \tau = \sqrt{1/2 \sigma' : \sigma'}. \) This tensor specifies the direction of the deviatoric part of Kirchhoff stress as well as the plastic deformation rate \( D^p \) for materials with a Von Mises type yield surface, with isotropic hardening.

A Von Mises type yield surface, with isotropic hardening, can be written in the following form:

\[ f' = \tau - F(\gamma) = 0 \quad (5) \]

where \( \gamma = \int_0^1 \dot{\gamma} \, dt \) and \( \dot{\gamma} = \sqrt{D^p : D^p}. \)

Notice that for pure shear, \( \tau = \sigma_{12} = \sigma_{21} \) and \( \dot{\gamma} = \sqrt{2D_{12}} = \sqrt{2D_{21}}; \) and the effective stress and the effective plastic strain are defined as \( \tau_{\text{eff}} = \tau \) and \( \gamma_{\text{eff}} = \sqrt{2\gamma}. \)

In terms of the normalized deviatoric tensor \( \mu \) defined above, the plastic strain rate is

\[ D^p = \begin{cases} 0 & \text{in elastic loading or unloading} \\ \dot{\gamma} \mu & \text{in plastic loading}. \end{cases} \quad (6) \]

The loading condition is determined by

\[ \mu : C : D \begin{cases} > 0 & \text{plastic loading} \\ = 0 & \text{neutral} \\ < 0 & \text{elastic unloading} \end{cases} \quad (7) \]

where \( C : D \) denotes the "trial" value of the stress rate.

We eliminate the elastic deformation rate in Eq. (2) using Eq. (1) and Eq. (6). Then the stress rate becomes:

\[ \dot{\sigma} = C : (D - \dot{\gamma} \mu). \quad (8) \]

for plastic loading.

Since \( \mu \) is a normalized deviatoric tensor, \( \mu : \mu = 1 \) and \( \dot{\mu} : \mu = 0, \mu : \mu = 0. \) It can be verified from the definition of \( \tau \) that

\[ \mu : \dot{\sigma} = \mu : \dot{\sigma}' = \mu : \dot{\sigma} = \mu : \dot{\sigma}' = \sqrt{2\dot{\gamma}}. \quad (9) \]

Then, a scalar equation in rate form results from Eq. (8) and Eq. (9), i.e.

\[ \sqrt{2\dot{\gamma}} = \mu : C : D - \dot{\gamma} (\mu : C : \mu). \quad (10) \]
Two scalar quantities can be defined from Eq. (10). They are

\[
d = \frac{\mu : C : D}{\mu : C : \mu} \quad \text{and} \quad a = \frac{1}{\sqrt{2}} (\mu : C : \mu).
\]

For an isotropic material, \(a = \sqrt{2}G\) and \(d = \mu : D = \mu : D'\).

Consider continuous plastic loading. The evolution of the yield surface, Eq. (5), gives

\[
\dot{\gamma} = \ddot{\gamma} - F(\gamma) = 0.
\]

A hardening parameter is introduced from Eq. (11).

\[
h(\gamma) \equiv F(\gamma) = \ddot{\gamma}.
\]

We make use of these parameters to simplify the expressions. Eq. (10) and Eq. (12) give

\[
\dot{\gamma} = \frac{a + \dot{\gamma}}{a + h(\gamma)}
\]

where \(a\) is a constant for an isotropic material only; while \(d\) and \(h\) are, in general, functions of the deformation.

### 3 Radial return algorithm

Nikishkov and Atluri (1992) have derived a nonlinear equation for the scalar multiplier that determines the plastic strain for the generalized mid-point algorithm. Although the generalized mid-point-radial return algorithm results in a complicated nonlinear equation, the final-radial return algorithm may lead to a scalar equation which can be solved very easily. Actually, the change in hardening rate, \(h(\gamma)\), in a loading step is usually very much smaller than the shear modulus. Thus, the hardening rate can be taken as a constant in the integration step; and the equation can be solved explicitly. This one-step approximation is usually accurate enough for the resulting scalar equation.

