Computational mechanics advances

Recent advances in the alternating method for elastic and inelastic fracture analyses

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Abstract

The numerical-analytical alternating method (wherein finite elements or boundary elements are used for numerical analysis of an uncracked structure) is a very efficient and accurate method for fracture analysis. It saves both time in the computational analysis and human effort in preparing analysis models. In this paper we summarize the recent developments in this method, including: (1) the alternating method and its convergence for mixed boundary value problems, with the presence of both traction and displacement boundary conditions; (2) a non-iterative method to construct solutions for multiple arbitrarily located embedded cracks using the solution for a single embedded crack in an infinite body; (3) the analysis of elastic-plastic fracture mechanics problems; (4) the analysis of crack-growth in plane fracture situations; and (5) an efficient and accurate algorithm for the evaluation of elastoplastic stress state in a cracked structure, based on the generalized mid-point radial return for 3D constitutive laws and the stress subspace method for the plane stress analysis, and a study of link-up of multiple cracks in a wide-spread fatigue damage situation in an aircraft panel. Some numerical examples are also given.

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1. Introduction

Fracture mechanics problems can be solved using a number of different methods, including finite element and boundary element methods, singular/hybrid finite elements, alternating method, and the use of path-independent and domain-independent integrals, etc. (See [2, 4] for comprehensive discussions and detailed summaries). The finite element (or boundary element) alternating method is considered to be a very efficient and accurate method. The Finite Element Alternating Method (FEAM) or the Boundary Element Alternating Method (BEAM) solves for the cracks (including surface cracks) in finite bodies by iterating between the analytical solution for an embedded crack in an infinite domain, and the finite element (or boundary element) solution for the uncracked finite body. The cohesive tractions at the locations of the cracks in the finite element (or boundary element) model of the uncracked body, and the residuals at the far field boundaries in the analytical solution for the infinite body, are corrected through the iteration process. Essentially, the alternating method is a linear superposition method. Fracture mechanics parameters can be found accurately because the near crack tip fields are captured exactly by the analytical solution. Coarser meshes can be used in the finite element analysis because the cracks are not modeled explicitly. In a crack growth analysis, or in conducting a parametric analysis of various crack sizes, the stiffness of the uncracked body remains the same for all crack sizes. Thus, the global stiffness matrix of the finite element model is decomposed only once. In the most common finite element analysis for fracture problems, it is necessary to use very fine meshes (or adaptive mesh refinements) around the crack tips, and to decompose the global stiffness matrix every time a crack size changes. Thus, the alternating method is very efficient in saving both time in the computational analysis and human effort in the mesh generation.

2. A historic review

Schwartz\(^1\) invented the alternating method to solve the Dirichlet problem for harmonic functions on a domain that is the union of two other partially overlapping domains, if the solutions on these two domains can be obtained. Neumann\(^1\) pointed out that a similar method can be used to obtain the solution on a domain that is the intersection of two other overlapping domains. Kantorovich and Krylov \([15]\) called this method the Schwartz–Neumann alternating method in their book about mathematical approximation methods. They used it to solve an arbitrary linear homogeneous partial differential equation of the second order.

The Schwartz–Neumann alternating method is an iterative method based on the superposition principle. As seen in Fig. 1, each of the overlapping domains contains a part of the imaginary boundaries, compared with the original domain. The solution on the given domain is assumed to be the sum of a series of solutions on these two overlapping domains. The solution series is constructed such that the residuals on the imaginary boundaries on one domain are corrected by the next solution on the other domain, in which those imaginary boundaries are indeed the real ones. Alternating between these two overlapping domains with boundary conditions being corrected in each iteration, this procedure converges when the residuals on the imaginary boundaries vanish.

This procedure can be applied to fracture mechanics problems. In general it is much easier to handle a finite body without any crack, and complementarily, an infinite body with a crack (or cracks). Thus, the Schwartz–Neumann alternating method can be used to obtain the solutions of cracked finite bodies, by iterating between the solutions for the uncracked finite bodies, and the solutions for the cracks in an infinite region.

Various methods can be used to solve the sub-problems, i.e. the uncracked body and the cracks in an infinite body, in the Schwartz–Neumann alternating method. Analytical solutions for uncracked bodies can be found for simple geometries. Standard numerical methods, such as the finite element method or the boundary element method, can be used to solve the uncracked body of complicated geometries subjected to any loading.

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\(^1\) See Kantorovich and Krylov \([15]\).
subjected to arbitrary loadings. Analytical solutions are available for embedded cracks in the infinite domains, subjected to arbitrary crack surface tractions. The singular integral equation method [50, 51] can also be used to obtain the solution for the cracks in an infinite domain. Combinations of all these different approaches lead to various flavors of Schwartz–Neumann alternating method, such that finite element alternating method, boundary element alternating method, etc.

Early applications of Schwartz–Neumann alternating method in the fracture mechanics involved solving edge crack problems in plane strain [17] and surface circular/elliptical cracks in 3-D bodies [45, 46, 49, 52]. These earlier works utilized analytical solutions for both the uncracked body, as well as for cracks in an infinite body. Such an analytical alternating method was also used later by various authors to solve problems of simple geometries. Chen et al. [11] presented an analysis of surface cracks in a pump shaft. A Fourier series analysis was used to solve the uncracked shaft, which was assumed to be infinitely long, while the analytical solutions for the embedded elliptical crack, developed by Vijayakumar and Atluri [53] and Nishioka and Atluri [28], were used to model the flaws in the shaft. Zhang and Hasebe [58] studied the interactions between rectilinear and circumferential cracks using the alternating method. Solutions for multiple embedded cracks in an infinite domain can be found by using the alternating method and the solution for a single embedded crack in the infinite domain. O'Donoghue et al. [31] used the alternating method to solve multiple embedded empirical cracks in an infinite three-dimensional solid, subjected to arbitrary crack face tractions. Chen and Chang [10] used it to analyze multiple 2-D mixed-mode cracks. However, it is not necessary to use the alternating technique to construct the multiple crack solutions from that for a single crack. A simple non-iterative
method can be used to achieve that. This idea will be presented in detail later in this work, for the first time.

A standard displacement-based finite element method is generally regarded as the most powerful numerical method. It can handle complicated geometries and loading conditions. It is possible to solve the fracture mechanics problems by directly using the finite element method, by using an adaptive mesh refinement strategy, or by using certain path-independent and domain-independent integrals based on conservative laws of continuum mechanics, etc. But, this requires an explicit modeling of cracks. Very fine meshes are needed in the vicinities of crack tips in order to capture the singularities around the crack tips. Thus, an accurate analysis can be very expensive in terms of both human efforts in the mesh generations, as well as computational resources involved in solving a finite element model with a large number of degrees of freedom. On the other hand, the finite element method can be used to solve the uncracked body problem more easily. In the absence of cracks, the stress gradients within the uncracked body are much smoother. Thus, the uncracked bodies can be modeled with a relatively coarser mesh with much smaller number of degrees of freedom. The use of the finite element method in solving the uncracked bodies enables the alternating method to solve realistic problems with complicated geometries. The finite element alternating method is now considered to be one of the most accurate and efficient methods in solving fracture mechanics problems in built-up structures such as aircraft fuselages, wings, etc.

The earlier alternating method [45], for the analysis of surface flaws, was not able to give accurate results, because of the lack of a general analytical solution in an infinite body for an embedded crack subjected to arbitrary crack face tractions. The analytical solution used by Shah and Kobayashi [45] was for an elliptical crack subjected to cubic variations of normal tractions on the surface. This form of load distribution was not general enough to capture accurately the fully arbitrary crack surface tractions involved in the alternating method. Vijayakumar and Atluri [53] derived a completely general solution in an infinite body, for an embedded elliptical crack, subjected to normal and tangential tractions of polynomial distribution of an arbitrary degree. This solution was refined by Nishioka and Atluri [28], who also devised a systematic way to evaluate the elliptical integrals and partial derivatives involved in the analytical solution. Thus, any smooth distribution of crack surface traction can be captured accurately. Using these analytical solutions, now popularly referred to as the VNA solutions, Nishioka and Atluri [28] implemented a finite element alternating method, which is both efficient and accurate, for the analysis of elliptical cracks as well as part-elliptical surface cracks. O'Donoghue et al. [30, 31] used the alternating method to find solutions for multiple embedded elliptical cracks in an infinite solid subjected to arbitrary crack face tractions, and used it in the finite element alternating method for the analysis of multiple elliptical cracks in finite bodies. Simon et al. [48] used the alternating method to evaluate mixed mode stress intensity factors for part-elliptical surface cracks. Using the analytically derived crack-surface displacements for an embedded elliptical crack, and their first order variations due to variations in the crack shapes, Nishioka and Atluri [29] presented a very efficient ‘weight function methodology’ for elliptical and partial elliptical cracks. Liao and Atluri [18, 19] and Mu and Reddy [22] revisited the semicircular surface crack problem with the Finite Element Alternating Method, and presented robust and efficient methods for computing weight functions for 3-D surface flaws. Following the lines of Vijayakumar and Atluri's solution procedure [53] for the isotropic material, Rajiyah and Atluri [40] derived a general analytical solution for an elliptical crack embedded in an infinite transversely isotropic solid, and oriented perpendicular to the axis of elastic symmetry. The finite element alternating method based on this solution is very efficient in the analysis of flaws in laminated composite structures.

Until recently, handbook solutions and explicit finite element models were the main methods in the 2-D analysis of fracture mechanics problems. Driven by the demand for an efficient and accurate method for routine analysis of fracture problems, and partially encouraged by the success of the finite element alternating method in solving surface flaws in the 3-D analysis, the development of finite element alternating method for 2-D analysis was the focus of considerable recent research. It is relatively easier to obtain 2-D analytical solutions for embedded cracks in infinite bodies, in contrast to solving a 3-D embedded crack, subjected to arbitrary surface traction. Different types of analytical solutions were used in the literature. Polynomial or Chebyshev polynomial distributions were used as an analogy to the load case of a surface flaw under a distributed load. Rajiyah and Fichter [41] used polynomial nodal distribution for their 2-D analytical solutions for a crack under a uniformly distributed load. The analytical solutions were used in mixed-mode fracture mechanics problems of the second kind. Rajiyah and Fichter [41] proposed the cracks emanating from an opening in an arbitrary way.

The boundaries of the analyzed regions have advantages over the application of the VNA solutions. Alternating Method is more general and is capable of handling the stress intensity factors of an arbitrary crack, handled by alternating method of crack in conjunction with Krishnamurthy [53].

The finite element alternating method has been used in conjunction with VNA solutions to obtain the superpositioning of stress intensity factors for the elastic-plastic analysis of cracks in practical situations. The elastic-plastic analysis of cracks in solids, 2-D elastic-plastic analysis of cracks in solids, and the stress intensity factors and stress fields of reports [55-58] have been successfully used to determine stresses and strains in the elastic-plastic regions.

Recently, a finite element analysis [49] has been carried out to determine the stress and strain field of cracked stiffened plates subjected to cyclic thermal cyclic fatigue loading [36]. The finite element analysis was conducted using the finite element alternating method.

In this paper, a new alternating method and its implementation, for the analysis of arbitrary crack in a structure, is presented. It is considered that (1) the analytical solution for the stress intensity factor at the crack tip will be localized in the vicinity of the crack tip, (2) the analytical solution for the stress intensity factor at the crack tip will be accurate near the crack tip, and (3) the analytical solution for the stress intensity factor at the crack tip will be accurate near the crack tip. The stress intensity factor at the crack tip is a widely used parameter in fracture mechanics.

3. An analytical solution for an elliptical crack

Following suggestions of Nishioka and Atluri [28], we derived a total of 12 polynomial solutions for the distribution of stress intensity factors around an elliptical crack. These solutions were later refined to a set of 12 simplified solutions. These 12 solutions form the basis of the solution. The solutions for the stress intensity factors for an elliptical crack with arbitrary orientation are given here, as a prerequisite for analyses of cracked structures.
analogy to the three-dimensional analysis. Localized distributions, such as point loads or piecewise distributed loads, were used recently to capture discontinuously distributed crack surface tractions. Raju and Fichler presented a finite element alternating method for Mode I crack configurations where polynomial normal pressures were used to represent the crack face load. It was extended for the analysis of crack under mixed-mode loads [16]. Chen and Chang [7, 8] presented an alternating procedure for cracks subjected mixed-mode loads and under mixed boundary conditions with similar analytical solutions. Chen and Atluri [9] studied the alternating method for orthotropic materials in mixed-mode fracture, in which, the crack face tractions were fitted into Chebyshev polynomials of the second kind. Park and Atluri [34] presented a finite element alternating method for arbitrary sized cracks emanating from a row of fastener holes in a fuselage lap joint, where crack surface tractions were arbitrary. Wang and Atluri [54] used piecewise linear representations of crack surface tractions.

The boundary element method reduces the dimension of geometry modeling by one and has certain advantages over the finite element method. This led to the development of Boundary Element Alternating Method (BEAM). Rajiyah and Atluri [38, 39] first presented a BEAM for the evaluation of stress intensity factors and weight functions for 2-D mixed-mode problems, where multiple cracks were handled by alternating technique. More details on numerical implementation were presented by Raju and Krishnamurthy [43]. Chen and Tu [12] performed thermal analysis using BEAM.

The finite element alternating method can be applied to the elastic-plastic analysis of cracks when it is used in conjunction with the initial stress method, even though the alternating method itself is based on the superposition principle which is valid only for linear problems. The initial stress approach converts the elastic-plastic analysis into a series of linear elastic steps, in which the superposition principle holds. The elastic-plastic finite element method was first presented by Nikishov and Atluri [27]. It was used in 2-D elastic-plastic analyses of wide-spread fatigue damage in ductile panels [37]. The method was successfully used to simulate the stable crack growth in the presence of multiple site damage in a series of reports [55–57].

Recently, a series of analyses of residual strength of built-up structures such as aging aircraft panels has been carried out successfully, using the finite element alternating method. This includes the analysis of cracked stiffened panels with/without composite-patch repairs [33], 2-D analysis of multi-site fatigue damage [36], 2-D elastic-plastic analysis of wide-spread fatigue damage in ductile panels [37], and 3-D fatigue analysis of aircraft components [32], etc. The success of all these applications shows that the finite element alternating method is a highly efficient and accurate method for fracture mechanics analysis.

In this paper we summarize the recent developments in this method, including: (1) the alternating method and its convergence for mixed boundary value problems, with the presence of both traction and displacement boundary conditions; (2) a non-iterative method to construct solutions for multiple arbitrarily located embedded cracks using the solution for a single embedded crack in an infinite body; (3) the analysis of elastic-plastic fracture mechanics problems; (4) the analysis of crack-growth in plane fracture situations; and (5) an efficient and accurate algorithm for the evaluation of elastoplastic stress state in a cracked structure, based on the generalized mid-point radial return for 3D constitutive laws and the stress subspace method for the plane stress analysis, and a study of link-up of multiple cracks in a wide-spread fatigue damage situation in an aircraft panel. Some numerical examples are also given.

