DYNAMIC ANALYSIS OF PLANAR FLEXIBLE BEAMS WITH FINITE ROTATIONS BY USING INERTIAL AND ROTATING FRAMES

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Abstract—An efficient formulation for dynamic analysis of planar Timoshenko’s beam with finite rotations is presented. Both an inertial frame and a rotating frame are introduced to simplify computational manipulation. The kinetic energy of the system is obtained by using the inertial frame so that it takes a quadratic uncoupled form. The rotating frame together with the small strain assumption is employed to derive the strain energy of the system. Since the exact solutions for linear static theory of Timoshenko’s beam are used to obtain the strain energy, the present stiffness operator is free from the locking problem without using any special technique. The resulting equations of motion of the system are defined in terms of a fixed global coordinates system. Nonlinear effects appear only in the transformation of displacement components between global and local coordinates. This results in a drastic simplification of nonlinear dynamic analysis of flexible beams. Numerical examples demonstrate the accuracy and efficiency of the present formulation.

I. INTRODUCTION

The dynamics of a flexible beam undergoing finite rotations has been formulated by several approaches: an inertial approach [1–6], a floating approach (see the references in [6, 7]), a co-rotational approach (e.g. [8–10]), and a convected coordinate approach (e.g. [11–13]).

A simple expression for the kinetic energy may be obtained by employing the inertial frame approach. This is a main reason why the inertial frame approach has been frequently employed in dynamic analysis of solids. In this approach, even if the elastic deformations are small, the nonlinear effects including the finite rigid displacements have to be taken into account correctly in the strain–displacement relationships. Therefore, when the inertial frame is used, a highly nonlinear beam theory is necessary to simulate the motion of beams undergoing finite rotations. Furthermore, when Timoshenko’s beam theory is used, a special technique (e.g. a selectively reduced integration method) should be introduced to remedy the shear locking problem. It shall be shown later that the introduction of such a technique would lead to an inferior rate of convergence for finite element solutions.

The use of the floating frame, relative to which the strains of the system are measured, is motivated by the assumption of small strains of the body undergoing finite rigid displacements (see e.g. [6, 7]). On the basis of this assumption, a linear beam theory has been often employed to obtain the strain energy function. The drawback of using the floating frame lies in the complicated expression for the kinetic energy. As pointed out by Simo and Vu-Quoc [6], the equations of motion are highly coupled in the inertia terms due to the presence of Coriolis and centrifugal effects, as well as inertia due to rotation of the floating frame. They have also shown that the floating frame approach leads to the highly nonlinear constraint equation. Due to the presence of those constraints, the discretization in space variables leads to a differential algebraic equation (DAE) system in time, which might be solved by a special computer code [14].

The co-rotational approach is defined herein as the approach in which a rotating frame is introduced to describe the elastic motion of the body and also to derive the kinetic energy function. A transformation of displacement components between global and local coordinates is used to define the equations of motion in terms of a fixed global coordinates system. The resulting equations of motion are highly coupled in the inertia terms. The co-rotational approach is different from the floating frame approach in that the constraint equations do not appear explicitly [9]. Therefore, the discretization in space variables leads to an ordinary differential equation (ODE) system instead of a DAE system. In a static analysis, the co-rotational approach has often been used because of simple manipulation for deriving the tangent stiffness matrix [15–19]. As pointed out by Crisfield [15],

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the consistent derivations for the out-of-balance force-vector and tangent stiffness matrices are still the main issue in this approach. Hsiao and Jang [8] have formulated the beam dynamics by using the same procedure as that of static analysis developed by Hsiao and Hou [16]. Though the mass matrix used has been constructed first in a rotating frame, the effects of inertia due to rotation of the rotating frame have been neglected in the mass matrix. The correct mass matrix has been given by Iura and Iwakuma [9]. Even in the static case, according to Crisfield [15], the procedure used by Hsiao and Hou [16] and Hsiao and Jang [8] does not correctly account for the variation of the transformation matrices.