The generalized mid-point-radial return algorithm involves the integration of Eq. (8) to find the stress components at the next time step. For isotropic materials, we can find the deviatoric and hydrostatic parts of stress separately. The deviatoric part is integrated as follows:

\[
\int_0^{\Delta t} \dot{\sigma} \, dt = C : D' \Delta t - C : \mu^* \int_0^{\Delta t} \dot{\gamma} \, dt. \tag{15}
\]

We invoked the approximation that \(\mu^* = \dot{\gamma} [(1 - \beta) \mu(t) + \beta \mu(t + \Delta t)]\) where \(0 \leq \beta \leq 1\). \(\dot{\gamma}\) is a parameter that is used to keep \(\mu^*\) normalized. Since \(\mu^* : \mu^* = 1\), we have

\[
\Delta_{\gamma} = \dot{\gamma} \Delta_{\gamma}' \sqrt{(1 - \beta)^2 + \beta^2 + 2(1 - \beta) \mu(t + \Delta t) \mu(t + \Delta t)}. \tag{17}
\]

Let's assume, for simplicity, that there is no spin here, i.e., \(\Omega = 0\). Thus, \(\sigma = \dot{\sigma}\). Spin will be treated objectively in Sect. 5. Now, Eq. (16) can be written in the following form.

\[
\dot{\gamma} [(1 + \Delta t) \mu(t) + \Delta t] = \Delta_{\gamma} \sqrt{2 \dot{\gamma} \mu(t) + C : D' \Delta t - \Delta_{\gamma} \mu [(1 - \beta) \mu(t) + \beta \mu(t + \Delta t)]}. \tag{18}
\]

Since \(C : D' = 2G D'\) and \(C : \mu(t + \Delta t) = 2G \mu(t + \Delta t)\) for an isotropic material (note that \(\sqrt{2G} = a\)), we can solve for the direction of stress at time \(t + \Delta t\), i.e., \(\mu(t + \Delta t)\), from Eq. (18).

\[
[(\mu(t + \Delta t) + \Delta_{\gamma}) \mu(t + \Delta t) = [(\mu(t) + \Delta t) \mu(t) + \Delta \mu(t)] \mu(t) + \Delta D' \Delta t. \tag{19}
\]

Using \(\mu(t + \Delta t) : \mu(t + \Delta t) = 1\), we can solve for \(\mu(t + \Delta t)\) and \(\Delta(t + \Delta t)\) in terms of \(\Delta_{\gamma}\) from Eq. (19).
\[ \frac{\tau(t + \Delta t)}{a} = -\beta \lambda \Delta \gamma + \sqrt{\left(\frac{\tau(t)}{a}\right)^2 + 2 \frac{\tau(t)}{a} \mathbf{D} : \mathbf{d}(t) \Delta t + (\mathbf{D} : \tilde{\mathbf{D}} \Delta t)^2}; \]  

(20)

\[ \mu(t + \Delta t) = \frac{\frac{\tau(t)}{a} \mathbf{d}(t) + \mathbf{D}^t \Delta t}{\sqrt{\left(\frac{\tau(t)}{a}\right)^2 + 2 \frac{\tau(t)}{a} \mathbf{D} : \mathbf{d}(t) \Delta t + (\mathbf{D} : \tilde{\mathbf{D}} \Delta t)^2}} \]  

(21)

where \( \xi = \tau(t) - a(1 - \beta \lambda \Delta \gamma); \) and \( \tilde{\mathbf{D}} = \mathbf{D}^t / \sqrt{\mathbf{D} : \mathbf{D}}. \)

Substituting Eq. (21) into Eq. (17), we have \( \Delta \gamma \) in terms of \( \lambda \Delta \gamma. \) Thus, \( \gamma(t + \Delta t) \) and \( \tau(t + \Delta t) \) are in terms of the plastic multiplier \( \lambda \Delta \gamma. \) The equation for the yield surface, Eq. (5), becomes a scalar nonlinear equation for plastic multiplier \( \lambda \Delta \gamma. \) We solve this scalar equation for \( \lambda \Delta \gamma; \) then, we have the updated stress in Eq. (20) and Eq. (24).