3. An analytical solution for an embedded elliptical crack

Following Shah and Kobayashi's [44] choice of potential functions, Vijayakumar and Atluri [53] derived a totally general solution for an embedded elliptical crack, subjected to arbitrary polynomial distributions of normal and shear crack surface tractions, in an infinite body. Nishioka and Atluri [28] later refined the solution and devised a systematic method of evaluating the elliptic integrals involved in the solution. Using this general solution they showed that the finite element alternating method for 3-D surface flaw analysis is not only efficient but also accurate. We present a brief summary of this solution here, as a precursor to developing a three-dimensional elastic-plastic finite element alternating method for analyses of embedded elliptical or part-elliptical surface flaws in elastic-plastic solids.
3.1. Trefftz’s formulation

The analytical solution for an embedded crack is based on Trefftz’s formulation for a plane surface of discontinuity. A complete set of potential functions is first chosen such that a linear combination of these potential functions can represent any polynomial distribution of crack surface traction. Once the crack surface tractions are curve fitted into polynomial distributions, the coefficients for the linear combination of potential functions are determined by solving a system of linear equations. Stress intensity factors are found in terms of these coefficients. Displacements and stresses are expressed in terms of the derivatives of these potentials.

The Navier’s equations of equilibrium in the absence of body forces, for a homogeneous isotropic linear elastic solid, are

\[ u_{i,j} + (1 - 2\nu)u_{i,j} = 0 \quad i = 1, 2, 3 \quad (1) \]

where \( u_i \) is the displacement in the direction of \( x_i \)-axis, and \( \nu \) is the Poisson’s ratio.

The displacements can be expressed by four harmonic functions \( \psi \) and \( \phi_i \) (\( i = 1, 2, 3 \)).

\[ u_i = \phi_i + x_i \psi_i \quad (2) \]

The Navier’s equations (Eq. (1)) will be satisfied if the four harmonic functions \( \psi \) and \( \phi_i \) (\( i = 1, 2, 3 \)) satisfy Eq. (3).

\[ \phi_{i,j} + (3 - 4\nu)\phi_i = 0 \quad (3) \]

Eq. (3) will be satisfied identically if these four harmonic functions \( \psi \) and \( \phi_i \) (\( i = 1, 2, 3 \)) are expressed by another three harmonic functions \( f_i \) (\( i = 1, 2, 3 \)) in the following fashion.

\[ \psi = f_{i,j} \quad (4) \]

\[ \phi_1 = (1 - 2\nu)(f_{1,1} + f_{1,2}) - (3 - 4\nu)f_{1,3} \quad (5) \]

\[ \phi_2 = (1 - 2\nu)(f_{2,1} + f_{2,2}) - (3 - 4\nu)f_{2,3} \quad (6) \]

\[ \phi_3 = -(1 - 2\nu)(f_{3,1} + f_{3,2}) - 2(1 - \nu)f_{3,3} \quad (7) \]

Thus, the problem of solving the Navier’s equation [Eq. (1)] for given boundary conditions is reduced to finding harmonic functions \( f_i \) (\( i = 1, 2, 3 \)) that satisfy given the boundary conditions.

In terms of the potential functions \( f_i \) the displacements are

\[ u_1 = (1 - 2\nu)(f_{1,1} + f_{1,2}) - (3 - 4\nu)f_{1,3} + x_1 f_{1,j} \quad (8) \]

\[ u_2 = (1 - 2\nu)(f_{2,1} + f_{2,2}) - (3 - 4\nu)f_{2,3} + x_2 f_{2,j} \quad (9) \]

\[ u_3 = -(1 - 2\nu)(f_{3,1} + f_{3,2}) - 2(1 - \nu)f_{3,3} + x_3 f_{3,j} \quad (10) \]

Using the above equations in the generalized Hooke’s law for an isotropic material, the stress components \( \sigma_{ij} \) in terms of these potential functions \( f_i \) are given by

\[ \sigma_{11} = 2G[f_{1,11} + 2f_{1,22} - 2f_{1,31} - 2\nu f_{2,32} + x_3 f_{1,j}] \quad (11) \]

\[ \sigma_{22} = 2G[f_{2,12} + 2f_{2,11} - 2f_{2,31} - 2\nu f_{1,32} + x_3 f_{2,j}] \quad (12) \]

\[ \sigma_{33} = 2G[f_{3,11} + 2f_{3,22} - x_3 f_{3,31} + f_{3,j}] \quad (13) \]

\[ \sigma_{12} = 2G[1 - 2\nu]f_{3,12} - (1 - \nu)(f_{1,32} + f_{2,31}) + x_3 f_{1,j} \quad (14) \]

\[ \sigma_{13} = 2G[1 - 2\nu](f_{1,11} + f_{1,22}) + \nu f_{1,11} + f_{1,22} + x_3 f_{1,j} \quad (15) \]

\[ \sigma_{23} = 2G[1 - 2\nu](f_{2,11} + f_{2,22}) + \nu f_{2,11} + f_{2,22} + x_3 f_{2,j} \quad (16) \]

where \( G \) is the shear modulus.

3.2. Ellipsoidal solution

We assume that the crack surface is described by the equation

\[ \frac{x_1}{a_1} \right)^2 + (\frac{x_2}{a_2})^2 + (\frac{x_3}{a_3})^2 = 1 \]

The ellipsoidal solution is

\[ \omega(s) = 1 - \frac{s}{Q(s)} \]

where \( a_i = 0 \). Then

\[ \omega(s) = \frac{P(s)}{Q(s)} \]

where

\[ P(s) = (s - s_0) \quad Q(s) = (s + s_0) \]

and \( s_0 \geq 0 \). \( s_0 \) is equal to the distance from the tip of the crack to the singularity point, and \( s \leq s_0 \). It is simply \( s_0 = 0 \). The elliptic solution is

\[ \omega(s) = \frac{1}{Q(s)} \]

It is noticed that

\[ -\frac{\partial Q(s)}{\partial s_0} = \frac{s_0}{Q(s)} \]

The Jacobian of the transformation can be found by setting \( s = s_0 \) (i.e. the transformation point coinciding with the crack tip) and evaluating the determinant of the Jacobian matrix. The Jacobian of the transformation can be found by setting \( s = s_0 \) (i.e. the transformation point coinciding with the crack tip) and evaluating the determinant of the Jacobian matrix.
3.2. Ellipsoidal potentials

We assume that the crack is in the $x_3 = 0$ plane, as shown in Fig. 2. The major axis of the elliptical crack surface is on the $x_1$-axis; and the minor axis is on the $x_2$-axis. The crack front is defined by

$$\left(\frac{x_1}{a_1}\right)^2 + \left(\frac{x_2}{a_2}\right)^2 = 1 \quad a_1 > a_2$$

(17)

The ellipsoidal coordinates $\xi_i$ ($i = 1, 2, 3$) are the roots of the cubic equation

$$\omega(s) = 1 - \frac{x_1^2}{a_1^2 + s} - \frac{x_2^2}{a_2^2 + s} - \frac{x_3^2}{a_3^2 + s} = 0$$

(18)

where $a_3 = 0$. The cubic equation (Eq. (18)) can be rewritten in the ellipsoidal coordinates as in the following:

$$\omega(s) = \frac{P(s)}{Q(s)} = \frac{(s - \xi_1)(s - \xi_2)(s - \xi_3)}{(s + a_1^2)(s + a_2^2)(s + a_3^2)} = 0$$

(19)

where

$$P(s) = (s - \xi_1)(s - \xi_2)(s - \xi_3)$$

$$Q(s) = (s + a_1^2)(s + a_2^2)(s + a_3^2)$$

and $\xi_3 > 0 > \xi_2 > -a_2^2 > \xi_1 > -a_1^2$. In the ellipsoidal coordinates, the surface of the elliptical crack is simply $\xi_3 = 0$. From Eqs. (18) and (19) we have

$$\frac{P(s)}{Q(s)} = 1 - \frac{x_1^2}{a_1^2 + s} - \frac{x_2^2}{a_2^2 + s} - \frac{x_3^2}{a_3^2 + s}$$

(20)

It is noticed that

$$\left[\frac{\partial P(s)}{\partial \xi_i}\right]_{s = \xi_i} = -P'(\xi_i) \quad i = 1, 2, 3$$

(21)

The Jacobian of the transformation, from the rectangular system to the ellipsoidal coordinate system, can be found by first taking the partial derivative of Eq. (20) with respect to $x_j$ ($j = 1, 2, 3$) and then setting $s = \xi_i$ ($i = 1, 2, 3$).

![Fig. 2. An elliptical crack in an infinite body.](image)
\[
\frac{\partial \xi_j}{\partial x_i} = \frac{2 \gamma_j Q(\xi_j)}{(a_j^2 + \xi_j)P(\xi_j)}
\]

Since
\[
\omega^\alpha = \left[ 1 - \frac{x_1^2}{a_1^2 + s} - \frac{x_2^2}{a_2^2 + s} - \frac{x_3^2}{a_3^2 + s} \right]^{\alpha}
\]
\[
= \sum_{p=0}^{2} \sum_{q=0}^{2} \sum_{r=0}^{2} (-1)^p C_p^\alpha C_q^\beta C_r^\gamma \left( \frac{x_1}{a_1} + s \right)^{\alpha-p} \left( \frac{x_2}{a_2} + s \right)^{\beta-q} \left( \frac{x_3}{a_3} + s \right)^{\gamma-r}
\]
where
\[
C_i^j = \frac{i!}{(i-j)!j!}
\]

the generic integral of the type
\[
P^{\alpha}_{i \beta k l} = \int_{\xi_j}^{\infty} \frac{\partial^{i+k+l} \omega^\alpha}{\partial x_1^i \partial x_2^k \partial x_3^l} \sqrt{Q(s)}
\]
can be found as in the following:
\[
P^{\alpha}_{i \beta k l} = \sum_{p=0}^{2} \sum_{q=0}^{2} \sum_{r=0}^{2} (-1)^p C_p^\alpha C_q^\beta C_r^\gamma \left( \frac{x_1}{a_1} + s \right)^{\alpha-p} \left( \frac{x_2}{a_2} + s \right)^{\beta-q} \left( \frac{x_3}{a_3} + s \right)^{\gamma-r}
\]
\[
\times x_1^{2p-2q-2r} x_2^{2q-2r-2k} x_3^{2r-2l}
\]
where \( J_{i \beta k l} \) is a generic elliptic integral defined as
\[
J_{i \beta k l}(\xi_j) = \int_{\xi_j}^{\infty} \frac{ds}{(s + a_1^2)(s + a_2^2)(s + a_3^2) \sqrt{Q(s)}}
\]
The following ellipsoidal potentials are used in the analytical solution.
\[
F_{i \beta} = \frac{\partial^{i+k} \omega^\alpha}{\partial x_1^i \partial x_2^k} \int_{\xi_j}^{\infty} \omega^{\alpha+k+1} \frac{ds}{\sqrt{Q(s)}}
\]
Since \( \omega(\xi_j) = 0 \), the basic potential functions \( F_{i \beta} \) can be expressed as in the following:
\[
F_{i \beta} = \int_{\xi_j}^{\infty} \frac{\partial^{i+k} \omega^\alpha}{\partial x_1^i \partial x_2^k} \frac{ds}{\sqrt{Q(s)}} + I^{i+k+1}_{i \beta k l}
\]
where \( I = 0 \).

The first partial derivatives of \( F_{i \beta} \) are
\[
F_{i \beta \alpha} = \int_{\xi_j}^{\infty} \frac{\partial^{i+k+1} \omega^\alpha}{\partial x_1^i \partial x_2^k \partial x_3} \frac{ds}{\sqrt{Q(s)}} = I^{i+k+1}_{i \beta \alpha}
\]
where \( j_1 = j + \delta_{1 \alpha}, k_1 = k + \delta_{2 \alpha}, \) and \( l_1 = \delta_{3 \alpha} \), and \( \delta_{1 \alpha} \) is the Kronecker delta.

The second partial derivatives of \( F_{i \beta} \) are
\[
F_{i \beta \alpha \beta} = I^{i+k+1}_{j_1 \beta k_1} + F^{(0)}_{i \beta \alpha \beta}
\]
where \( j_1 = j + \delta_{1 \alpha} + \delta_{2 \beta}, k_1 = k + \delta_{2 \alpha} + \delta_{2 \beta}, l_1 = \delta_{3 \alpha} + \delta_{3 \beta} \), and
\[
F^{(0)}_{i \beta \alpha \beta} = (k + l + 1)! \left[ \frac{\rho_j \rho_2 \rho_1}{P(s)} \sqrt{Q(s)} \right] s_j = \xi_j
\]
\[
\rho_j = \frac{\partial w_j}{\partial x_j} = \frac{1}{a_j^2}
\]
The third partial derivative
\[
F_{i \beta \alpha \beta \gamma} = I^{(0)}_{i \beta \alpha \beta \gamma}
\]
where \( j_3 = j + \delta_{1 \alpha} + \delta_{2 \beta} \), \( k_3 = k + \delta_{2 \alpha} + \delta_{2 \beta} \), and \( \delta_{1 \alpha} \) is the Kronecker delta.

The partial derivative
\[
F^{(0)}_{i \beta \alpha \beta \gamma} = (k + l + m + 1)! \left[ \frac{\rho_j \rho_2 \rho_1}{P(s)} \sqrt{Q(s)} \right] s_j = \xi_j
\]

3.3. The generic elliptic integral

Nishioka and Aihara (1980) appears repeated here. The generic elliptic integral is summarized as follows:

Using the Jacobi's elliptic functions
\[
\text{sn}^2 u = \frac{a_1^2}{a_2^2} + \frac{a_2^2}{a_3^2}
\]
\[
\text{sn}^2 u + \text{cn}^2 u = 1
\]
\[
k^2 \text{sn}^2 u + \text{dn}^2 u = 1
\]
\[
k^2 = \frac{a_1^2 - a_2^2}{a_1^2}
\]
The generic elliptic integral
\[
J_{i \beta}(\xi_j) = \int_{\xi_j}^{\infty} \frac{ds}{(s + a_1^2)(s + a_2^2)(s + a_3^2) \sqrt{Q(s)}}
\]
can be rewritten as
\[
J_{i \beta} = \frac{\rho_j \rho_2 \rho_1}{P(s)} \sqrt{Q(s)}
\]
Denote
\[
L_{i \beta} = \int_{\xi_j}^{\infty} \frac{ds}{(s + a_1^2)(s + a_2^2)(s + a_3^2) \sqrt{Q(s)}}
\]
The recursive formula
\[
\rho_j = \frac{\partial w_j}{\partial x_j} = \frac{1}{a_j^2}
\]
where
\[ \rho_i = \frac{\partial w}{\partial x_i} = \frac{-2x_i}{a_i^2 + s} \quad i = 1, 2, 3 \]  \hspace{1cm} (31)

The third partial derivatives of \( F_{jk} \) are
\[ F_{jko\bar{\beta}y} = F_{jko\bar{\beta}y}^0 \left( G_{jko\bar{\beta}y}^0 \right) \]  \hspace{1cm} (32)

where \( j = j + \delta_{j} \), \( k = k + \delta_{k} \), \( \beta = \beta + \delta_{\beta} \), \( \gamma = \gamma + \delta_{\gamma} \), \( l_3 = l_3 + \delta_{l_3} + \delta_{\gamma} \), and
\[ \begin{align*}
G_{jko\bar{\beta}y}^0 &= (k + l + 1)! \left[ \frac{\rho_1^2 \rho_2^4 \rho_3^4 \sqrt{Q(s)}}{P'(s)} \left( \frac{j_2(j_2 - 1)}{2\rho_1 x_1} + \frac{k_2(k_2 - 1)}{2\rho_2 x_2} + \frac{l_2(l_2 - 1)}{2\rho_3 x_3} \right) \right] + \xi_3 \]  \hspace{1cm} (33)

The partial derivative \( F_{jko\bar{\beta}y}^0 \) is
\[ F_{jko\bar{\beta}y}^0 = F_{jko\bar{\beta}y}^0 \left[ \frac{j_2 \delta_{\xi_3}}{\rho_1} + \frac{k_2 \delta_{\xi_3}}{\rho_2} + \frac{l_2 \delta_{\xi_3}}{\rho_3} + \frac{Q'(s)}{2Q(s)} \left( \frac{P''(s)}{P'(s)} - \frac{j_2}{a_1 + s} - \frac{k_2}{a_2 + s} - \frac{l_2}{a_3 + s} \right) \right] \]  \hspace{1cm} (34)

3.3. The generic elliptic integral

Nishioka and Atluri [28] devised a systematic way to evaluate the generic elliptic integral \( J_{jm} \), which appears repeatedly in the ellipsoidal potentials and their partial derivatives. The procedure is briefly summarized here.