Belytschko and Hsieh [11] and Belytschko et al. [12] have developed the convected coordinate approach and a direct nodal force computational scheme. The inertial and external forces have been evaluated in the fixed global coordinates, while the internal forces have been calculated from the stress components measured in the convected coordinates. The discretized equations of motion have been derived under the condition that the local nodal forces of an element must be self-equilibrated for a simply connected element. From computational viewpoint, this condition does not always hold during the iterative procedure in which the out-of-balance forces do not vanish. Dowser et al. [13], following the works of Belytschko et al. [11, 12], have introduced the Cauchy stress tensor to evaluate the internal work. This formulation may belong to the Euler formulation. Though the simple expressions are obtained with the use of the Euler formulation, the integration scheme may become more complicated than that of the Lagrangian formulation; the upper limit of integration or the deformed beam length is unknown in general. The convected coordinate approach is different from the other approaches in that the inertial and the internal forces are defined in the different frames. This results in a drastic simplification for the formulation of beam dynamics.

In this paper, following the works of Belytschko et al. [11, 12], we introduce both an inertial frame and a rotating frame; the inertial frame is used to derive the kinetic energy function and the rotating frame to derive the strain energy function. The kinetic energy function of the system, therefore, is reduced to a standard quadratic uncoupled form. The condition used by Belytschko et al. [11, 12] is not introduced herein. On the basis of the assumption of small strains, a linear theory may be available in the rotating frame to derive the strain energy function of the system. Since the strain energy function is expressed first in terms of the displacement components referred to the rotating frame, the kinematical relationships are used to rewrite the strain energy function in terms of the displacement components referred to a fixed global coordinates system. Then, with the help of Hamilton's principle, the equations of motion are expressed in terms of the fixed global coordinates system. In the present approach, the nonlinear effects appear in the transformations of displacement components between global and local coordinates. This results in a drastic simplification of nonlinear dynamic analysis of flexible beams. In the case of static analysis, the present approach becomes identical to the rotational approach. Though a variety of procedures have been proposed to formulate the co-rotational elements, there seem to be few consistent co-rotational formulations [15]. Since the present formulation is entirely based on the energy principle, the consistent method to derive the out-of-balance force-vector and tangent stiffness matrices is presented.

It has been well known that Timoshenko's beam theory is superior to Bernoulli-Euler's beam theory, especially in dynamic problems [20]. When Timoshenko's beam theory is used for developing a finite element, the $e^6$-shape functions are available and also a simple expression for the rotary inertia is obtained even for the beam with finite rotations. The drawback of using Timoshenko's beam theory lies in the shear locking problem [21]; the poor numerical results are obtained when the beam becomes very thin. A variety of methods, such as a reduced integration method [21], have been proposed to overcome the locking problem. In this paper, we employ the assumption of small strains of the body which undergoes finite displacements. Because of this assumption, the exact solutions for linear static theory of Timoshenko's beam are available for deriving the strain energy function. Therefore, the shear locking problem can be avoided without using any special technique. We shall show that the present formulation gives a better rate of convergence for finite element solutions than the existing formulation, in which the exact nonlinear kinematical relations are used together with the selectively reduced integration method.

The following is an outline of this paper. The kinetic energy function is derived in Section 2, on the basis of the exact nonlinear kinematical relations. The strain energy function is presented in Section 3. The discretized equations of motion for the system are obtained in Section 4, in which a time integration scheme is also discussed. In Section 5, numerical examples including a planar mechanism are presented. Concluding remarks are given in Section 6.

2. KINETIC ENERGY

We consider Timoshenko's beam shown in Fig. 1, the plane sections of the beam remain plane after the deformation. Let $\{I_1, I_2\}$ be the unit base vectors of the fixed global coordinate system $(x_0, y_0)$. An inertial frame $\{E_1, E_2\}$ is attached to the undeformed beam element. Then we have

$$
\begin{bmatrix}
E_1 \\
E_2
\end{bmatrix} =
\begin{bmatrix}
\cos \phi_0 & \sin \phi_0 \\
-\sin \phi_0 & \cos \phi_0
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}
$$

where $\phi_0$ is the angle between $I_1$ and $E_1$. After the deformation, we introduce the moving frame $\{\bar{I}_1, \bar{I}_2\}$.
near effects appear only as a result of the coordinate system. This results in the static analysis of the system to be identical to the coordinate system. By rotating the coordinate system, the equation of motion becomes identical to the coordinate system. The solution of the equation of motion is presented in the form of a matrix equation.