The final-radial return algorithm is a special case of the generalized mid-point algorithm. In the final-radial return algorithm, \( \beta = \lambda = 1 \) and, as a result, \( \xi = \tau(t). \) Thus, Eq. (20) can be simplified as

\[ \frac{\tau(t + \Delta t)}{a} + \Delta \gamma = \sqrt{\left(\frac{\tau(t)}{a}\right)^2 + 2 \frac{\tau(t)}{a} \mathbf{D} : \mathbf{d}(t) \Delta t + (\mathbf{D} : \tilde{\mathbf{D}} \Delta t)^2}. \]  

(22)

Equation (22) can be used to estimate the time step needed for the stress state to reach yield surface in elastic loading or to return back to yield surface in elastic unloading. In elastic unloading, we can set \( \Delta \gamma = 0 \) and \( \tau(t + \Delta t) = \tau(t) \) to solve for this time step. It is

\[ \Delta t_e = -\frac{2 \frac{\tau(t)}{a} \mathbf{D} : \mathbf{d}(t)}{(\mathbf{D} : \tilde{\mathbf{D}} \Delta t)^2}. \]  

(23)

In initial elastic loading, one can solve \( \Delta t_e \) with case by setting \( \Delta \gamma = 0 \) and \( \tau(t + \Delta t) = \tau(t), \) where \( \tau(t) \) is the initial effective yield shear stress. If the estimated time step \( \Delta t_e \) is smaller than the marching step \( \Delta t, \) it is used first, as an elastic loading time step, to bring the stress state to the yield surface. The rest of the marching step \( \Delta t - \Delta t_e \) is a plastic loading step.

Since the right-hand side is known, Eq. (22) is easy to solve. Note that \( \tau(t) = F(\gamma) \) and \( \tau(t + \Delta t) = F(\gamma + \Delta \gamma) \) in plastic loading, a first order expansion of \( \tau(t + \Delta t) \) gives

\[ \frac{a + h}{a} \Delta \gamma = -\frac{\tau(t)}{a} + \sqrt{\left(\frac{\tau(t)}{a}\right)^2 + 2 \frac{\tau(t)}{a} \mathbf{D} : \mathbf{d}(t) \Delta t + (\mathbf{D} : \tilde{\mathbf{D}} \Delta t)^2}. \]  

(24)

4 Nemat-Nasser's (1991) explicit algorithm, and the presently proposed modifications

Nemat-Nasser (1991) integrated Eq. (13) to find the increment in effective stress in plastic loading. He found the updated effective stress and plastic strain increment as follows.

\[ \tau(t + \Delta t) = \frac{a F(\gamma + d^* \Delta t) + h(t + \Delta t) \tau(t)}{a + h(t + \Delta t)} \], \quad \Delta \gamma = \gamma(t + \Delta t) - \gamma(t) = (d^* \Delta t - \tau(t)) / (a + h(t + \Delta t)). \]  

(25, 26)

where \( d^* \Delta t \) is some time average of \( d \) during this time integration step.

Notice that the hardening rate, \( h, \) is usually an order of magnitude smaller than the shear modulus, \( a, \) and the change in \( h \) is even much smaller than \( a. \) Thus, \( a + h \) can be taken as a constant in an integration step. Integrating Eq. (14) directly, we have

\[ \Delta \gamma = \int_{t}^{t + \Delta t} \frac{ad(t) dt}{a + h(t)} \approx \frac{1}{a + h(t)} \int_{t}^{t + \Delta t} \frac{ad(t) dt}{a}. \]  

(27)
Nemat-Nasser's approximation Eq. (26) is equivalent to Eq. (27) up to the first order in \( d^* \Delta t \) from Taylor series expansion.

Assume that the deformation rate is constant, Nemat-Nasser (1991) used the following estimation for an isotropic material.

\[
d^* \Delta t = \int_t^{t+\Delta t} d(t) dt = D : \int_t^{t+\Delta t} \mu(t) dt = D : \left[ \frac{\theta}{2} \mu(t) + \left(1 - \frac{\theta}{2}\right) \bar{D} \right] \Delta t
\]

(28)

where \( \theta = h/(a + h) \). Note that \( \theta \ll 1 \) for most moderately hardening materials. He argued that if the deformation rate is kept constant, the deviatoric stress will finally have the same direction as that of the deviatoric part of the deformation rate, i.e., \( \bar{D} \). This is true only if the spin effect is insignificant (see Eq. (35)).