Using the Jacobian elliptical functions,
\[ \begin{align*}
\sin^2 u &= \frac{a_1^2}{a_1^2 + s} \\
\sin^2 u + \cos^2 u &= 1 \hspace{1cm} \text{nc} = \frac{1}{\csc} \\
k^2 \sin^2 u + \cos^2 u &= 1 \hspace{1cm} \text{and} = \frac{1}{\csc} \\
k^2 &= \frac{a_1^2 - a_2^2}{a_1^2} \hspace{1cm} k'^2 = 1 - k^2
\end{align*} \]  \hspace{1cm} (35) \hspace{1cm} (36) \hspace{1cm} (37) \hspace{1cm} (38)

The generic elliptic integral
\[ J_{jm}(\xi_3) = \int_{\xi_3}^{\infty} \frac{ds}{(s + a_1^2)(s + a_2^2)s^m \sqrt{Q(s)}} \]  \hspace{1cm} (39)

can be rewritten as
\[ J_{jm} = \frac{2}{a_1^2 l_{j+m+1} + 1} \int_0^{a_1} \sin^{2l_{j+m}} u \eta \Delta_{j:m} u \nabla_{2m} u du \]  \hspace{1cm} (40)

Denote
\[ L_{jm} = \int_0^{a_1} \sin^{2l_{j+m}} u \eta \Delta_{j:m} u \nabla_{2m} u du \]  \hspace{1cm} (41)

The recursive formula for the evaluation of \( L_{jm} \) is
\[ L_{ijm} = \frac{1}{(2m - 1)k^{2i}} \left\{ \left[ s_{n}^{2i} u_{nc}^{2m-1} u \right] u_{i}^{*} \right. \]
\[ + [2(m-1) + 2(j-2m-2)k]L_{ij,m-1} + k^{2}(2j + 2m - 3)L_{ij,m-2} \} \] (42)

The starting terms in the evaluation of \( L_{ijm} \) are
\[ L_{ij,-1} = \int_{0}^{\alpha} s_{n}^{2i} u_{nc}^{2m-1} u \, du \] (43)
\[ L_{ij,-2} = \int_{0}^{\alpha} s_{n}^{2i} u_{nc}^{2m-1} u \, du \] (44)

They can be found by using
\[ L_{ij,-1} = \frac{1}{k^{2i+2}} \sum_{a=0}^{i} \sum_{\beta=0}^{1} (-1)^{a + \beta + 1} k^{2(1 - \beta) \alpha} M_{2(i - \alpha - \beta)} \] (45)
\[ L_{ij,-2} = \frac{2}{k^{2i+2}} \sum_{a=0}^{i} \sum_{\beta=0}^{1} (-1)^{a + \beta + 1} k^{2(2 - \beta) \alpha} M_{2(i - \alpha - \beta)} \] (46)

where
\[ M_{2m} = \int_{0}^{\alpha} u_{nc}^{2m-1} u \, du \quad m > 0 \] (47)
\[ M_{-2m} = M_{2m} = \int_{0}^{\alpha} u_{nc}^{2m-1} u \, du \quad m > 0 \] (48)

The recursive formulae for \( M_{2m} \) and \( M_{-2m} \) are
\[ M_{2m+2} = \frac{2m(2 - k^{2}) M_{2m} + (1 - 2m) M_{2m-2} - k^{2} s_{nu} cnu_{i} nd^{2m-1} u_{i}^{*}}{(2m + 1) k^{2}} \] (49)
\[ M_{-2m+2} = \frac{2m(2 - k^{2}) M_{2m} + (1 + 2m) k^{2} M_{-2m-2} + k^{2} s_{nu} cnu_{i} nd^{2m-1} u_{i}^{*}}{(2m + 1)} \] (50)

The starting terms for the evaluation of \( M_{2m} \) and \( M_{-2m} \) are
\[ M_{0} = M_{0} = F(u_{1}) = u_{1} \] (51)
\[ M_{2} = \left( \frac{E_{0} - k^{2} s_{nu} cnu_{i} nd_{i}}{k^{2}} \right) \] (52)
\[ M_{2} = E(u_{1}) \] (53)

where \( F(u) \) and \( E(u) \) are incomplete elliptic integrals of the first and the second kinds.

3.4. An analytical solution for an embedded elliptical crack

Since \( x_{1} = 0 \) at the crack surface, the crack surface tractions can be represented by potential functions \( f_{i} (i = 1, 2, 3) \) as in the following:
\[ \sigma_{x1} = 2G[(1 - v)(f_{1,11} + f_{1,22}) + v(f_{1,11} + f_{2,22})] \] (54)
\[ \sigma_{y1} = 2G[(1 - v)(f_{2,11} + f_{2,22}) + v(f_{1,11} + f_{2,22})] \] (55)
\[ \sigma_{z1} = 2G(f_{3,11} + f_{3,22}) \] (56)

The crack surface displacements can be expressed in the following fashion:
\[ f_{a} = \sum_{i=0}^{1} \sum_{j=0}^{1} f_{a}(u_{i}) \] (41)

The ellipsoidal potential system of equations can be obtained by comparing terms and comparing the coefficients of the displacement and intensity factors
\[ K_{1} = KH_{1} \] (57)
\[ K_{11} = K(H_{1}) \] (58)
\[ K_{111} = (1 - \nu^{2}) \] (59)

where
\[ K = \frac{8G}{a^{2}a_{1}} \] (59)
\[ A = (a_{1}^{2} - a^{2}) \] (59)
\[ H_{1} = \sum_{i=0}^{1} \sum_{j=0}^{1} - \frac{1}{a_{i}a_{j}} \] (59)
\[ H_{2} = \sum_{i=0}^{1} \sum_{j=0}^{1} \frac{1}{a_{i}a_{j}} \] (59)
\[ H_{3} = \sum_{i=0}^{1} \sum_{j=0}^{1} \frac{1}{a_{i}a_{j}} \] (59)

where \( i = 1 - i \) and \( j = 1 - j \)

4. Solutions for the embedded elliptical crack

Complex potentials can be obtained for situations of this type. Solutions for most of the cases, especially linear loads, are of a fairly simple nature. The constant/linear intensity factor must be evaluated for each load case. Since computing the intensity factors has the advantage to considerably reduce the evaluations of displacements.
The crack surface traction of polynomial distribution can be expressed as

$$
\vec{\sigma}_{s\alpha} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} A_{ijmn}^a x_1^{2m-2n-i} x_2^{2n+j} \alpha = 1, 2, 3.
$$

(57)

We express the potential functions $f_i$ ($i = 1, 2, 3$) in terms of the basic potential functions $F_{ij}$ in the following fashion.

$$
f_i = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} C_{ijmn}^{a} F_{2m-2n-i, 2n+j} \alpha = 1, 2, 3.
$$

(58)

The ellipsoidal potentials $F_{ij}$ are polynomials in $x_1$ and $x_2$ at the crack surface $\xi = 0$. Thus, a linear system of equations for $C_{ijmn}^{a}$ can be obtained by substituting Eqs. (57) and (58) into Eqs. (54)-(56) and comparing the coefficients of the power series. Solving the system of linear equations, we obtain the coefficients of the potential series, $C_{ijmn}^{a}$, for the given boundary condition (Eq. (57)). Stresses and displacements are given in terms of the potentials functions in Eq. (8) through Eq. (16). The stress intensity factors [53] are

$$
K_1 = KH_1 A
$$

(59)

$$
K_{11} = K(H_1 a_2 \cos \theta + H_2 a_1 \sin \theta) A^{-1}
$$

(60)

$$
K_{111} = (1 - v)K(H_1 a_2 \cos \theta - H_1 a_1 \sin \theta) A^{-1}
$$

(61)

where

$$
K = \frac{8G}{a_1 a_2} \sqrt{\frac{\pi}{a_1 a_2}}
$$

(62)

$$
A = (a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta)^{1/4}
$$

(63)

$$
H_1 = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} \frac{T_{ijm}}{a_1} \left( \frac{\cos \theta}{a_1} \right)^{2m-2n+i} \left( \frac{\sin \theta}{a_2} \right)^{2n+j} C_{ijmn}^1
$$

(64)

$$
H_2 = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} \frac{T_{ijm}}{a_2} \left( \frac{\cos \theta}{a_1} \right)^{2m-2n+i} \left( \frac{\sin \theta}{a_2} \right)^{2n+j} C_{ijmn}^2
$$

(65)

$$
H_3 = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} \frac{T_{ijm}}{a_1} \left( \frac{\cos \theta}{a_1} \right)^{2m-2n+i} \left( \frac{\sin \theta}{a_2} \right)^{2n+j} C_{ijmn}^3
$$

(66)

where $i = 1 - i$ and $j = 1 - \bar{j}$, and

$$
T_{ijm} = (-2)^{2n-i-j}(2m + i + j + 1)!
$$

4. Solutions for 2-D cracks

Complex potentials are in terms of Cauchy integrals. Explicit solutions in infinite bodies can be obtained for situations of simple crack surface tractions, using the complex potential theory. Here, the solutions for multiple collinear cracks subjected to point loads, or alternatively, piecewise constant/linear loads, are presented. Also, solutions for a single crack subjected to point loads, and piecewise constant/linear loads, are presented. In general, using the complex potential theory, displacements must be evaluated using numerical integration schemes. But explicit solutions are also available for a single crack subjected to simple loads, such as point loads, or piecewise constant/linear loads, etc. Since computing displacements using numerical integration is both expensive and inaccurate, it is of advantage to construct the multiple crack solutions using that for a single crack, especially when the evaluations of displacements is necessary.
These localized loading forms have the advantage of being able to capture complicated crack surface tractions, in contrast to some smooth and continuously distributed loads, such as polynomial distributions. Point loading gives the simplest solution. But point loading causes artificial redistribution of crack surface tractions. This leads to poor stress solutions at the crack surface. Point loading can be used in the linear elastic analysis, where the stress solution around crack surface traction is not important, as long as the stress intensity factor, which is of primary interest, can be obtained directly. However, piecewise constant/linear crack-face loads should be used in an elastoplastic analysis, since capturing the stress-state near the crack is essential, in order to compute the elastic-plastic fracture parameters, as discussed later in this paper.

4.1. Mushkelishvili’s fundamental solution

It is assumed that the n collinear multiple cracks (see Fig. 3), \( \bar{a}_k \bar{b}_k, k = 1, 2, \ldots, n \), are on the real axis. The \( x \)-coordinates of the left and the right crack tips of the \( k \)-th crack are \( a_k \) and \( b_k \). The stresses at the upper crack surfaces are \( \sigma^+ \) and \( \sigma^- \), while the stresses at the lower crack surfaces are \( \sigma^+ \) and \( \sigma^- \).

The load functions \( p(x) \) and \( q(x) \) are defined on the real axis \( x \).

\[
p(x) = \frac{1}{2} [\sigma^+(x) + \sigma^-(x)] - i \frac{1}{2} [\sigma^+(x) - \sigma^-(x)]
\]

\[
q(x) = \frac{1}{2} [\sigma^+(x) - \sigma^-(x)] + i \frac{1}{2} [\sigma^+(x) + \sigma^-(x)]
\]

where \( i = \sqrt{-1} \).

The complex potential functions \( \Phi(z) \) and \( \Omega(z) \) are defined in terms of load functions,

\[
\Phi(z) = \Phi_0(z) + \frac{P_n(z)}{X(z)} - \alpha
\]

\[
\Omega(z) = \Omega_0(z) + \frac{P_n(z)}{X(z)} + \alpha
\]

where

\[
\Phi_0(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X^*(t) p(t)}{t - z} dt + \frac{1}{2\pi i} \int_L \frac{q(t)}{t - z} dt
\]

\[
\Omega_0(z) = \frac{1}{2\pi i X(z)} \int_L \frac{X^*(t) p(t)}{t - z} dt - \frac{1}{2\pi i} \int_L \frac{q(t)}{t - z} dt
\]

and \( P_n \) is a polynomial of order \( n \), i.e.

\[
P_n(z) = \sum_{k=0}^n c_k z^k
\]

The integration path \( L \) for \( \Phi_0 \) and \( \Omega_0 \) is the union of all the cracks \( \bar{a}_k \bar{b}_k, k = 1, 2, \ldots, n \). The + sign in

\[
X^*(t) \text{ indicates a function } X(z) \text{ is complex conjugate of } X(z).
\]

\[
X(z) = \prod_{k=1}^n \frac{1}{z - a_k}
\]

\[
\kappa = \frac{3 - \nu}{1 + \nu} \text{ and the rigid body displacement } \text{ is}
\]

\[
\kappa \int_{L_k} \Phi(z)
\]

where \( L_k \) is that for the \( k \)-th crack.

Once the constant \( \alpha \) is determined, the solution is given by

\[
\sigma^+ = \sigma^-, \ \sigma^- = \sigma^+
\]

\[
2\mu(u + w)
\]

where \( w \) is the dilatation and \( \mu \) is the shear modulus.

The above complex load function method. The complex potential satisfies the governing differential equation of equilibrium from the finite crack problem. The principle of superposition and the reciprocity, the stress state at the crack surfaces, i.e. \( \sigma^+ \) and \( \sigma^- \), and (68) can be written as

\[
\Phi(z) = \Omega(z)
\]

where

\[
I_0(z) = \int_{L_k} \Phi(z)
\]

The uniqueness theorem for the complex potential function \( \Phi(z) \) indicates the solution

\[
\sum_{j=0}^{n-1} K_{ij} c_i = 0
\]
$X^i(t)$ indicates that the upper surface value of the function $X(t)$ is taken in the integration. The function $X(z)$ is defined as

$$X(z) = \prod_{k=1}^{n} \sqrt{z-a_k} \sqrt{z-b_k}$$

The constant $a$ and the coefficient $c_k (k = n)$ in the potential functions are determined by the stress and the rigid body rotation at infinity. The other coefficients $c_k, k = 0, 1, \ldots, n-1$ are determined by the uniqueness conditions of the displacements (Eq. (69)).

$$\kappa \oint_{\Gamma_k} \Phi(z) \, dz - \oint_{\Gamma_k} \Omega(\bar{z}) \, d\bar{z} = 0 \quad k = 1, 2, \ldots, n \quad (69)$$

where $\Gamma_k$ is the contour surrounding the $k$th crack $\bar{a_k} \bar{b_k}, k = 1, 2, \ldots, n$. The material constant $\kappa$ is defined as

$$\kappa = \begin{cases} 
3 - 4\nu & \text{plane strain} \\
\frac{3 - \nu}{1 + \nu} & \text{plane stress}
\end{cases}$$

where $\nu$ is the Poisson’s ratio.