In Timoshenko's Bernoulli–Euler's beam theory, the equation of motion is given by:

\[ \frac{d^2}{dt^2} \left( I_1 + \gamma I_2 \right) + \frac{d^2}{dt^2} \left( I_1 + \gamma I_2 \right) = \sum F(t) \]

where \( I_1 \) and \( I_2 \) are the moments of inertia about the neutral axis, \( \gamma \) is the axial strain, and \( F(t) \) is the external force applied on the beam.

The solution of the equation of motion is given by the characteristic equation:

\[ (\lambda^2 + \alpha)^2 = 0 \]

where \( \lambda \) is the natural frequency of the beam and \( \alpha \) is the damping ratio.

The energy of the beam is given by:

\[ E = \frac{1}{2} \int \left( \frac{d}{dt} \left( I_1 + \gamma I_2 \right) - \sum F(t) \right) \cdot \left( \frac{d}{dt} \left( I_1 + \gamma I_2 \right) - \sum F(t) \right) \, dt \]

The total energy is the sum of the kinetic energy and the potential energy.

\[ E = E_k + E_p \]

where \( E_k \) is the kinetic energy and \( E_p \) is the potential energy.

The kinetic energy is given by:

\[ E_k = \frac{1}{2} \int \left( \frac{d}{dt} \left( I_1 + \gamma I_2 \right) \right)^2 \, dt \]

The potential energy is given by:

\[ E_p = \int \sum F(t) \cdot \left( \frac{d}{dt} \left( I_1 + \gamma I_2 \right) \right) \, dt \]

In the case of Bernoulli–Euler's beam, the kinetic energy takes a more complicated form than eqn (6). Because of the assumption that the shear deformation can be neglected, we have [22]:

\[ E_k = \frac{1}{2} \int \left( \frac{d}{dt} \left( I_1 + \gamma I_2 \right) \right)^2 \, dt \]

The total energy is the sum of the kinetic energy and the potential energy.

\[ E = E_k + E_p \]

where \( \alpha \) is the axial strain and \( \gamma = \frac{d}{dt} \gamma \) is the strain rate. It follows from eqn (10) that the time derivative of \( \Phi \) is expressed as:

\[ \dot{\Phi} = \frac{\dot{\gamma} (U' + \cos \Phi) - \dot{U} (V' + \sin \Phi)}{(1 + \alpha)^2} \]

Though the rotary inertia has often been neglected in a dynamic analysis of Bernoulli–Euler's beam, its effect on the phase velocity may become more important with an increase in the frequency of vibration [20, 23].
3. POTENTIAL ENERGY

Let us introduce a rotating frame \( \{e_1, e_2\} \) so that the \( x \)-axis possessing \( e_1 \) passes through the two end nodes of the beam element (see Fig. 1). Then we have

\[
\begin{bmatrix}
e_1
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta 
\end{bmatrix} \begin{bmatrix}
1
\end{bmatrix},
\]

(12)

where

\[
\sin \theta = \frac{V_i - V_i' + l \sin \Phi_0}{(U_i - U_i' + l \cos \Phi_0)^2 + (V_i - V_i' + l \sin \Phi_0)^2}^{1/2},
\]

(13)

\[
\cos \theta = \frac{U_i - U_i' + l \cos \Phi_0}{(U_i - U_i' + l \cos \Phi_0)^2 + (V_i - V_i' + l \sin \Phi_0)^2}^{1/2},
\]

(14)

Let \( d = e_1 \times e_2 \) denote the displacement vector at the beam axis, referred to the rotating frame. The relations between the components of \( D \) and \( d \) are obtained from geometrical consideration in the form:

\[
U = U_i - \xi \cos \Phi_0 + (\xi + u) \cos \theta - v \sin \theta,
\]