We now propose a modification to the algorithm to estimate \( d^* \Delta t \). Using an implicit scheme to estimate \( d^* \Delta t \) (in a similar way as that used to estimate \( \mu^* \)), as shown in Eq. (29), \( \Delta \gamma \) and \( d^* \) can be solved from Eq. (27) and Eq. (19), assuming that \( h \) is constant in a time step.

\[
d^* \Delta t = D : \int_t^{t+\Delta t} \mu(t) dt \approx D : \mu(t + \Delta t) \Delta t = D : \mu(t + \Delta t) \Delta t
\]

(29)

\[
d^* \Delta t = \frac{1}{2} \left[ -\frac{\tau(t)}{a} + \sqrt{\left(\frac{\tau(t)}{a}\right)^2 + 4 \frac{\tau(t)}{a} D : \mu(t) \Delta t + 4(D : \bar{D} \Delta t)^2} \right]
\]

(30)

Comparing Eq. (30) (the presently proposed modified algorithm), Eq. (24) (the final-radial return algorithm), and Eq. (28) (Nemat-Nasser's (1991) explicit algorithm), we find the following features of the explicit approach of Nemat-Nasser (1991). Regardless of the size of time step, his approach puts a huge weight on the limiting state, \( D : \bar{D} \) (see Eq. (28)), puts a tiny weight on the initial state \( D : \mu(t) \), and ignores the effect of elastic effective strain, \( \tau(t)/a \). Based on this, we may conclude that this approach can give good results for problems where the plastic deformation is dominant, provided \( D : \mu(t) \) close to 1, which means that the direction of the stress doesn't change too much during the loading step.

It is interesting to notice that the presently proposed modified algorithm Eq. (30), and the final-radial return algorithm Eq. (24), have the same structure in the expressions of the plastic strain increment. The present modified algorithm puts more weight on the limiting state than the final-radial return algorithm does.

5 Spin

Rubinstein and Atluri (1983) have proposed an objective approach to integrate the spin. In this approach, we first find out the stress state in a rotating coordinate system as specified by the spin. Then, we calculate the stress components in the current coordinate system by applying appropriate transformation laws.

Consider the stress, \( \dot{\sigma} \), in a rotating coordinate system. The transformation law gives \( \dot{\sigma} = R^T \sigma R \). The stress rate is

\[
\dot{\sigma} = R^T \sigma R + R^T \sigma \bar{R} + R^T (\sigma + \Omega \sigma - \sigma \Omega) \bar{R}.
\]

Specify the rotation of the system such that \( \Omega = \bar{R} R^T \). We have \( \dot{\sigma} = R^T \sigma R \), using the identity that \( R R^T + R R^T = 0 \).

For an isotropic material, transform Eq. (8) into the one in the rotating system

\[
\dot{\sigma} = R^T \sigma R = R^T C : (D - \gamma \mu) R
\]

\[
= C : (R^T D R - \gamma R^T \mu R)
\]

\[
= C : (\bar{D} - \gamma \mu)
\]

(31)
where $\mathbf{D} = R^T \mathbf{DR}$ and $\mathbf{\mu} = R^T \mathbf{\mu} R$.

It is noticed that $\tau = \sqrt{2} \sigma' : \sigma'$ is an invariant under the coordinate rotation. We have,

$$
\frac{\mathbf{\Omega}}{\sqrt{\sigma' : \sigma'}} = \frac{\mathbf{\Omega}}{\sqrt{2} \tau} = \frac{R^T \sigma' R}{\sqrt{2} \tau} = R^T \mathbf{\mu} R = \mathbf{\mu},
$$

which means $\mathbf{\mu}$ is the direction of $\mathbf{\sigma}$. Now, the stress in the rotating system, $\mathbf{\sigma}$, can be solved using the algorithms presented earlier for no spin cases, if we can find the time integration of $\mathbf{D}$ as needed in Eq. (16).