Once the complex potential functions $\Phi$ and $\Omega$ are determined, the stresses and displacements are given by

$$\sigma_{xy} + \sigma_{yx} = 2[\Phi(z) + \sqrt{\Phi(z)}] \quad (70)$$

$$\sigma_{xy} = i\sigma_{yx} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\Phi'(z) \quad (71)$$

$$2\mu(\sigma_{xy} + i\sigma_{yx}) = \kappa \Phi(z) - \omega(z) - (z - \bar{z})\Phi'(z) \quad (72)$$

where $\Phi'(z) = \Phi(z)$ and $\omega(z) = \Omega(z)$.

The above described solution can be simplified in the context of the finite element alternating method. The constants $a$ and $c_n$ must be zero if we assume that there are no stresses and no rigid body rotation at infinity. The analytical solution is used to remove the crack-face cohesive stress obtained from the finite element solution in the alternating method. Therefore, by Newton’s third law of traction reciprocity, the stresses $\sigma_{xy}$ and $\sigma_{yx}$ are the same for the upper crack surfaces and the lower crack surfaces, i.e., $\sigma_{xy}^+ = \sigma_{xy}^-$ and $\sigma_{yx}^+ = \sigma_{yx}^-$. Consequently, $q(t) = 0$. Thus, the complex potentials in Eqs. (67) and (68) can be rewritten as

$$\Phi(z) = \Omega(z) = \frac{\Gamma(z) + iP_n(z)}{2\pi i X(z)} \quad (73)$$

where

$$\Gamma(z) = \oint_{\Gamma_k} \frac{X^i(t)p(t)}{t-z} \, dt \quad (74)$$

The uniqueness condition of the displacements (Eq. (69)) can be simplified as

$$\oint_{\Gamma_k} \Phi(z) \, dz = 0 \quad k = 1, 2, \ldots, n \quad (75)$$

if the contours $\Gamma_k, k = 1, 2, \ldots, n$ symmetrical about the real axis. If we substitute Eq. (73) into uniqueness condition equation (Eq. (75)) and expand the polynomial $P_n(z)$, we have the following linear system for the coefficients $c_k (k = 0, 1, \ldots, n-1)$.

$$\sum_{j=0}^{n-1} K_{kj}c_j = r_k \quad k = 1, 2, \ldots, n \quad (76)$$
where the coefficients \( K_{kj} \) and \( r_j \) are determined by the following contour integrals.

\[
K_{kj} = \oint_{\Gamma_k} \frac{z^j}{X(z)} \, dz \quad \text{and} \quad r_j = \int_{\Gamma_k} \frac{I_{\Omega}(z)}{X(z)} \, dz .
\]

The stresses, displacement gradients and displacements are determined by the following equations, once the potential functions \( \Phi(z) \) or \( \Omega(z) \) are determined from the crack surface loading functions \( p(t) \), \( q(t) \) and the uniqueness condition of displacements.

\[
\sigma_x + i\sigma_y = 2[\Omega(z) + \Omega(\bar{z})] \quad \text{(77)}
\]

\[
\sigma_x - i\sigma_y = \Omega(z) + \Omega(\bar{z}) + (z - \bar{z})\Omega'(\bar{z}) \quad \text{(78)}
\]

\[
2\mu(u + iv) = \kappa \omega(z) - \omega(\bar{z}) - (z - \bar{z})\Omega'(\bar{z}) \quad \text{(79)}
\]

\[
2\mu(u + iv) = \kappa \Omega(z) - \Omega(\bar{z}) - (z - \bar{z})\Omega'(\bar{z}) \quad \text{(80)}
\]

\[
2\mu(u - iv) = \kappa \Omega(z) + \Omega(\bar{z}) + (z - \bar{z})\Omega'(\bar{z}) - 2\Omega(z) \quad \text{(81)}
\]

where \( \omega'(z) = \Omega(z) \), and the derivative of \( \Omega(z) \) is

\[
\Omega'(z) = \frac{i\Gamma(z) + iP'_\mu(z)}{2\pi iX(z)} = \frac{\Omega(z)}{2} \sum_{k=1}^{n} \left( \frac{1}{z - a_k} + \frac{1}{z - b_k} \right) \quad \text{(82)}
\]

In general, \( \omega(z) \) can be obtained only by using numerical integration. However, explicit expressions can be found for the single crack in an infinite domain, subjected to certain crack surface loading of a simple pattern, such as point loads, piecewise constant or piecewise linear loads, etc. Details will be given in the following sections.

The stress intensity factors are defined as

\[
K_i - iK_{ii} = \lim_{x \to a_k} \sqrt{2\pi(x - a_k)}(\sigma_x - i\sigma_y) \quad \text{for the left crack tip} \ a_k \quad \text{(83)}
\]

\[
K_i - iK_{ii} = \lim_{x \to b_k} \sqrt{2\pi(x - b_k)}(\sigma_x - i\sigma_y) \quad \text{for the right crack tip} \ b_k \quad \text{(84)}
\]

The following expressions for stress intensity factors are found by substituting Eqs. (78) and (73) into these definitions.

\[
K_i - iK_{ii} = \frac{i\gamma_{ab}}{X_k(x_{ab})} \sqrt{\frac{2}{\pi(x_{ab} - a_k)}} \left[ \Gamma(x_{ab}) + iP_{\mu}(x_{ab}) \right] \quad \text{(85)}
\]

where

\[
i_{ab} = \begin{cases} 1 & \text{for the left crack tip} \ a_k \\ -1 & \text{for the right crack tip} \ b_k \end{cases}
\]

\[
x_{ab} = \begin{cases} a_k & \text{for the left crack tip} \ a_k \\ b_k & \text{for the right crack tip} \ b_k \end{cases}
\]

and

\[
\tilde{X}_k(t) = \prod_{l=1,l \neq k}^{n} \sqrt{t - a_l} \sqrt{t - b_l}
\]

### 4.2. Approximations of the Cauchy integral

The evaluation of the Cauchy integral in Eq. (74) is very important in this analytical solution. The accuracy and efficiency of the finite element alternating method relies on the implementation of the algorithm for its numerical evaluation.
Any crack surface load can be represented by the linear combination of a set of linearly independent basis functions. Once the solutions for cracks subjected to a loading in term of these basis functions are obtained, the solution for cracks subjected to arbitrary load can be approximated by the linear combination of these solutions. Park and Atluri [34] used a set of Delta functions as basis functions. This may be the simplest and most efficient method to approximate the Cauchy integral. However, the use of Delta functions corresponds to replacing the distributed load with point loads. Point loads redistribute the crack surface traction artificially. Thus, the stress solutions around the load points are poor. This method can be used in the linear elastic fracture mechanics analysis, where an accurate solution of the actual stress field immediately around the crack surface is not required. The stress intensity factors, which represent the amplitudes of the singular solutions, are evaluated directly from the analytical solution in the linear elastic analysis. However, accurate solutions of the stress and strain fields around the crack surface are important in the elastic-plastic analysis in order to obtain fracture mechanics parameters, such as \( T^* \), accurately. Because the point loads will induce undesired artificial yielding around the loading points along the crack surfaces, the Delta functions are not suitable in the elastic-plastic analysis.

To improve the accuracy, Park [35] used a set of approximate piecewise constant functions. This may be the best choice when the analytical solution is used in conjunction with linear finite elements, where the stresses are constants within each element. Piecewise linear basis functions are used by Wang and Atluri [54], which may be the best when the quadratic elements are used.

Here, we present the solutions for point loadings and approximated piecewise constant/linear loadings.

### 4.2.1. Point loads

The point load at \( t = d \) can be expressed as

\[
p(t) = \delta(t - d)
\]

where \( \delta \) is Dirac's delta function. The Cauchy integral (Eq. (74)) for point load is simply

\[
\Gamma(z) = \int_L \frac{X^+(t)p(t)}{i - z} \, dt = \frac{X^+(d)}{d - z}
\]

(87)

\[
\Gamma'(z) = \frac{X^+(d)}{(d - z)^2} = \frac{\Gamma(z)}{d - z}
\]

(88)

### 4.2.2. Approximated piecewise constant loads

The approximated piecewise constant loads are described below:

\[
p(t) = \begin{cases} 
\sqrt{t - a_k} \sqrt{i - b_k} X^+(d) & t \in [d - e, d + e] \subset [a_k, b_k] \\
\sqrt{d - a_k} \sqrt{d - b_k} X^+(t) & \text{elsewhere}
\end{cases}
\]

(89)

They have non-zero values only on a small interval \([d - e, d + e]\) on the \( k \)th crack. The function \( p(t) \) equals one at \( t = d \). It is approximately one on the interval \([d - e, d + e]\). \( p(t) \) tends to one as the size of the interval tends to zero. These loads, defined on intervals that are small enough, can be used to approximate any distributed load to any desired accuracy. For simplicity, denote \( a = a_k \) and \( b = b_k \). The Cauchy integral (Eq. (74)) for this load is

\[
\Gamma(z) = \left[ \Gamma(d + e, z) - \Gamma(d - e, z) \right] \frac{X^+(d)}{\sqrt{d - a} \sqrt{d - b}}
\]

(90)

where
\[ \bar{F}(t, z) = \int_0^1 \frac{\sqrt{t-a} \sqrt{1-b}}{t-z} \, dt \]
\[ = \sqrt{t-a} \sqrt{1-b} + (2z-a-b) \ln(\sqrt{t-a} + \sqrt{1-b}) \]
\[ - \sqrt{a-z} \sqrt{b-z} \ln \frac{\sqrt{t-a} \sqrt{b-z} - \sqrt{1-b} \sqrt{a-z}}{\sqrt{t-a} \sqrt{b-z} + \sqrt{1-b} \sqrt{a-z}} \]
(91)

The derivative of \( \bar{F}(z) \) is
\[ \bar{F}'(z) = 2 \ln(\sqrt{t-a} + \sqrt{1-b}) + \frac{\sqrt{t-a} \sqrt{1-b}}{z-t} \]
\[ + \frac{1}{2} \left[ \frac{\sqrt{b-z}}{\sqrt{a-z} + \sqrt{b-z}} + \frac{\sqrt{a-z}}{\sqrt{a-z} + \sqrt{b-z}} \right] \ln \frac{\sqrt{t-a} \sqrt{b-z} - \sqrt{1-b} \sqrt{a-z}}{\sqrt{t-a} \sqrt{b-z} + \sqrt{1-b} \sqrt{a-z}} \]
(92)

It is noticed that \( X(t) = \sqrt{t-a} \sqrt{1-b} \) for a single crack. Thus, \( p(t) = 1 \) on the interval \([d-e, d+e] \). In this case we have the exact piecewise constant load functions.

An alternative choice of approximated piecewise constant load functions is Eq. (93).
\[ p(t) = \begin{cases} 
\sqrt{t-c} X'(d) & t \in [d-e, d+e] \subset [a_k, b_k] \\
0 & \text{elsewhere}
\end{cases} \]
(93)

where \( c \) is the \( x \)-coordinate of the closest crack tip to \( d \). So, \( c \) is either \( a_k \) or \( b_k \), depending of which is closer to \( d \). The Cauchy integral (Eq. (74)) for this load is
\[ f(t) = \left[ \bar{F}(d+e, z) - \bar{F}(d-e, z) \right] \frac{X'(d)}{\sqrt{d-c}} \]
(94)

where
\[ \bar{F}(t, z) = \int \frac{\sqrt{t-c}}{t-z} \, dt \]
\[ = 2\sqrt{t-c} + \sqrt{z-c} \ln \frac{\sqrt{t-c} - \sqrt{z-c}}{\sqrt{t-c} + \sqrt{z-c}} \]
(95)

The derivative of \( \bar{F}(z) \) is
\[ \bar{F}'(z) = \frac{\sqrt{t-c}}{z-t} + \frac{1}{2\sqrt{t-c}} \ln \frac{\sqrt{t-c} - \sqrt{z-c}}{\sqrt{t-c} + \sqrt{z-c}} \]
(96)

This alternative choice leads to a simpler form of analytical solution. But it is noticed that \( p(t) \) is an approximated step function even for a single crack in an infinite domain. The difference between \( p(t) \) and the step function for a single crack can be estimated as the following. Let \( c = b \). The load function
\[ p(t) = \frac{\sqrt{d-a}}{\sqrt{t-a}} = \left[ 1 + \frac{t-d}{d-a} \right]^{-1/2} \]
\[ = \left[ 1 - \frac{1}{2} \frac{t-d}{d-a} + O \left( \frac{t-d}{d-a} \right) \right] \]
\[ \approx 1 \]
It can be seen that \( p(t) \) is closer to 1 if \( b \) is the closer tip to \( d \). This indeed is the very reason why \( c \) should be the closest crack tip.

4.2.3. Approximated piecewise linear loads

The approximated piecewise linear basis functions are shown in Eq. (97).

\[
p(t) = \begin{cases} 
\sqrt{t-a} \sqrt{t-b} X^+(d_e) \quad (t-d_e) \\
\sqrt{d_a-a} \sqrt{d_a-b} X^+(t) \quad (d_a-d_e) \\
0 & \text{elsewhere}
\end{cases} 
\quad t \in [d-e, d+e] \subset [a_k, b_k] 
\]

where \( d_e = d \pm e \) and \( d_a = d \mp e \). They have zero values on the most part of the crack surfaces. The basis functions are approximately linear on the interval \([d-e, d+e]\). It can be easily verified that \( p(d_e) = 0 \) and \( p(d_a) = 1 \). On the interval \([d-e, d+e]\), the difference between the basis function \( p(t) \) and the linear basis function \((t-d_e)/(d_a-d_e)\) is determined by

\[
f(t) = \frac{p(t)}{(t-d_e)/(d_a-d_e)} = \frac{\sqrt{t-a} \sqrt{t-b} X^+(d_e)}{\sqrt{d_a-a} \sqrt{d_a-b} X^+(t)} 
\]

The factor \( f(t) \) is one at \( t = d_e \). It tends to one on the interval \((t-d_e)/(d_a-d_e)\) as the size of the interval shrinks to zero. Thus, the crack surfaces can be broken down to a number of intervals, on which the load can be approximated by the given basis functions. Better accuracy is achieved by reducing the size of the intervals, although this will increase the number of intervals and increase the computational time.