(15)

\[
V = V_i - \xi \sin \Phi_0 + (\xi + u) \sin \theta + v \cos \theta.
\]

(16)

Let \( \phi \) denote the slope of the beam, referred to the rotating frame. Then the nodal displacement components referred to the rotating frame are related with the global displacement components by

\[
\begin{bmatrix}
\phi_i \\
u_i \\
\phi_i
\end{bmatrix} = \begin{bmatrix}
(U_i - U_i' + l \cos \Phi_0) \cos \theta + (V_i - V_i' + l \sin \Phi_0) \sin \theta
\end{bmatrix}.
\]

(17)

Note that, due to the definition of the rotating frame, we have \( u = v = \phi = 0 \). The above equation is expressed symbolically by \( [d_i] = [d_i, (D_i)] \), in which \( [d_i] = [\phi_i, u_i, \phi_i] \). Using eqns (13) and (14), the variation of eqn (17) leads to

\[
\begin{bmatrix}
\delta d_i
\end{bmatrix} = \begin{bmatrix}
-\sin \theta & \cos \theta & 1
-\sin \theta & \cos \theta & 1
\cos \theta & \sin \theta & 0
\cos \theta & \sin \theta & 0
\end{bmatrix} \begin{bmatrix}
\delta U_i
\delta V_i
\delta \phi_i
\end{bmatrix}.
\]

(18)

It should be noted that the effects of axial displacements \( u_i \) are included in the transformation matrix of eqn (18). Though \( u_i \) is negligibly small in comparison with \( l \), the importance for including \( u_i \) has been pointed out by Oran [17].

The strain energy function of the beam \( \Pi \), may be expressed as [22, 24]:

\[
\Pi = \int_0^l \left[ \frac{EA}{2} (\epsilon)^2 + \frac{EI}{2} (\kappa)^2 + \frac{GA}{2} (\gamma)^2 \right] \, d\xi,
\]

(19)

where \( EA \) is the axial stiffness, \( EI \) the flexural stiffness, \( GA \) the shear stiffness, \( \kappa \) the curvature and \( \gamma \) the shear strain. The exact relations between the strains and the displacements are expressed as [22]

\[
\epsilon = (d_i' + \cos \Lambda_0) \cos \lambda + (d_i + \sin \Lambda_0) \sin \lambda - 1 \quad (20)
\]

\[
\kappa = \lambda' \quad (21)
\]

\[
\gamma = (d_i' + \sin \Lambda_0) \cos \lambda - (d_i + \cos \Lambda_0) \cos \lambda \quad (22)
\]

where

\[
id_i = U_i, \quad d_i = V_i, \quad \lambda = \Phi, \quad \Lambda_0 = \Phi_0
\]

(23)

\[
id_i = u_i, \quad d_i = v_i, \quad \lambda = \phi, \quad \Lambda_0 = 0
\]

(24)

When the kinetic energy function is discretized, the global displacement components are interpolated by linear functions, as shown in eqn (8). Using these linear interpolations and eqns (15)-(17), we obtain the shape functions, associated with the rotating frame, in the form:

\[
u = Su, \quad \phi = (1 - S) \phi_i + S \phi_i.
\]

(25)

Note that the above shape functions are consistent with the linear interpolation in the rotating frame.

\[
\begin{bmatrix}
\phi_i - \theta \\
u_i \\
\phi_i - \theta
\end{bmatrix} = \begin{bmatrix}
(U_i - U_i' + l \cos \Phi_0) \cos \theta + (V_i - V_i' + l \sin \Phi_0) \sin \theta - l
\end{bmatrix}.
\]

(17)

since \( u_i = v_i = \phi = 0 \).