We can solve the orientation of the rotating coordinate system at the end of a time step $\Delta t$ exactly, if we assume that spin is constant during the loading step. We solve for this rotation $\mathbf{R(\Delta t)}$ from $\mathbf{R} = \mathbf{\Omega R}$.

$$
\mathbf{R(\Delta t)} = e^{\mathbf{\Omega} \Delta t} = I + \frac{\sin \omega \Delta t}{\omega} \mathbf{\Omega} + \frac{1 - \cos \omega \Delta t}{\omega^2} \mathbf{\Omega}^2; \quad = I \cos \omega \Delta t + (1 - \cos \omega \Delta t)P + \frac{\sin \omega \Delta t}{\omega} \mathbf{\Omega}
$$

(32)

where

$$
\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = (\omega_1, \omega_2, \omega_3)^T; \quad P = (\omega_1, \omega_2, \omega_3)^T \cdot (\omega_1, \omega_2, \omega_3)/\omega^2
$$

and

$$
(\omega_1, \omega_2, \omega_3) = (\Omega_{23}, \Omega_{31}, \Omega_{12})
$$

Since all the tensors at the right side of Eq. (32) are constants, the integration of the rate of deformation can be easily carried out.

$$
\int_0^t \mathbf{\dot{D}} dt = \int_0^t R^T \mathbf{DR} dt
$$

$$
= \left( \frac{\Delta t}{4\omega} \sin 2\omega \Delta t \right) \mathbf{D} + \left( \frac{\Delta t}{2} - \frac{2 \sin \omega \Delta t}{\omega} \right) \mathbf{PDP} + \left( \frac{\Delta t}{2} \sin \omega \Delta t + \frac{\sin 2 \omega \Delta t}{4 \omega} \right) (PD + DP)
$$

$$
+ \frac{3 - 4 \cos \omega \Delta t}{4 \omega} (P \mathbf{D} \mathbf{\Omega} - \mathbf{\Omega} \mathbf{D} P) + \left( \frac{\Delta t}{4 \omega} - \frac{2 \sin \omega \Delta t}{\omega} \right) (PD + P \mathbf{D} \mathbf{\Omega} - \mathbf{\Omega} \mathbf{D} P) (33)
$$

where $\mathbf{\Omega} = \mathbf{\Omega}/\omega$ is the normalized spin. We approximate Eq. (33) to the first order in $\omega \Delta t$, using Taylor series if the rotation is small in the loading step. The result is

$$
\int_0^t \mathbf{\dot{D}} dt \approx \mathbf{\dot{D}} \Delta t = \int_0^t \mathbf{\dot{D}} dt
$$

(34)

which means that we can simply use the deformation rate in the current coordinate system for the stress calculation in the rotating system, in a first order approximation.

After finding the stress components $\mathbf{\sigma}$ using the algorithms described above and Eq. (33), the transformation law gives the stress in the current system, i.e. $\mathbf{\sigma} = R^T \mathbf{\sigma} R^T$.

It's interesting to look at the role of the spin in the evaluation of the stress state. As derived from Eq. (2) and Eq. (8), the following relations hold for an isotropic Von Mises material.

$$
\mathbf{D'} = \frac{h + a}{a} \mathbf{D} + \frac{\tau}{a} \mathbf{\mu} = \frac{h + a}{a} \mathbf{D} + \frac{\tau}{a} (\mathbf{\mu} - \mathbf{\Omega} \mathbf{\mu} + 2 \mathbf{\mu} \mathbf{\Omega})
$$

(35)

In general, $(h + a)/a$ is close to 1 while elastic strain, $\sqrt{2} \tau / a$, is much smaller than 1. Thus, only if the spin is one order of magnitude larger than the rate of deformation, will the spin affect the evaluation of the stress significantly. Also, $\mathbf{\mu}$ is insignificant in proportional loading.
6 Numerical results

A problem with a large spin is studied numerically to compare the performances of the three algorithms. They are the Explicit Algorithm (EA) of Nemat-Nasser (1991) Eq. (28); the presently proposed Modified Algorithm (MA) Eq. (30); and the Final-Radial Return Algorithm (RRA) Eq. (24).

The solution strategy for a plastic loading is following. First, transfer the problem into the rotating system to eliminate the spin. Second, use one of the three algorithms to calculate the effective plastic strain and the effective stress. Then, use the approach outlined in Nemat-Nasser

Figs. 1, 2. 1a the normalized stress component $\sigma_{xy}/\tau_0$, b the normalized stress component $\sigma_{xx}/\tau_0$, and c the effective plastic strain $\gamma_{pl}$ solved by Nemat-Nasser's (1991) Algorithm without normalization. 2a the normalized stress component $\sigma_{xy}/\tau_0$, b the normalized stress component $\sigma_{xx}/\tau_0$, and c the effective plastic strain $\gamma_{pl}$ solved by the Presently Proposed Algorithm without normalization.
(1991) to find the stress components in the rotating system where the spin is zero. Finally, calculate the stress components in current system using the transformation law.