Denote \( a = a_k \) and \( b = b_k \). The Cauchy integral in Eq. (74) can be found explicitly as for this approximated piecewise linearly distributed load.

\[
\bar{I}(z) = \left[ \bar{P}(d+e, z) - \bar{P}(d-e, z) \right] \frac{X^+(d_e)}{(d_a-d_e) \sqrt{d_a-a} \sqrt{d_a-b}} 
\]

where

\[
\bar{P}(t, z) = \int \sqrt{t-a} \sqrt{t-b} \frac{t-d_e}{t-z} dt 
= \sqrt{t-a} \sqrt{t-b} \left[ \frac{(t-a)+(t-b)}{4} + (z-d_e) \right] 
+ (d_e-z) \sqrt{a-z} \sqrt{b-z} \ln \frac{\sqrt{t-a} \sqrt{b-z} - \sqrt{t-b} \sqrt{a-z}}{\sqrt{t-a} \sqrt{b-z} + \sqrt{t-b} \sqrt{a-z}} 
+ \left[ (z-d_e)(2z-a-b) - \frac{1}{4} (a-b)^2 \right] \ln(\sqrt{t-a} + \sqrt{t-b}) 
\]

The derivative of \( \bar{I}(z) \) is

\[
\bar{I}'(z) = \sqrt{t-a} \sqrt{t-b} \left[ 1 + \frac{z-d_e}{x} \right] 
- (a+b+2d_e-4z) \ln(\sqrt{t-a} + \sqrt{t-b}) 
- \left[ \sqrt{a-z} \sqrt{b-z} + \frac{d_e-z}{2} \left( \frac{\sqrt{b-z}}{\sqrt{a-z}} + \frac{\sqrt{a-z}}{\sqrt{b-z}} \right) \right] \ln \frac{\sqrt{t-a} \sqrt{b-z} - \sqrt{t-b} \sqrt{a-z}}{\sqrt{t-a} \sqrt{b-z} + \sqrt{t-b} \sqrt{a-z}} 
\]
An alternative simpler choice of approximated piecewise linear load is in Eq. (102).

\[
p(t) = \begin{cases} 
\frac{\sqrt{t-c}}{d_e-c} \frac{X^+(d_e)}{(d_e-d_c)\sqrt{d_e-c}} & t \in [d-e, d+e] \subseteq [a_k, b_k] \\
0 & \text{elsewhere}
\end{cases} 
\tag{102}
\]

where \(c\) is the \(x\)-coordinate of the closest crack tip to \(d\). Denote \(c'\) the other tip of the \(k\)th crack. The Cauchy integral (Eq. (74)) is

\[
I(z) = [\tilde{I}(d+e, z) - \tilde{I}(d-e, z)] \frac{X^-(d_e)}{(d_e-d_c)\sqrt{d_e-c}} 
\tag{103}
\]

where

\[
\tilde{I}(t, z) = \int \frac{\sqrt{t-c}}{t-z} \frac{t-d_e}{t-d_c} \, dt = 2[(z-d_e) + (t-c)/3] \sqrt{t-c} + (z-d_e) \sqrt{z-c} \ln \frac{\sqrt{t-c} - \sqrt{z-c}}{\sqrt{t-c} + \sqrt{z-c}} 
\tag{104}
\]

The derivative is

\[
\tilde{I}'(z) = \frac{3z - 2t - d_e}{z - t} \sqrt{t-c} + \frac{3z - 2c - d_e}{2\sqrt{z-c}} \ln \frac{\sqrt{t-c} - \sqrt{z-c}}{\sqrt{t-c} + \sqrt{z-c}} 
\tag{105}
\]

This choice of load (Eq. (102)) leads to a simpler analytical solution. But it is not exactly piecewise linear for a single crack in the infinite domain. On the other hand, Eq. (97) reduces to an exact piecewise linear function for a single crack in the infinite domain.

4.3. Solutions for a single crack in the infinite domain

The solution can be simplified significantly for a single crack in the infinite domain. The simplified solutions, including the closed form expressions for \(w(z)\) needed in the evaluation of displacements \(u\) and \(v\), are presented here for the point loads and piecewise constant/linear loads.

The complex potential functions become

\[
\Phi(z) = \Omega(z) = \frac{I(z)}{2\pi i X(z)} 
\tag{106}
\]

The derivatives of the complex potential functions are

\[
\Phi'(z) = \Omega'(z) = \frac{I'(z)}{2\pi i X(z)} = \frac{\Omega(z)(2z-a-b)}{2(z-a)(z-b)} 
\tag{107}
\]

4.3.1. Point loads

The complex potential functions are

\[
\Phi(z) = \Omega(z) = \frac{X^-(d)}{2\pi i (d - z) X(z)} 
\tag{108}
\]

Their derivatives are

\[
\Phi'(z) = \Omega'(z) = \Omega(z) \left[ \frac{1}{d-z} - \frac{2z-a-b}{2(z-a)(z-b)} \right] 
\tag{109}
\]

The function \(w(z)\) is

\[
w(z) = \frac{1}{\pi i} \ln \frac{\sqrt{z-d}}{\sqrt{d-a} \sqrt{z-b} - \sqrt{d-b} \sqrt{z-a}} 
\tag{110}
\]
4.3.2. Piecewise constant loads

The basis functions in Eq. (89) become piecewise constant functions

\[ p(t) = \begin{cases} 1 & t \in [d - e, d + e] \subset [a, b] \\ 0 & \text{elsewhere} \end{cases} \]  

(111)

The Cauchy integral in Eq. (106) is

\[ \tilde{I}(z) = \left[ \tilde{I}(d + e, z) - \tilde{I}(d - e, z) \right] \]

(112)

\( \tilde{I}(t, z) \) and \( \tilde{I}'(z) \) and are defined in Eqs. (91) and (92). The function \( \omega(z) \) used to find the displacement field can be found as the following:

\[ \omega(z) = \frac{\tilde{\gamma}(d + e, z) - \tilde{\gamma}(d - e, z)}{2\pi i} \]  

(113)

where

\[ \tilde{\gamma}(t, z) = 2\sqrt{z - a} \sqrt{z - b} \ln(\sqrt{t - a} + \sqrt{t - b}) \\
- t \ln(z - t) + 2t \ln(\sqrt{t - a} \sqrt{z - b} + \sqrt{t - b} \sqrt{z - a}) \\
+ z \ln \frac{\sqrt{t - a} \sqrt{b - z} - \sqrt{t - b} \sqrt{a - z}}{\sqrt{t - a} \sqrt{b - z} + \sqrt{t - b} \sqrt{a - z}} \]  

(114)

The basis functions in Eq. (93) become

\[ p(t) = \begin{cases} \sqrt{d - c'} & t \in [d - e, d + e] \subset [a, b] \\ \sqrt{t - c'} & \text{elsewhere} \end{cases} \]  

(115)

where \( c \) is the \( x \)-coordinate of the crack tip that is closer to \( d \), and \( c' \) is the other one. The Cauchy integral and the function \( \omega(z) \) become

\[ \tilde{I}(z) = \sqrt{d - c'} \left[ \tilde{I}(d + e, z) - \tilde{I}(d - e, z) \right] \]

(116)

\[ \omega(z) = \frac{\sqrt{d - c'} \left[ \tilde{\gamma}(d + e, z) - \tilde{\gamma}(d - e, z) \right]}{2\pi i} \]  

(117)

where

\[ \tilde{\gamma}(t, z) = 2\sqrt{z - c'} \ln \frac{\sqrt{t - c} - \sqrt{z - c}}{\sqrt{t - c} + \sqrt{z - c}} \\
+ 4\sqrt{t - c'} \ln \left( \frac{\sqrt{t - c} \sqrt{z - c'} + \sqrt{t - c'} \sqrt{z - c}}{\sqrt{z - t}} \right) \]  

(118)

\( \tilde{I}(t, z) \) and \( \tilde{I}'(z) \) are given in Eqs. (95) and (96).

4.3.3. Piecewise linear loads

The basis functions in Eq. (97) become exact piecewise linear functions.

\[ p(t) = \begin{cases} \frac{t - d_c}{d_{c'} - d_e} & t \in [d - e, d + e] \subset [a, b] \\ 0 & \text{elsewhere} \end{cases} \]  

(119)

The Cauchy integral in Eq. (106) is

\[ I(z) = \frac{\tilde{I}(d + e, z) - \tilde{I}(d - e, z)}{(d_c - d_e)} \]  

(120)
\( \tilde{I}(t, z) \) and \( \tilde{I}'(z) \) are defined in Eqs. (100) and (101). The function \( \omega(z) \) used to find the displacement field can be found explicitly as:

\[
\omega(z) = \frac{\tilde{\gamma}(d + e, z) - \tilde{\gamma}(d - e, z)}{2\pi i(d_a - d_e)}
\]  
(121)

where

\[
\tilde{\gamma}(t, z) = \sqrt{t - a} \sqrt{z - b} \left[ \sqrt{t - a} \sqrt{t - b} + (a + b + 2z - 4d_e) \ln(\sqrt{t - a} + \sqrt{t - b}) \right] \\
- (t - z)(t + z - 2d_e) \ln \frac{\sqrt{t - a} \sqrt{b - z} - \sqrt{t - b} \sqrt{a - z}}{\sqrt{t - a} \sqrt{b - z} + \sqrt{t - b} \sqrt{a - z}}
\]

The basis functions in Eq. (102) reduce to the following form.

\[
\rho(t) = \begin{cases} 
\sqrt{d_a - c'} \frac{(t - d_e)}{(d_a - d_e)} & t \in [d - e, d + e] \subset [a, b] \\
0 & \text{elsewhere}
\end{cases}
\]  
(122)

where \( c \) is the \( x \)-coordinate of the crack tip that is closer to \( d \) and \( c' \) is the other one. The Cauchy integral and the function \( \omega(z) \) become

\[
\tilde{I}(z) = \frac{\sqrt{d_a - c'} \tilde{I}(d + e, z) - \tilde{I}(d - e, z)}{(d_a - d_e)} 
\]  
(123)

\[
\omega(z) = \frac{\sqrt{d_a - c'} \tilde{\gamma}(d + e, z) - \tilde{\gamma}(d - e, z)}{2\pi i(d_a - d_e)}
\]  
(124)

where

\[
\tilde{\gamma}(t, z) = \frac{2}{3} \left[ 2\sqrt{v - c} \sqrt{z - c'} \sqrt{t - c} - \sqrt{z - c'}(3d_e - 2c' - t) \ln \frac{\sqrt{t - c} - \sqrt{z - c}}{\sqrt{t - c} + \sqrt{z - c}} \right. \\
+ 2\sqrt{t - c'}(3d_e - 2c' - t) \ln \frac{\sqrt{z - t}}{\sqrt{t - c} \sqrt{z - c'} + \sqrt{t - c'} \sqrt{z - c}}
\]

\( \tilde{I}(t, z) \) and \( \tilde{I}'(z) \) are given in Eqs. (104) and (105).

5. **Schwartz–Neumann alternating method**

The Schwartz–Neumann alternating method is based on the superposition principle. The solution on a given domain is the sum of the solutions on two other overlapping domains, with part of the boundary conditions as unknowns. The alternating method can be viewed as the fixed-point iteration scheme used to solve these unknown boundary conditions. Based on this point of view, we can perform a convergence study. The alternating method converges unconditionally when there are only traction boundary conditions specified on the body. The convergence criterion for mixed boundary value problems, where there are applied displacement boundary conditions as well as traction boundary conditions, is discussed in the following. Compare the work done by the applied forces in the following two cases. In the first case, arbitrary displacement conditions exist on the surfaces of the cracks in the cracked finite body, while all the boundary conditions elsewhere are replaced by homogeneous boundary conditions, i.e. remove all tractions and reduce all the applied displacements to zero magnitude. In the second case, the same displacement conditions exist on the surfaces of the cracks in the infinite domain. If the work done in the cracked finite body is always smaller than twice the work done in the infinite domain, the alternating method converges. Otherwise, it does not. For most
practical problems, this ratio is close to one. Thus, the alternating method converges rapidly, as discussed in detail in the following section.

5.1. Superposition principle and the alternating method

Consider \( n \) cracks in a body of a finite size. The crack surfaces which are traction free, are denoted collectively as \( \Gamma_c \). Let the boundary of the finite domain (not including the crack surface) be \( \Gamma'_c \), which the boundary with prescribed tractions \( \mathbf{t}^{(0)} \) is \( \Gamma'_c \), and the boundary with prescribed displacements \( \mathbf{u}^{(0)} \) is \( \Gamma'_c \). It is clear that \( \Gamma = \Gamma'_c \cup \Gamma_c \).

The alternating method uses the following two simpler problems to solve the original one. The first one, denoted as \( P_{ANA} \) [shown\(^2\) in Fig. 4(c)], is that of the same \( n \) cracks in the infinite domain subjected to the unknown crack surface loading \( T \). The second one, denoted as \( P_{FEM} \) [shown in Fig. 4(b)], has the same finite geometry as in the original problem except that the cracks are ignored. The boundary \( \Gamma'_c \) of \( P_{FEM} \) has the prescribed displacement \( \mathbf{u} \), while the boundary \( \Gamma'_c \) has the prescribed traction \( \mathbf{t} \). The prescribed displacements and tractions are different from those in the original problem in general. Because of the absence of the cracks, the problem \( P_{FEM} \) can be solved much easier by the finite element method (or the boundary element method).

To solve the original problem, \( P_{ORG} \) (shown in Fig. 4(a)), the crack surface loading \( T \), the prescribed displacement \( \mathbf{u} \) and the traction \( \mathbf{t} \) must be found such that the superposition of the two alternative

\[ \mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(b)} \]

\[ \mathbf{t} = \mathbf{t}^{(0)} + \mathbf{t}^{(b)} \]

![](image.png)

Fig. 4. Superposition principle for finite element alternating method.

\(^2\) Fig. 4 only illustrates one crack. Many cracks may be present.
problems \( P_{\text{ANA}} \) and \( P_{\text{FEM}} \) yields the original one, \( P_{\text{ORG}} \). The detailed procedures to find these boundary conditions are described as the following.

In the uncracked body problem \( P_{\text{FEM}} \), the tractions \( T \) at the location of the cracks in the cracked body \( P_{\text{ORG}} \), can be solved, for any given boundary loads \( u \) and \( t \), using the finite element method. Due to the linearity of the problem, the solution can be denoted as

\[
T = K' u + K' t
\]

(125)

where \( K' \) and \( K' \) are linear operators.

Similarly, the tractions \( t' \) on boundary \( \Gamma' \) and the displacements \( u' \) on boundary \( \Gamma_u \) can be found in the infinite domain \( P_{\text{ANA}} \), for the given crack surface load \( T \), which is the same as the crack surface traction obtained in the \( P_{\text{FEM}} \). The solution can be denoted as

\[
u' = K' T
\]

(126)

\[
t' = K' T
\]

(127)

where \( K' \) and \( K' \) are also linear operators. Subtract the solution for \( P_{\text{ANA}} \) from the one for \( P_{\text{FEM}} \). The resulting solution has zero tractions at the location of the crack surfaces. To ensure that the resulting solution has the same boundary conditions on \( \Gamma \), Eqs. (128) and (129) must be satisfied.

\[
u = u_0 + u'
\]

(128)

\[
t = t_0 + t'
\]

(129)

The unknown tractions \( T, t \) and unknown displacement \( u \) can be solved using these equations (Eqs. (125)–(129)). Eliminate \( u, u' \) and \( t, t' \), by substituting Eqs. (128), (129), (126) and (127) into Eq. (125) to obtain the following equation for the traction \( T \).

\[
(I - (K' K u + K' K') )T = (K' u_0 + K' t_0)
\]

(130)

Eliminate \( u', t' \) and \( T \) to obtain the following equation for the unknown traction \( t \) and unknown displacement \( u \).

\[
(I - A) \begin{bmatrix} u \\ t \end{bmatrix} = \begin{bmatrix} u_0 \\ t_0 \end{bmatrix}
\]

(131)

where

\[
A = \begin{bmatrix} K' K u & K' K' \\ K' K' & K' K' \end{bmatrix}
\]

and \( I \) is the identity operator.