In order to show that the present shape functions yield the invariant strain components under the coordinate transformation, we substitute eqn (25) into eqns (20)-(22) to obtain

\[
\epsilon = \left( \frac{u_i}{l} + 1 \right) \cos \left( (1 - S) \phi_i + S \phi_i \right) - 1 \quad (26)
\]

\[
\kappa = \frac{\phi_i - \phi}{l} \quad (27)
\]

\[
\gamma = \left( \frac{u_i}{l} + 1 \right) \sin \left( (1 - S) \phi_i + S \phi_i \right), \quad (28)
\]
where \( \phi \) denotes the value associated with the rotating frame. Substituting eqn (8) into eqns (20)–(22), we obtain the strain components associated with the rotating frame in the form:

\[
\epsilon_i = \left( \frac{U_i - U_j}{l} + \cos \Phi_j \right) \cos \left( (1 - S) \Phi_i + S \Phi_j \right) + \left( \frac{V_i - V_j}{l} + \sin \Phi_j \right) \sin \left( (1 - S) \Phi_i + S \Phi_j \right) - 1
\]

(29)

\[
\kappa_i = \frac{\Phi_i - \Phi_j}{l}
\]

(30)

\[
\gamma_i = \left( \frac{V_i - V_j}{l} + \sin \Phi_j \right) \cos \left( (1 - S) \Phi_i + S \Phi_j \right) - \left( \frac{U_i - U_j}{l} - \cos \Phi_j \right) \sin \left( (1 - S) \Phi_i + S \Phi_j \right),
\]

(31)

where \( \phi \) denotes the value associated with the inertial frame.

Introducing the geometrical relationships given by eqn (17) into the right side of eqns (26)–(28) and using eqns (13) and (14), one can show that the strain components associated with the rotating frame are exactly the same as those with the inertial frame; the strain components are invariant under the coordinate transformation.

When eqns (29)–(31) are introduced into eqn (19) and the full integration scheme is used, the poor numerical results may be obtained in the case of thin beams. This is the well-known problem called shear locking, which has been a main issue for an analysis of Timoshenko’s beam. The selective reduced integration method has been frequently used to overcome the shear locking problem. However, the use of the selectively reduced integration method gives an inferior rate of convergence for finite element solutions when the beam becomes very thin. It is better, therefore, to propose another method to remedy the shear locking problem. This will be discussed again in Section 5.

In this paper, we will present an alternative approach to obtain the strain energy function and show that the present approach is free from the shear locking problem. The introduction of the rotating frame is motivated by the conventional assumption of infinitesimal strains. When a sufficient number of finite elements are used, the displacements in each element can be assumed to be small. On the basis of these assumptions, a linear theory might be available in each finite element. Then, as shape functions, we introduce the exact solutions for linear static theory of Timoshenko’s beam in which no distributed forces are applied. Then, noting that \( u = v = \gamma = 0 \), we obtain the exact solutions for linear Timoshenko’s beam in the form:

\[
u = (1 - S)u,
\]

(32)

\[
v = \frac{l^2}{12 \beta} \left[ S^3 - (2 + 6 \beta)S^2 + (1 + 6 \beta)S \right]
\]

(33)

\[
\phi = \frac{\Phi}{1 + 12 \beta} \left[ 3S^2 - (4 + 12 \beta)S + 1 + 12 \beta \right]
\]

(34)

where \( \beta = EI/AG^2 \). Then, based on the linear theory, we obtain the strain energy function expressed as

\[
\Pi = \frac{1}{2} \left( \frac{d_s}{l} \right)^T [k_s] \left( \frac{d_s}{l} \right).
\]

(35)

Note that the present strain energy function is derived through the exact integration. Since the exact linear solutions are introduced for the strain energy, no shear locking is observed when the beam becomes very thin. Furthermore, in the case of thick beams, the present element captures the effects of shear deformations. It will be shown later that the present approach yields the satisfactory rate of convergence for finite element solutions in the case of geometrically nonlinear problems.

In order to express the strain function in terms of the global displacement components, we introduce the geometrical relationships of eqn (17) into eqn (35) and obtain

\[
\Pi_e = \frac{1}{2} \left( \frac{d}{l} \right)^T [k_e] \left( \frac{d}{l} \right).
\]

(36)

In this approach, the geometrically nonlinear effects appear only in eqn (17). This results in a drastic simplification of finite element implementation.