Nemat-Nasser (1991) proposed a normalize procedure in order to improve the accuracy. After find the stress components \( \sigma(t + \Delta t) \), he normalized it with the effective stress \( \tau \) found by Eq. (25).

\[
\sigma(t + \Delta t) = \tau(t + \Delta t) \frac{\sigma(t + \Delta t)}{\sqrt{1 + 2\sigma(t + \Delta t) \tau}}
\]  

(36)

Although this approach works well for some loading conditions, it causes problems in this example. Thus, we compare the results obtained with and without the normalization.

Figs. 3, 4. 3a the normalized stress component \( \sigma_{xx} / \tau_0 \), b the normalized stress component \( \sigma_{yy} / \tau_0 \), and c effective plastic strain \( \dot{\gamma}_{eff} \) solved by the Radial Return Algorithm without normalization. 4a the normalized stress component \( \sigma_{yy} / \tau_0 \), b the normalized stress component \( \sigma_{yy} / \tau_0 \), and c the effective plastic strain \( \dot{\gamma}_{eff} \) solved by Nemat-Nasser’s (1991) Algorithm with normalization.
The 3-D algorithm can be reduced to a 2-D algorithm for plane strain problems, by specifying that the velocity change along the third direction is zero.

The yield surface for the problems is

\[ f = \tau - \tau_0 \left(1 + \frac{\gamma}{\gamma_0}\right)^n \]

where \( \tau_0 = 1, \gamma_0 = 0.005 \), and \( n = 0.1 \). The shear modulus is \( G = 100 \). The initial state of the material is that \( \sigma_{xx} = -\sigma_{yy} = 1.1745, \sigma_{xy} = 0 \) and \( \gamma = 0.019979 \) at time \( t = 0 \mu s \).

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**Figs. 5, 6**

5a the normalized stress component \( \sigma_{xx}/\tau_0 \), b the normalized stress component \( \sigma_{yy}/\tau_0 \), and c the effective plastic strain \( \gamma_{eff} \) solved by the Presently Proposed Algorithm with normalization. 6a the normalized stress component \( \sigma_{xx}/\tau_0 \), b the normalized stress component \( \sigma_{yy}/\tau_0 \), and c the effective plastic strain \( \gamma_{eff} \) solved by the Radial Return Algorithm with normalization.
The material is subjected to a constant deformation rate and a constant spin as the following.

\[ D = \begin{bmatrix} 0 & 6,800 \\ 6,800 & 0 \end{bmatrix} \text{s}^{-1} \quad \text{and} \quad \Omega = \begin{bmatrix} 0 & -500,000 \\ 500,000 & 0 \end{bmatrix} \text{s}^{-1} \quad \text{for} \quad 0 \leq t \leq 50 \mu \text{s}. \]

Figures 1–3 show the results obtained without applying the normalize procedure, from which we can see that EA converges slower than the other two, while RRA oscillates at large time steps (see the 50-step solution). Figures 4–6 show the results obtained with the normalization. With the normalization, all of the three Algorithms give bad 50-step solutions, which are not plotted in these figures. The normalization essentially has no effect on RRA. The behavior of EA is changed significantly by the normalization. We can see from Fig. 4 that EA with the normalization does not converge correctly.

7 Summary

The explicit algorithm of Nemat-Nasser (1991) is reasonable for the problems where the direction of stress follows well the direction of the rate of deformation, such as in proportional loading or for low hardening rate materials. With the normalization, the solutions of the explicit algorithm may deviate from the true solutions in problems where the direction of the stress and that of the deformation rate differ. In such problems, the final-radial return algorithm may oscillate for large steps, while the presently proposed modified algorithm can give reasonable solutions. For an isotropic Von Mises material, the final-radial return algorithm and the presently proposed modified algorithm can be as efficient as the explicit algorithm, while they are more reliable. Our numerical experience shows that the presently proposed modified algorithm has the same convergence rate as the final-radial return method and is the best among the above three algorithms.

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