Similarly, we can obtain the following linear system for traction \( t' \) and displacement \( u' \).

\[
(I - A) X = Y
\]

(132)

where

\[
X = \begin{bmatrix} u' \\ t' \end{bmatrix}
\]

\[
Y = A \begin{bmatrix} u' \\ t' \end{bmatrix} = \begin{bmatrix} K' K u & K' K' \\ K' K' & K' K' \end{bmatrix} \begin{bmatrix} u' \\ t' \end{bmatrix}
\]

It is possible to solve these equations directly to obtain the tractions \( T, t \) and displacement \( u \). But this involves the evaluation of \( K' \) and \( K' \), which requires solving the traction \( T \) at the location of the uncracked body subjected to all different loading patterns \( u \) and \( t \). We have to solve the uncracked body problem a larger number of times, as in the case of the boundary values problem. An alternative is solving directly the linear system. The solution is

\[
X = \sum_{i=1}^{n} X^{(i)}
\]

where \( X^{(i)} \) is the iterative scheme

\[
\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} K' K u \\ K' K' \end{bmatrix} = \begin{bmatrix} u_0 \\ t_0 \end{bmatrix}
\]

for \( i = 0, 1, 2, \ldots \) until the fixed point is reached.

5.2. Uniqueness of Solution

First it is shown that \( I - A \) is singular, i.e., there exist non-zero \( \begin{bmatrix} u' \\ t' \end{bmatrix} \) such that

\[
T = K' u' + K' t'
\]

\[
\begin{bmatrix} u' \\ t' \end{bmatrix} = -A^{-1} \begin{bmatrix} u_0 \\ t_0 \end{bmatrix}
\]

In this case the analytical solution on the boundary \( \Gamma_u \) and the solution, we obtain are not the same. Indeed, we obtain non-zero \( T \), while the analytical solution yields non-zero \( T \). This leads to a contradiction because the condition \( A \) is not singular.

The fixed point, \( X^{(i)} \), is achieved for an interval \((-1, 1) \).
problem a larger number of times, of the same order as that of the total number of degrees of freedom of the boundary nodes, using the finite element method. Thus, it can be very expensive to find \( X \) by solving directly the linear system \((I - A)X = Y\). A fixed point iteration scheme can be used to solve this linear system. The iterative scheme can be devised as:

\[
X^{(i+1)} = AX^{(i)} \quad i = 0, 1, 2, \ldots, \infty \tag{133}
\]

where \( X^{(0)} = \{u_0, t_0\}^T \). If this procedure converges, the solution is

\[
X = \sum_{i=1}^{\infty} X^{(i)}
\]

Since

\[
A = \begin{bmatrix} K_u' \\ K_t' \end{bmatrix} \cdot \begin{bmatrix} K_u & K_t' \end{bmatrix}
\]

the iterative scheme (Eq. (133)) is equivalent to the following alternating scheme

\[
T^{(i+1)} = K_u u^{(i)} + K_t t^{(i)}
\]

\[
\begin{bmatrix} u^{(i+1)} \\ t^{(i+1)} \end{bmatrix} = \begin{bmatrix} K_u & K_t' \end{bmatrix}^{-1} T^{(i)}
\]

for \( i = 0, 1, 2, \ldots, \infty \). In this case, the uncracked body problem is solved only a few times, because this fixed point iteration scheme converges quickly for practical problems. Therefore, the alternating method is much more efficient than solving the linear system directly. But it should be noticed that it may not be necessary to use the alternating method in some cases. It can be more efficient and accurate to solve directly when multiple crack solutions are constructed from that for a single crack. This will be discussed in detail in a later section.

5.2. **Uniqueness of solution and convergence**

First it is shown that \( I - A \) is not singular and the linear system (Eq. (132)) has a unique solution. Suppose \( I - A \) is singular. Then, there must exist a non-zero \( X \) such that \((I - A)X = 0\), which means that there exist non-zero \( u^{\circ} \) and \( t^{\circ} \), and therefore a non-zero \( T \), such that

\[
T = K_u u^{\circ} + K_t t^{\circ}
\]

\[
u^{\circ} = K_u T
\]

\[
t^{\circ} = K_t T
\]

In this case the analytical solution and the finite element solution have the same displacement \( u^{\circ} \) on boundary \( \Gamma_u \) and the same traction \( t^{\circ} \) on boundary \( \Gamma_t \). Subtracting the analytical solution from the FEM solution, we obtain the solution for the following problem. The entire boundary \( \Gamma \) is free of external loadings as well as the crack surfaces. But, the FEM solution gives zero displacements for the crack surfaces, while the analytical solution gives non-zero displacements for the crack surfaces because of the non-zero \( T \). Thus, the resulting solution has non-zero displacements at the crack surfaces. This is a contradiction because the cracks cannot be opened without any external load. Consequently, \( I - A \) is not singular.

The fixed point iteration scheme (Eq. (133)) converges if all the eigenvalues of \( A \) are in the open interval \((-1, 1)\). The scheme of Eq. (133) converges since the eigenvalues of \( A \) are in \((-1, 1)\) for most
problems of practical interest. The eigenvalues of $A$ are smaller than 1. Let $X_i$ be an eigenvector of $A$ corresponding to the eigenvalue $\lambda$.

$$T = K'(u^i) + K'(t^i)$$

$$\lambda u^i = K'T$$

$$\lambda t^i = K'T$$

The solution $P_{\text{RES}}$, shown in Fig. 5(a), is obtained by subtracting $\lambda$ times the FEM solution (Fig. 5(c)) from the analytical solution (Fig. 5(b)). Here, $u = 0$ and $t = 0$ on $\Gamma$ and the crack surface loading is $(1 - \lambda)T$, while the displacements at the crack surfaces are the same as those in the analytical solution. If the work done in opening the cracks in the infinite domain is $W$, the work done in opening the cracks in the finite domain (with the boundary condition $u = 0$ and $t = 0$) is $(1 - \lambda)W$, which is equal to the strain energy stored in the body. It must be positive. Thus, $\lambda < 1$.

It can be shown that $\lambda > 0$ in the absence of the prescribed displacement boundary conditions. In this case, the resulting solution from the subtraction has zero tractions at the boundary $\Gamma$. Apply additional load $\lambda t$ to the boundary $\Gamma$ with the crack surfaces fixed. The stress state in the body will be the same as that in the analytical solution described above, after this additional loading is applied. This procedure of adding load on the boundary $\Gamma$ is exactly the same as that in the FEM solution for the uncracked body, except that the load level is $\lambda$ times that in the FEM solution, because the crack surfaces are fixed. Therefore, the work done by the additional load is positive. Consequently, $(1 - \lambda)W < W$ and $\lambda > 0$. So, the alternating method converges for cracks in finite domains with arbitrary shapes and arbitrary traction boundary conditions.

In general, the eigenvalue $\lambda$ can be smaller than zero for mixed boundary problems. It is greater than $-1$ only if $(1 - \lambda)W < 2W$. Thus, the convergence criterion for the alternating method for the general case with mixed boundary conditions can be stated as follows. The alternating method [Eq. (133)] converges if the crack surface loads do less work in the finite domain, with the homogeneous boundary condition $u = 0$ and $t = 0$ on $\Gamma$, than twice as much as they do in the infinite domain for any arbitrary distribution of crack surface displacements (see Fig. 6).

Fig. 5. Subtract $\lambda P_{\text{FEM}}$ from $P_{\text{ANA}}$ to obtain the solution for $P_{\text{RES}}$, which has homogeneous boundary condition on $\Gamma$. 
Quick convergence can be expected for most of the practical applications. For any crack surface displacements, the displacements and stresses at a point decay rapidly as the point moves away from the cracks. Thus, the work done in the finite domain with the homogeneous boundary condition is very close to the work done in the infinite domain. This implies that the eigenvalues of $A$ are very small and the fixed point iteration converges rapidly. Indeed, all mixed boundary value problems we have solved (for both 2D and 3D problems) to date using finite element alternating method have converged.

5.3. Summary of FEAM procedure

The alternating procedure defined in Eq. (133) can be translated into the following simple procedure. Refer to Fig. 4.

1. Solve $P_{\text{TEM}}$ with the given load on the boundary $\Gamma$. Solve for the tractions, which are used to close the cracks. Denote the solution as $S_{1}^{\text{TEM}}$, where 1 indicates that this is the solution for the first iteration.

$$S_{1}^{\text{TEM}}: \quad T^{(1)} = K^{u}u^{(0)} + K^{t}t^{(0)}$$

2. Reverse the crack surface traction obtained in the previous step and apply it as the load on the crack surfaces and solve the $P_{\text{ANA}}$. Denote the solution as $S_{1}^{\text{ANA}}$.

$$S_{1}^{\text{ANA}}: \quad \begin{bmatrix} u^{(1)} \\ t \end{bmatrix} = \begin{bmatrix} K^{u} \\ K^{t} \end{bmatrix} T^{(1)}$$

3. Find the tractions on the boundary $\Gamma$ and the displacements on the boundary $\Gamma_{n}$ from the analytical solutions obtained in the previous step. Reverse them as the load for $P_{\text{TEM}}$. Find the crack closing tractions from the solution $S_{2}^{\text{TEM}}$.

$$S_{2}^{\text{TEM}}: \quad T^{(2)} = K^{u}u^{(1)} + K^{t}t^{(1)}$$
(4) Repeat steps 2 and 3 until the residual load is small enough to be ignored.

\[ -S^{\text{ANA}}_i = \left[ u^{(t)} \right] = \left[ K^u \right] T^{(t)} \]

\[ S^{\text{FEM}}_{i+1} = T^{(t) + 1} = K^u u^{(t)} + K^f f^{(t)} \]

for \( i = 2, 3, \ldots \).

The solution to the original problem is the summation of all those obtained in the alternating procedure, i.e.

\[ S = \sum_{i=1}^{n} (S^{\text{FEM}}_i + S^{\text{ANA}}_i) \quad (136) \]

5.4. Solutions for multiple embedded cracks

Solutions for multiple embedded cracks in an infinite body, subjected to arbitrary crack surfaces tractions, can be constructed using the solution to a single crack in an infinite body subjected to arbitrary crack surface loading. Analytical solutions for multiple embedded cracks in an infinite body are available only for some special configurations, such as multiple collinear cracks subjected to arbitrary crack surface tractions [23]. There are several implementations of finite element alternating method based on such analytical solutions [34, 54]. It is, in general, easier to construct the solution of multiple embedded cracks in an infinite body using the solution for a single embedded crack. Solutions for arbitrary distributions of cracks can be obtained using this approach. It can be more accurate and efficient to build the multiple crack solutions from that for a single crack even when the analytical solution is available, such as for the multiple collinear cracks in an infinite domain.

In the context of the finite element alternating method, it seems natural to use the Schwartz–Neumann alternating method to obtain the analytical solution iteratively. This approach has been used by many authors, such as O’Donoghue et al. [10] and Chen and Chang [31], etc. Using the alternating method and the solution for the single crack in the infinite domain, residuals induced by closing the other cracks are erased by reversing them and applying them as loads on the crack surfaces. Consider \( n \) arbitrarily distributed cracks. Let the given crack surface load on \( k \)th crack be \( T^{(0)}_k \), \( (k = 1, 2, \ldots, n) \). The alternating procedure is outlined as follows.

1. Consider a single crack, located at the same position as that of the \( k \)th crack in the original problem, in the infinite domain. Apply the load \( T^{(0)}_k \) on the crack surface. Denote the analytical solution for this loading as \( S^{(0)}_k \).

2. Compute the cohesive traction \( t^{(0)}_k \), that is used to close the \( j \)th crack in the original problem, using the solution \( S^{(0)}_k \).

\[ S^{(0)}_k = t^{(0)}_k = K^{(1)}_k T^{(0)}_k \quad j = 1, 2, \ldots, n \text{ and } j \neq k \]

where \( T^{(0)}_k = T^{(0)}_0 \) and \( K^{(1)}_j \) \((j = 1, 2, \ldots, n \text{ and } j \neq k)\) are linear operators. The superscript \( [k] \) indicates that the crack in the single-crack solution is at the same location as the \( k \)th crack in the original multiple-crack problem.

3. Sum the crack closure tractions due to all solution \( S^{(0)}_k \), \( (k = 1, 2, \ldots, n) \) to find the residual traction. Reverse the residual tractions as loads.

\[ T^{(1)}_j = - \sum_{k=1, k \neq j}^{n} t^{(0)}_k \quad j = 1, 2, \ldots, n \]

(4) Apply the load \( T^{(1)}_j \) on the crack surface as in the first step. Denote the solution as \( S^{(1)}_k \). Repeat the process of finding residual tractions until the residual tractions are small enough to be ignored.
$$S_{ik}^{(t)}: \quad t_{ik}^{(t)} = K_j^{[k]} T_k^{(t)} \quad j = 1, 2, \ldots, n \quad j \neq k$$

$$T_j^{(t+1)} = - \sum_{k=1, k \neq j}^{n} t_{ik}^{(t)} \quad j = 1, 2, \ldots, n$$

for \(t = 1, 2, \ldots\).

The solution to the multiple cracks subjected to the given load \(T_k^0 (k = 1, 2, \ldots, n)\) is the sum of all the solutions involved in the alternating procedure, i.e.

$$S = \sum_{t=1}^{\infty} \sum_{k=1}^{n} S_{ik}^{(t)}$$

However, the solution can be obtained using a non-iterative approach in a simpler and more efficient fashion. As we have shown in the previous section, an alternating method is essentially a fixed point iteration scheme used to solve a linear system. In the analysis of a cracked body, solving the linear system directly needs crack closure tractions for all possible boundary loadings. Thus, the uncracked body problem has to be solved a large number of times, of the same order as that of the total number of degrees of freedom of the nodes at the boundary, for different loadings. Only a few iterations are needed when finite element alternating method is used. Thus the uncracked body is solved only a few times.

On the other hand, these tractions can be easily evaluated in the analysis of multiple cracks using the analytical solutions. Since the number of degrees of freedom involved is small, the linear system can be solved directly.

The linear system for solving the multiple cracks problem is derived from the superposition principle. Consider the superposition of \(n\) solutions of single cracks in the infinite body. Each of these \(n\) solutions involves only one crack. Denote the \(k\)th solution as \(S_k\), where the crack is at the same location as that of the \(k\)th crack of the original multiple-crack problem. The crack surface traction \(T_k\) for the problem \(S_k\) \((k = 1, 2, \ldots, n)\) is to be determined (see Fig. 7).

The traction at the location of the \(j\)th crack in the problem \(S_i\) can be found for any load \(T_i\), i.e.

$$t_{jk} = K_j^{[k]} T_k \quad j, k = 1, 2, \ldots, n.$$  \hspace{1cm} (137)

It is noticed that \(K_j^{[k]} = I\) \((k = 1, 2, \ldots, n)\) are identity operators, because the tractions at the crack surfaces are the same as the applied loads.

The superposition of the \(n\) solutions should give back the original problem, i.e., the tractions at the locations of the crack surfaces should be the same as the given crack surface loads. Thus, the linear system to be solved is

$$\sum_{k=1}^{n} t_{jk} = \sum_{k=1}^{n} K_j^{[k]} T_k = T_k^0 \quad j = 1, 2, \ldots, n.$$  \hspace{1cm} (138)

We can denote collectively the undetermined crack surface loads, \(T_k\) \((k = 1, 2, \ldots, n)\), as \(T\). Similarly, denote collectively the given loads, \(T_k^0\) \((k = 1, 2, \ldots, n)\), as \(T^0\). Thus, Eq. (138) can be rewritten as

$$KT = T^0$$  \hspace{1cm} (139)

where \(K\) is a linear operator. Once the linear operator \(K\) is evaluated numerically, we can solve the linear system directly instead of using alternating method.

Crack surface tractions have to be discretized by a set of linearly independent basis functions, such as polynomial functions, Chebyshev polynomials, or certain piecewise continuous functions, since arbitrary crack surface tractions can not be handled directly by numerical methods.