4. EQUATIONS OF MOTION AND TIME INTEGRATION SCHEME

The equations of motion of the system are obtained with the help of Hamilton’s principle which states that

\[
\delta H = 0, \quad H = \int_{t_0}^{t_f} \left( T - \Pi_e + \Pi_c \right) dt.
\]

(37)

where \( \Pi_c \) is the potential energy of the external forces and the variables subjected to variation are the global displacement components \( \{D_s\} \). When we assume
that the external forces are conservative and that the equivalent forces are applied at each node, we obtain the potential energy of the external forces as
\[
\Pi = P, U, + P, U, + Q, V, + Q, V, + M, \Phi, + M, \Phi, ,
\]
where \(P, P, Q, Q, M, M, \) are the equivalent nodal forces.

In this paper, we assume that there exists no damping force. Substituting eqns (9), (35) and (38) into eqn (37), and using eqn (17), we obtain the equations of motion for one element as
\[
\begin{align*}
\{M\} \{\ddot{\delta}\} + \{K\} - \{F\} &= \{0\},
\end{align*}
\]
where \{\delta\} is the internal force vector, \{\delta\} the external force vector. From a computational point of view, it is convenient to take a variation of eqn (35) first with respect to the local displacement components, and then introduce eqns (17) and (18) into the resulting equations.

Following the standard finite element assembling procedure, we obtain the equations of motion for the system, expressed as
\[
\{M\} \{\ddot{\delta}\} = \{F\},
\]
where \{\delta\} = \sum_{\text{elements}} \{M\} \{\ddot{\delta}\} \text{ and } \{\delta\} = \sum_{\text{elements}} \{K\} - \{F\}. There have been a variety of methods to solve the equations of motion expressed by eqn (40) [25]. An explicit method is introduced herein to integrate eqn (40). At first, eqn (40) is rewritten in the form of \{\dot{\delta}\} = \{M\}^{-1} \{F\}.

Then, introducing the new notations defined by \(z = U, \rho = V, \) and \(\rho = \Phi, \) we obtain the first-order differential equations expressed by
\[
\{\dot{\delta}\} = \{G\},
\]
where \{\dot{\delta}\} = \{U, V, \Phi, \rho, \rho, \rho, \ldots, \rho, \rho\} is a function of \{\delta\}. Since eqn (41) is the first-order ordinary differential equation, any standard scheme is available to integrate eqn (41). In this paper, the fourth-order Runge-Kutta method is used for the integration.

5. NUMERICAL EXAMPLES

5.1. Static case: reduced integration method vs present formulation

At first, we consider a static problem in order to compare the numerical results obtained by the present formulation with those by the conventional one, in which the exact kinematic relations together with the selectively reduced integration method is used. When the inertial forces are neglected the present approach becomes identical to the co-rotational approach. According to Crisfield [15], when the co-rotational approach is employed, there seem to be few consistent ways to derive the out-of-balance force-vector and tangent stiffness matrices. In this paper, the equation of motion for a static problem are given by
\[
\begin{align*}
\{f\} &= 0, \quad \{f\} = \frac{\delta (\Pi, - \Pi,)}{\delta D^n},
\end{align*}
\]
where \{\delta\} and \(D^n\) are the components of \{\delta\} and \{D^n\}, respectively. There have been many methods to solve the nonlinear algebraic equations. When we use the full Newton-Raphson method to solve the nonlinear algebraic eqn (42), the incremental displacements \(\Delta D\) are obtained by
\[
\{DK\} \{\Delta D\} = -\{f\},
\]
where the tangent stiffness matrix is given by
\[
\{DK\} = \begin{bmatrix}
\frac{\partial^2 \Pi,}{\partial D^n \partial D^n} & \frac{\partial^2 \Pi,}{\partial D^n \partial D^m} \\
\frac{\partial^2 \Pi,}{\partial D^m \partial D^n} & \frac{\partial^2 \Pi,}{\partial D^m \partial D^m}
\end{bmatrix}
\]
It is noted that the above formulation presents a consistent method to derive the out-of-balance force-vector and tangent stiffness matrices [26]. Since the present formulation is entirely based on the energy principle, the resulting tangent stiffness matrix is always symmetric. It is easy to incorporate the arc-length method [27, 28] in the above procedure.