Let the undetermined load \(T\) be approximated by \(N\) basis functions \(B_j\) \((j = 1, 2, \ldots, N)\).

$$T = \sum_{j=1}^{N} T_j B_j$$  \hspace{1cm} (140)
Similarly, the given load $T^0$ which can be approximated as in the following:

$$T^0 \approx \sum_{j=1}^{N} T_{j}^0 B_j$$  \hspace{1cm} (141)

We apply load $B_j$ on the cracks. Close all other cracks except the single crack on which $B_j$ has non-zero values. Find the tractions at the locations of the $n$ cracks of the original problem (see Fig. 8), using the analytical solution for a single crack in an infinite domain.

$$t_j = KB_j \quad j = 1, 2, \ldots, N$$  \hspace{1cm} (142)

where $K$ is a linear operator. The tractions $t_j$ ($j = 1, 2, \ldots, N$) are also approximated by these basis functions.

$$t_j \approx \sum_{i=1}^{N} t_i B_i \quad j = 1, 2, \ldots, N$$  \hspace{1cm} (143)

---

Footnote: Point loads (Delta functions) are used to illustrate the base functions in the figure. Only some of the loading cases are illustrated in the figure.
Once the magnitudes $t_{ij}$ are evaluated, the traction at the location of $k$th crack can be determined for a crack surface load $T_k$. Since

$$ t = KT $$

$$ = K \left( \sum_{j=1}^{N} T_j B_j \right) $$

$$ = \sum_{j=1}^{N} T_j K B_j $$

$$ \approx \sum_{j=1}^{N} \sum_{i=1}^{N} t_{ij} T_j B_i $$

the linear system (Eq. (139)) can be approximated as

$$ \sum_{j=1}^{N} \sum_{i=1}^{N} t_{ij} T_j B_i = \sum_{i=1}^{N} T_i^0 B_i $$

which leads to the following linear system of equations for the magnitudes $T_j$ ($j = 1, 2, \ldots, n$).

$$ \sum_{j=1}^{N} t_{ij} T_j = T_i^0 \quad i = 1, 2, \ldots, N $$
The alternating method essentially solves the same approximated linear system with a fixed point iteration scheme.

The non-iterative procedure to solve the multiple crack problem is summarized as the following.

1. Apply loads in terms of unit basis functions on one of the cracks, ignoring the other cracks. Use the analytical solution for a single crack to solve the tractions at the locations of all the other cracks (see Fig. 8).

2. Approximate these tractions in terms of the linear combination of the basis functions. Find the magnitude of each component. Thus, the coefficients of the linear system is determined as in Eq. (143).

3. Approximate the given crack surface load \( T_i \) in terms of the linear combination of the basis functions. Find the magnitude of each component.

4. Solve the linear system (Eq. (145)) to obtain the loads applied on each single crack solution.

5. Superpose the \( n \) single crack solutions to form a solution for the original problem.

The coefficients of the linear system remain the same in the analysis of the same cracks under different loadings, because they depend only on the crack configuration and the basis functions. Thus, the linear system can be solved for different loads without recomputing the coefficients of the system. This feature is particularly useful when the constructed multiple crack solution is used in the finite element alternating method, where it is necessary to evaluate the solution for the same cracks under different loadings during the alternating procedure.

5.5. Elastoplastic analysis of multiple cracks in a finite body

The linear elastic fracture mechanics approach can lead to non-conservative results in many fracture problems. Therefore, it is often necessary to perform elastoplastic analysis. The success of finite element alternating method in the linear elastic fracture analysis led to the development of Elastic-Plastic Finite Element Alternating Method (EPFEAM) [27]. EPFEAM has been successfully used in the analysis of builds-up aircraft structures [36, 37].

Elastoplastic analysis can be carried out by the Initial Stress Method [24], which reduces the non-linear analysis into a series of linear analyses, for which the principle of superposition holds. Thus, the finite element alternating method can be used to perform these linear analyses.

The initial stress method can be described as the following. Assuming no body forces, the virtual work principle is

\[
\int_{\Omega} \sigma : \delta \nabla u \, d\Omega = \int_{\Gamma} t^i : \delta u \, d\Gamma
\]  

(146)

where \( \sigma \) is the elastoplastic stress, \( t^i \) is the prescribed surface traction, \( \Omega \) is the domain of the body, and \( \Gamma_i \) is the boundary with prescribed tractions.

First, the elastic prediction is found by assuming that the deformation is entirely elastic. The elastoplastic stress \( \sigma^p \) within the body is found by using the displacements obtained in the linear elastic analysis. But \( \sigma^p \) may not satisfy the equilibrium equations. Let \( \sigma^+ \) be the undetermined correction for the stress, i.e., \( \sigma = \sigma^p + \sigma^+ \). Substituting this into Eq. (146), we find that \( \sigma^+ \) satisfies

\[
\int_{\Omega} \sigma^+ : \delta \nabla u \, d\Omega = \int_{\Gamma} t^i : \delta u \, d\Gamma - \int_{\Omega} \sigma^p : \delta \nabla u \, d\Omega
\]  

(147)

The right-hand side of the equation can be viewed as the virtual work done by the unbalanced force. The left-hand side of Eq. (147) is the virtual work done by the correction stress. The elastic estimate of the correction stress \( \sigma^+ \) can be solved by the alternating method for the linear elastic analysis. The new elastic prediction for the displacements is the sum of the old one and the correction term. This correction procedure is repeated until the unbalanced force becomes negligible.

The elastoplastic analysis of the cracked structures, using initial stress method and finite element alternating method, can be outlined as the following.

(1) Solve the linear elastic analysis for the cracked structure.

(2) Compute the displacement increment \( \delta u \).

(3) Compute the stress increment \( \delta \sigma \) using the elastic solution.

(4) Apply the stress increment \( \delta \sigma \) to the structure.

(5) Compute the new displacement solution.

(6) Repeat this process until the unbalanced force is negligible.

The above procedure is performed for each increment of applied loading. The contribution of each crack loading to any load component is calculated by its incremental increases. This general approach is also applied to the multiple crack analysis.
(1) Perform a linear elastic analysis of the cracked structure using the finite element alternating method. Denote the solution as \( S'_{(0)} \).

(2) Compute the elastoplastic stress \( \sigma_{(1)} \) using the displacement gradients of solution \( S'_{(0)} \).

(3) Compute the residual body force due to the incorrectness of the elastoplastic stress as indicated in Eq. (147).

(4) Apply the residual body force on the body. Perform linear elastic analysis as in step 1. Denote the solution as \( S_{(1)} \).

(5) Compute the elastoplastic stress \( \sigma_{(2)} \) using the displacement gradients from the sum of the solution \( S'_{(1)} \) and \( S_{(0)} \).

(6) Compute the residual body force and perform linear elastic analysis as in steps 3 and 4. Repeat this process of reducing the residual body force until the residual is small enough to be ignored. 

The method described above involves two loops of iteration. The finite element alternating method is embedded in the iteration of correcting residual body force due to the incorrectness of the stress solution. Since the alternating method corrects the residuals at the boundary and the initial stress method corrects the error in the plastic zone, we can combine them to form a single loop, as follows:

(1) Solve the crack closure traction using finite element method, assuming that the material is elastic. Denote the solution of displacement gradients as \( F_{(1)}^{\text{FEM}} \).

\[
T^{(1)} = K'u^{(0)} + K'f^{(0)}
\]

\[
F_{(1)}^{\text{FEM}} = F'u^{(0)} + F'f^{(0)}
\]

where \( K' \), \( K' \), and \( F' \) are linear operators. \( T^{(1)} \) is the traction used to close the crack.

(2) Reverse the traction obtained in the previous step and apply it as the load on the crack surfaces. Denote the analytical solution of displacement gradient as \( F_{(1)}^{\text{ANA}} \).

\[
F_{(1)}^{\text{ANA}} = -F^{T}T^{(1)}
\]

where \( F^{T} \) is a linear operator.

(3) Compute the elastoplastic stress due to the displacement gradient \( F_{(1)}^{\text{FEM}} + F_{(1)}^{\text{ANA}} \).

\[
\sigma_{(1)} = R(F_{(1)}^{\text{FEM}} + F_{(1)}^{\text{ANA}})
\]

where \( R \) is a non-linear operator.

(4) Compute the boundary load \( u^{(1)} \), \( t^{(1)} \) and the distributed load \( f^{(1)} \) due to the incorrectness of the stress \( \sigma_{(1)} \).

(5) Apply the load \( u^{(1)} \), \( t^{(1)} \) and \( f^{(1)} \) on the uncracked body, assuming that the material is elastic. Repeat the procedure of finding residuals until the process converges

\[
T^{(i+1)} = K'\bar{u}^{(i)} + K'\bar{f}^{(i)}
\]

\[
F_{(i+1)}^{\text{FEM}} = F'\bar{u}^{(i)} + F'\bar{f}^{(i)}
\]

\[
F_{(i+1)}^{\text{ANA}} = F^{T}T^{(i+1)}
\]

\[
\sigma_{(i+1)} = R\left( \sum_{j=1}^{i+1} (F_{(j)}^{\text{FEM}} + F_{(j)}^{\text{ANA}}) \right)
\]

for \( i = 1, 2, \ldots \), where \( K' \) and \( F' \) are also linear operators.

The above procedure can be applied using the deformation theory of plasticity, which is valid for a cracked structure undergoing monotonic proportional loading. For a plastic material undergoing loading/unloading, it is only valid for the first loading step using a \( J_2 \) flow theory of plasticity, i.e., loading the unstrained body to the given level of boundary load. But similar procedures can be applied to any loading/unloading process: the deformation gradient in the above procedure should be replaced by its increment for the loading step. The stress is determined from the previous stress state and the increment of displacement gradient. In the analysis of crack growth, the newly created cracks surfaces in general experience plastic deformation. To remove the crack closure stress, a step of evaluating
elastoplastic stress at the crack surface must be added before the evaluation of the analytical solution for cracks in the infinite domain.

5.6. Crack growth analysis in elastic and elastic-plastic solids

The alternating method can also be used in a crack growth analysis. In the case of linear elastic materials, the solution is unique for different crack sizes. Therefore, we can simply restart the entire finite element alternating procedure, after the crack extension, to analyze the new crack subjected to the given load. On the other hand, one may also, alternatively, make use of the linear elastic solution obtained right before the crack extension. As shown in Fig. 9, the crack closure traction ahead of the crack tips can be removed using the analytical solution for a crack that has the same length as the new crack.

Denote the original crack length as $2a$. The amount of crack extension is $2\Delta a$. The crack closure traction $T$ ahead of the crack tip for the problem with crack length $2a$ is evaluated from the solution $S^0$ obtained for the crack before the crack extension. In order to create new additional traction-free crack surfaces of additional length $2\Delta a$, we reverse the traction $T$ and apply it on the surface of the crack of length $2a + 2\Delta a$. Boundary residuals can be computed from this analytical solution $S_0^{\text{ANA}}$. Then, the usual finite element alternating method can be used to remove these boundary residuals for the crack of length $2a + 2\Delta a$. We suppose all the solutions, including $S^0$ and $S_0^{\text{ANA}}$, to obtain the solution for the extended crack.

It is noted that crack closure traction is distributed only at the recently generated crack surface. This traction is actually singular around the original crack tip. This type of localized and non-smooth crack surface traction can be captured by localized special basis functions. It can not be captured correctly by smooth basis functions, such as polynomials.

![Diagram](image)

Fig. 9. Remove the tractions at newly created crack surface.

Due to the nature of the analysis, we assume that the crack is i.e. rectangular, and the body remains rigid.

The traction $T$ for the material case is only relevant at crack tips. The new tractions $T_2$ can be evaluated according to the crack extension by

1. $T_2 = T$ in the region $0 < r < \Delta a$.
2. $T_2 = T$ in the region $\Delta a < r < 2a$.
3. $T_2 = 0$ in the region $2a < r < 2a + 2\Delta a$.

for $\Delta a < \Delta a < 2a$.

6. Evaluation of the algorithm

In the evaluation of the algorithm, it has been confirmed that the stresses for $\Delta a < 0.75a$ are valid.

There are two usual radius cases for the proposed method. For $\Delta a < 0.75a$, the stresses are achieved for $\Delta a < 0.75a$.
Due to the singular nature of the crack closure traction ahead of the crack tip in the linear elastic analysis, it may be much easier and more accurate to use the first approach to analyze growing cracks, i.e., restart the entire alternating procedure. Crack size changes as the crack grows, but the uncracked body remains the same. The stiffness of the uncracked body is calculated and decomposed only once.

The first method does not work for an elastoplastic analysis, because of the history dependency of the material. The analysis must be based on the solution $S^0$. But the crack closure tractions is not singular or is only weakly singular in the elastoplastic analysis because of the plastic zone ahead of the crack tip. The new crack surface was in the plastic zone. Therefore, the elastoplastic crack closure tractions must be evaluated before the analytical solution is applied. The alternating procedure for an elastoplastic crack extension step is thus outlined as the following.

1. Compute the elastoplastic crack closure traction $T^0$ for the solution $S^0$ obtained for the crack of length $2a$.

2. Reverse the traction obtained in the previous step and apply it as the load on the crack surfaces. Denote the analytical solution of displacement gradient as $F_0^{ANA}$. 
   \[ F_0^{ANA} = -F^T T^0 \]
   where $F^T$ is a linear operator.

3. Compute the elastoplastic stress due to the increment of displacement gradient $F_0^{ANA}$.
   \[ \sigma_0^{(i)} = R(\sigma_0^0, F_0^{ANA}) \]
   where $R$ is a non-linear operator, $\sigma_0^0$ is the stress state in the solution $S_0$.

4. Compute the boundary load $u^{(i)}$, $t^{(i)}$ and the distributed load $f^{(i)}$ due to the incorrectness of the stress $\sigma_0^{(i)}$.

5. Apply the load $u^{(i)}$, $t^{(i)}$ and $f^{(i)}$ on the uncracked body, assuming that the material is elastic. Repeat the procedure of finding residuals until the process converges for the crack of length $2a + 2\Delta a$.

\[
F_{(i+1)}^{FEM} = F_0^{FEM} + F^{(i)} + F^{f^{(i)}}
\]

\[
T^{(i+1)} = R(\sigma_0^{(i)}, F_0^{ANA} + F_{(i+1)}^{FEM} + \sum_{j=2}^{i} (F_{(j)}^{FEM} + F_{(j)}^{ANA}))
\]

\[
F_{(i+1)}^{ANA} = F^T T^{(i+1)}
\]

\[
\sigma_0^{(i+1)} = R(\sigma_0^{(i)}, F_0^{ANA} + \sum_{j=2}^{i} (F_{(j)}^{FEM} + F_{(j)}^{ANA}))
\]

for $i = 1, 2, \ldots$, where $R'$ is a non-linear operator.

6. Evaluation of elastoplastic stresses in a cracked structure

In the elastoplastic analysis of the crack growth, the evaluation of the stress at Gauss points takes most of the CPU time in the EPFEAM method. It is very important to have an efficient and accurate algorithm to evaluate the stress state for a given increment of strain. The radial return algorithm has been considered as one of the most accurate and efficient methods in the evaluation of elastoplastic stresses for given strain increment.

There are several popular algorithms for evaluating the stress for the plane stress problems. The usual radial return method requires iterations to satisfy the plane stress condition. Simo and Taylor [47] proposed a stress subspace method, which is more efficient and accurate than the usual radial return method. Using the stress subspace method, Fuschi et al. [13] showed that the best accuracy can be achieved for the generalized mid-point algorithm when the integration parameter $\alpha$ is approximately 0.75.
Here, an alternative efficient and accurate method is presented for an isotropic material with/without isotropic hardening. The presented method is shown to be more accurate than the generalized mid-point algorithm.