Wen and Rahimzadeh [19] have proposed a similar method in which the tangent stiffness matrix is defined by
\[
\{DK\} = \begin{bmatrix}
\frac{\partial^2 \Pi,}{\partial D^n \partial D^n} & \frac{\partial^2 \Pi,}{\partial D^n \partial D^m} \\
\frac{\partial^2 \Pi,}{\partial D^m \partial D^n} & \frac{\partial^2 \Pi,}{\partial D^m \partial D^m} + \frac{\partial^2 \Pi,}{\partial D^n \partial D^m}
\end{bmatrix}
\]
where \(\Pi,\) is the function of the local displacement components \{\delta\}, and \(D^n\) is the component of \{\delta\}. From a mathematical point of view, eqn (45) is identical to eqn (44). It is advantageous, however, to use eqn (44) instead of eqn (45) when the tangent stiffness matrix derived from the co-rotational approach is compared with that from the inertial approach. If eqn (45) is used, the comparison becomes possible only when the explicit form for the tangent stiffness matrix is given, though it is cumbersome to derive the tangent stiffness matrix. When eqn (44) is used, on the other hand, the comparison is possible without deriving the explicit form for the tangent stiffness matrix, but is easily carried out by simply deriving the explicit form for the strain energy function.

As a first example, the beam with hinged ends, as shown in Fig. 2, is solved. The concentrated force is applied at the center of the beam. Because of symmetry, only half is discretized. The full Newton-Raphson method is used and the iterations continue until the Euclidean norm of residual forces is less than the prescribed value. The analytical solutions for this problem have been obtained by Goto et al. [29].
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\begin{equation}
\frac{\partial^2 \mathbf{D}_n}{\partial \mathbf{D}_n^2}
\end{equation}

... components of \( \mathbf{f} \) and \( \mathbf{g} \) have been many methods for solving these equations. When the method to solve (42), the incremental strain is given by

\begin{equation}
-\mathbf{f}_i
\end{equation}

... the out-of-balance forces matrices [26]. Since the proposed method to incorporate the above procedure, we propose a similar stiffness matrix is

\begin{equation}
\left[ \frac{\partial^2 \mathbf{D}_n}{\partial \mathbf{D}_n^2} \mathbf{D}_n \right]
\end{equation}

... local displacement component of \( \mathbf{d}_1 \)... of view, eqn (45) is advantageous, however. When the tangent is the co-rotational inertial component becomes zero for the tangent... the strain energy.

... with hinged ends, as concentrated force beam. Because of this. The full Newton-Raphson iterations continue until forces is less than 0.001 solutions for this into et al. [29]. The

... slenderess ratio of the beam is taken as 100 and the ratio of \( EA \) to \( GA \), as 3. In this case, the numerical solutions are insensitive to the value of \( EA/GA \). the effects of shear deformations may be neglected. Since this beam is very thin, the use of linear shape functions [eqn (8) or (25)], together with the full integration scheme, leads to poor numerical results—the shear locking problem. In order to overcome the shear locking problem, the selectively reduced integration method is used herein. Then we compare the numerical results obtained by the present formulation with those by using the exact kinematic relations together with the selectively reduced integration method. The converging processes of the finite element solutions for center deflections of the beam, in which \( P_f / EI \) is taken as 10.0, are shown in Fig. 3. It will be noticed that the convergence rate of numerical solutions obtained by the present formulation is more excellent than that by the conventional method.

The second example is the thick beam in which the slenderess ratio is taken as 10.0. The other conditions are the same as those of the first example. The numerical results for center deflections of the beam with 10 elements are plotted in Fig. 4, in which the exact solutions both for the finite and small strain theories are shown. Good agreement between the analytical solutions for Timoshenko's beam theory and the present ones is obtained. Note that, in this case, there is significant difference between the numerical results of Bernoulli-Euler beam and those of Timoshenko's beam. These numerical results, shown in Figs 3 and 4, indicate that the present formulation is available for a finite displacement analysis not only of thin beams but also of thick ones.

5.2. Dynamic case: flying spaghetti

Simo and Vu-Quoc [6] have first solved the free flight of the very flexible beam. The beam is subjected to a force and a torque applied simultaneously at one end. The problem data are given in Fig. 5. The number of elements used is 10 and the time step size is 0.005. The consistent mass matrix is used. The sequence of the beam motion is shown in Fig. 6, in which the solid lines indicate the present results and the dashed lines the results of Simo and Vu-Quoc. Excellent agreement between these two solutions is observed.

The numerical results obtained by using the lumped or diagonal mass matrix are shown in Fig. 7, in which the solid lines indicate the present results and the
dashed lines the results of Simo and Vu-Quoc. It is shown that the use of the lumped mass matrix can save computational time at very little expense of numerical accuracy.

5.3. Planar mechanism: a multibody system in free flight

As an application to a planar mechanism, we consider two links connected by the revolute joint. Song and Haug [7] have derived the constraint conditions for the revolute joint. In their approach, the constraint condition even at the rigid connection is constructed. Because of these constraint conditions, a DAE system has appeared instead of an ODE system.

In the present approach, the rigid connection is satisfied automatically by using the standard finite element assembling procedure. The constraint condition for the revolute joint is satisfied herein by introducing the additional DOF for the rotation at the node where the hinge exists. For example, as shown in Fig. 8, the DOF at the connection node is extended from \( \{U, V, \Phi_j\} \) into \( \{U, V, \Phi_j, \Phi_j^*\} \).
 Dynamic analysis of planar flexible beams

Fig. 8. Revolute joint.

Material properties:
EA = GA = 1.000,000
EI = 1,000

\[ I_p = \begin{cases} 1 & \text{for link A} \\ 10 & \text{for link B} \end{cases} \]

Fig. 9. Multibody system: problem data.

The rotations \( \Phi_j \) and \( \Phi_j^* \) are associated with the \( m \)th and \( n \)th elements, respectively. The remaining procedure is exactly the same as that of a standard finite element formulation. Since no Lagrange multiplier appears in this approach, the resulting equations are still the ODEs.

A multibody system to be solved is shown in Fig. 9, in which the problem data are also given. According to Simo and Vu-Quoc [6], the applied force and the torque are removed at \( t = 0.5 \). Hsiao and Jang [8] have also solved this problem, in which the force and the torque are removed at \( t = 2.5 \). Then they have obtained the sequence of motion which is similar to that of Simo and Vu-Quoc [6]. In the present paper, the force and the torque are removed at \( t = 2.5 \). Each link is discretized by two elements and the time step size used is 0.005. The consistent mass matrix is used. The sequence of motion is shown in Fig. 10, where the solid lines denote the present results and the dashed lines the results of Simo and Vu-Quoc [6]. Again good agreement between these results is obtained.

6. CONCLUDING REMARKS

A simple but efficient formulation is presented for dynamic analysis of flexible planar beams with finite rotations. The inertial frame and the rotating frame are used to derive the kinetic energy and the strain energy of the system, respectively. By rewriting the strain energy in terms of the global displacement components, we obtain the equations of motion of the system referred to a fixed global coordinate system. The advantage of this formulation lies in the simple manipulation; the nonlinear effects appear only in the transformations of displacement components between global and local coordinates. Though Timoshenko's beam theory is employed, the present stiffness operator does not yield the shear locking. The model used is available not only for thin beams but also for thick ones. Furthermore, the present formulation gives an excellent rate of convergence for finite element solutions. Since all matrices are given in explicit forms, no numerical integration is used. This results in a drastic reduction of computational time. The numerical results presented herein show the validity and the applicability of the present formulation.

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REFERENCES


Present (consistent mass) ~ Simo and Vu-Quoc (1986)

Fig. 10. Sequence of motion of multibody system.