6.1. Preliminaries

The 3-D isotropic elastic constitutive relation is

\[
\sigma_i = 2\mu\epsilon_i + \lambda(\epsilon : I)I \quad i = 1, 2, 3.
\]

(148)

where \(\mu\) is the shear modulus, \(\lambda\) is the Lamé's constant and \(I\) is the identity tensor (\(i = 1, 2, 3\) indicates that all the tensors in the equation are of order 3).

Since \(\sigma_{11} = \sigma_{22} = \sigma_{33} = 0\) in the plane stress problem, it is possible to reformulate the constitutive laws in the reduced stress space in the \(x-y\) plane.

\[
\sigma_{33} = 2\mu\epsilon_{33} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = 0
\]

The following results are derived from the above equation.

- \(\epsilon_{33} = -\frac{\lambda}{2\mu + \lambda}(\epsilon_{11} + \epsilon_{22})\)

- \(\lambda(\epsilon : I) = -2\mu\epsilon_{33} = \frac{2\mu}{2\mu + \lambda}(\epsilon_{11} + \epsilon_{22})\)

Define \(\tilde{\lambda} = 2\mu/(2\mu + \lambda)\) for the convenience. In the reduced space, the constitutive law (Eq. (148)) is rewritten as

\[
\sigma_i = 2\mu\epsilon_i + \tilde{\lambda}(\epsilon : I)I \quad i = 1, 2,
\]

where \(i = 1, 2\) indicates that all the tensors in the equation are of order 2.

6.2. The elastic step to the yield surface

Now, consider the time it takes to reach the yield surface at a given deformation rate. At the end of time \(t\),

\[
\sigma_n = \sigma_0 + 2\mu\dot{\epsilon}t + \tilde{\lambda}t(\epsilon : I)I \quad i = 1, 2
\]

where \(\sigma_n\) is the stress at end of the elastic step, \(\sigma_0\) is the stress at the beginning of the step. Now, rewrite this in terms of the deviatoric and hydrostatic part in the reduced space.

\[
\sigma_n = (\sigma_0' + 2\mu\dot{\epsilon}t') + [p_0 + (\mu + \tilde{\lambda})(\dot{\epsilon} : I)t]I \quad i = 1, 2
\]

Here, \(\sigma_n\) is

\[
\sigma_n = \begin{bmatrix}
p + x & \tau & 0 \\
\tau & p - x & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where

- \(\tau = \sigma_{12} + 2\mu\dot{\epsilon}_{12}t\)
- \(x = \sigma_{11}' + 2\mu\dot{\epsilon}_{11}' = (\sigma_{11}' - \sigma_{22}')/2 + \mu(\dot{\epsilon}_{11} - \dot{\epsilon}_{22})t\)
- \(p = p_0 + (\mu + \tilde{\lambda})(\dot{\epsilon} : I)t = (\sigma_{11} + \sigma_{22})/2 + (\mu + \tilde{\lambda})(\dot{\epsilon}_{11} + \dot{\epsilon}_{22})t\)

Since the yield surface is

\[
\sigma_n = \sigma_n', \quad \sigma_n' = \sigma_0' + 2\mu\dot{\epsilon}t'
\]

The Mises criterion

\[
J^2 = \sigma_n' : \sigma_n' = \tau^2 + x^2
\]

where the stress deviator is

\[
\tau = \sqrt{\sigma_n' : \sigma_n'}
\]

\[
x = \sqrt{x^2 + \sigma_{11}'^2 + \sigma_{22}'^2}
\]

where

\[
\tau = 2\mu\dot{\epsilon}_t,
\]

\[
x = 2\mu\dot{\epsilon}_x
\]

\[
p = (\mu + \tilde{\lambda})\dot{\epsilon}_t
\]

Thus, the time required to reach the yield surface is

\[
t = \frac{1}{\dot{\epsilon}_t}
\]

6.3. The plastic step

Decomposing the notation,

\[
\sigma = \sigma_p + \sigma_n,
\]

\[
\epsilon = \epsilon_p + \epsilon_n
\]

\[
N = \text{const.}
\]

where \(s = \|\dot{s}\|\) is the rate of plastic strain energy.

The usual rules

\[
p = K\dot{\epsilon}_t
\]

\[
s = 2\mu\dot{\epsilon}_t
\]
Since the deviatoric part of $\sigma_n$ is

$$\sigma_n' = \begin{bmatrix} p/3 + x & \tau & 0 \\ \tau & p/3 - x & 0 \\ 0 & 0 & -2p/3 \end{bmatrix},$$

$\sigma_n'$ is found as in the following:

$$J_2' = \sigma_n' : \sigma_n' = 2x^2 + \frac{2}{3} p^2 + 2\tau^2 \quad i = 1, 2, 3.$$  \hspace{1cm} (149)

The Mises yield criterion is

$$\sigma_n' : \sigma_n' = 2\tau_y^2 \quad i = 1, 2, 3,$$

where $\tau_y$ is the yield stress in the simple shear test. An equation for the time step $t$ of the increment where the stress state reaches the yield surface is obtained by substituting $\sigma_n'$ into the yield criterion.

$$\tau_y^2 = x^2 + \frac{p^2}{3} + \tau^2$$

$$= (x_0 + x_i)^2 + \frac{(p_0 + p_i)^2}{3} + (\tau_0 + \tau_i)^2$$

$$= \left[ x_0^2 + \frac{p_0^2}{3} + \tau_0^2 \right] + 2 \left[ x_0 x_i + \frac{p_0 p_i}{3} + \tau_0 \tau_i \right] t + \left[ x_i^2 + \frac{p_i^2}{3} + \tau_i^2 \right] t^2$$

$$= \frac{(\sigma_{11} - \sigma_{22})^2}{4} + \sigma_{12}^2 + \frac{p_i^2}{3} + 2 \left[ \frac{(\sigma_{11} - \sigma_{22})}{2} \right] x_i + \sigma_{12} \tau_i + \frac{p_0 p_i}{3} \right] t + \left[ x_i^2 + \frac{p_i^2}{3} + \tau_i^2 \right] t^2$$

$$= C + Bt + At^2$$

where

$$\tau_i = 2\mu \varepsilon_{12}$$

$$x_i = 2\mu \varepsilon_{11} = \mu (\dot{e}_{11} - \dot{e}_{22})$$

$$p_i = (\mu + \lambda)(\dot{e} : I) = (\mu + \lambda)(\dot{e}_{11} + \dot{e}_{22})$$

Thus, the time $t$ needed to reach the yield surface is

$$t = -\frac{B + \sqrt{B^2 - 4AC - \tau_y^2}}{2A}$$

6.3. The plastic step on the yield surface

Decompose the strain and stress into the deviatoric and hydrostatic parts, using the following notation.

$$\sigma = \sigma' + pl = s + pl \quad i = 1, 2, 3.$$  
$$\varepsilon = \varepsilon' + \theta I = e + \theta I \quad i = 1, 2, 3.$$  
$$N = \frac{s}{\|s\|} \quad i = 1, 2, 3.$$  

where $\|s\|$ is defined as

$$\|s\| = \sqrt{s : s}$$

The usual rate formulation is

$$\dot{p} = K \dot{\theta}$$

$$s = 2\mu (\dot{e} - \dot{\gamma} N) \quad i = 1, 2, 3.$$
Thus
\[ \sigma = (2\mu \dot{e} + K \dot{\theta} I) - 2\mu \gamma N \quad i = 1, 2, 3. \]
or
\[ \sigma = (2\mu \dot{e} + \lambda \dot{\theta} I) - 2\mu \gamma N \quad i = 1, 2, 3. \]
Because of the plane stress condition, \( \sigma_{33} = 0 \),
\[ \sigma_{33} = 2\mu \dot{\varepsilon}_{33} + \lambda (\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + \dot{\varepsilon}_{33}) - 2\mu \gamma N_{33} = 0 \]
which leads to
\[ \dot{\varepsilon}_{33} = \frac{2\mu}{2\mu + \lambda} \gamma N_{33} - \frac{\lambda}{2\mu + \lambda} (\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22}) \]
\[ \lambda \dot{\theta} = \frac{2\mu \lambda}{2\mu + \lambda} (\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + \gamma N_{33}) = \lambda (\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + \gamma N_{33}) \]
The following 2-D rate equation is obtained by substituting \( \lambda \dot{\theta} \) into the 3-D rate equation
\[ \sigma = 2\mu \dot{e} + \lambda (\dot{\varepsilon} : I + \gamma N_{33}) I - 2\mu \gamma N \quad i = 1, 2. \] (150)
where
\[ N_{33} = \frac{-2p/3}{J_2} = \frac{-2p/3}{\sqrt{2x^2 + 2p^2} / 3 + 2\tau} \]
\[ = -\frac{1}{3} \frac{\sigma_{11} + \sigma_{22}}{\sqrt{(\sigma_{11} - \sigma_{22})^2 / 2 + (\sigma_{11} + \sigma_{22})^2 / 6 + 2\sigma_{11}^2}} \]
Decomposing Eq. (150) into the deviatoric and hydrostatic parts in the reduced space, we obtain
\[ \dot{s} = 2\mu \dot{e} - 2\mu \gamma N' \quad i = 1, 2. \]
\[ \dot{\rho} = (\mu + \lambda) \dot{\varepsilon} : I - [\mu N : I - \lambda N_{33}] \gamma \quad i = 1, 2. \]
It is noticed that
\[ N' = \frac{s}{J_2} \quad i = 1, 2. \]
\[ N : I = \frac{2p}{3J_2} \quad i = 1, 2. \]
\[ N_{33} = -N : I = -\frac{2p}{3J_2} \quad i = 1, 2. \]
These rate equations can be rewritten as in the following, with the help of the above relations.
\[ \dot{s} = 2\mu \dot{e} - \frac{2\mu}{J_2} \gamma \dot{s} \]
\[ \dot{\rho} = (\mu + \lambda) \left[ \dot{\varepsilon} : I - \frac{2p\gamma}{3J_2} \right] \]
The following notation is used to simplify the equations in terms of components.
\[ \alpha = 1 + \frac{\frac{\dot{e}_s}{e_s} - \frac{\tau}{\mu}}{\frac{1}{1 - \nu}} \]

\[ J_2 = \sqrt{2x^2 + \frac{2}{3} p^2 + 2\tau^2} \]

\[
\begin{cases}
    e_s = (\epsilon_{e1} - \epsilon_{e2}) / 2 \\
    e_t = \epsilon_{t2} \\
    e_p = (\epsilon_{e1} + \epsilon_{e2}) / 2
\end{cases}
\]

\[
\begin{cases}
    x = (\sigma_{e1} - \sigma_{e2}) / 2 \\
    \tau = \sigma_{t2} \\
    p = (\sigma_{e1} + \sigma_{e2}) / 2
\end{cases}
\]

where \( \nu \) is the Poisson’s ratio. Thus, the rate equations can be written in the terms of the components in the following simple form.

\[
\begin{bmatrix}
    \dot{e}_s \\
    \dot{\epsilon}_s \\
    \dot{\epsilon}_p
\end{bmatrix}
= 2\mu \begin{bmatrix}
    \dot{e}_s \\
    \dot{\epsilon}_s \\
    \dot{\epsilon}_p
\end{bmatrix}
- \frac{2\mu \gamma}{J_2} \begin{bmatrix}
    x \\
    \tau \\
    p
\end{bmatrix}
\]

The stress can be solved explicitly if \( J_2 \) and \( \gamma \) are assumed to be constants. The result is

\[
\begin{bmatrix}
    x \\
    \tau \\
    p
\end{bmatrix}
= \begin{bmatrix}
    x_0 e^{-q} \\
    \tau_0 e^{-q} \\
    p_0 e^{-q\alpha/3}
\end{bmatrix}
+ \frac{J_2}{\Delta \gamma} \begin{bmatrix}
    e_s (1 - e^{-q}) \\
    e_t (1 - e^{-q}) \\
    3e_p (1 - e^{-q\alpha/3})
\end{bmatrix}
\]

where \( q = 2\mu \Delta \gamma / J_2 \). The evaluation equation for the yield surface is derived from the Mises’ yield criterion (Eq. (149))

\[ e \dot{x} + \frac{1}{3} \dot{p} p + \tau \dot{\tau} = \tau, h(\gamma) \dot{\gamma} \]

(151)

where \( h(\gamma) = \tau' \gamma(\gamma) \). The rate of the equivalent plastic strain, \( \dot{\gamma} \), is solved by substituting \( x, \tau \) and \( p \) into Eq. (151).

\[ \dot{\gamma} = \frac{x \dot{e}_s + \alpha \dot{e}_p / 3 + \tau \dot{e}_t}{J_2 h / 2\sqrt{2\mu + (x^2 + \alpha p^2 / 9 + \tau^2)} / J_2} \]

6.4. Summary of the algorithm

Given the strain increment

\[
\begin{bmatrix}
    \epsilon_{e1} \\
    \epsilon_{t2} \\
    \epsilon_{e2}
\end{bmatrix}
\]

compute the following parameters first.

\[
\begin{bmatrix}
    e_s = (\epsilon_{e1} - \epsilon_{e2}) / 2 \\
    e_t = \epsilon_{t2} \\
    e_p = (\epsilon_{e1} + \epsilon_{e2}) / 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x_i = 2\mu e_s \\
    \tau_i = 2\mu e_t \\
    p_i = 2\mu e_p
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x_0 = (\sigma_{e1} - \sigma_{e2}) / 2 \\
    \tau_0 = \sigma_{t2} \\
    p_0 = (\sigma_{e1} + \sigma_{e2}) / 2
\end{bmatrix}
\]

Then, the load step to reach the yield surface is

\[ t = -B + \sqrt{B^2 - 4A(C - \epsilon_y^2)} / 2A \]

where
\[ A = x_i^2 + \frac{p_i^2}{3} + \tau_i^2 \]
\[ B = 2\left(x_0v_1 + \frac{p_0p_1}{3} + \tau_0\tau_1\right) \]
\[ C = x_0^2 + \frac{p_0^2}{3} + \tau_0^2 \]

If \( t > 1 \), the entire step is elastic. Calculate the stress state using the elastic model. Otherwise, split the strain increment into the pure elastic part and the elastic-plastic part. Following is the method to find the final stress state for the elastic-plastic strain increment.

Use the limiting state to get an estimated plastic strain increment \( \Delta \gamma \):

\[
\begin{pmatrix}
\frac{x_i}{\tau_i} \\
p_i
\end{pmatrix} = \frac{\epsilon_i}{\sqrt{2}e_{\text{p}} + 6e_{\text{p}}^2 + 2e_{\text{p}}^3 + 3e_{\text{p}}}
\]

\[ \Delta \gamma = \frac{x_i e_{\text{p}} + \alpha p_i e_{\text{p}} / 3 + \tau_i e_{\text{p}}}{J_2^{1/2}} \]

Calculate \( q_1 \) and \( q_2 \):

\[ q_1 = e^{-q} = e^{-2\mu \Delta \gamma J_2^{1/2}} \]

\[ q_2 = e^{-\mu \Delta \gamma J_2^{1/2}} \]

The stress predicted is

\[
\begin{pmatrix}
x_\mu \\
t_\mu \\
p_\mu
\end{pmatrix} = \begin{pmatrix}
x_0 q_1 \\
t_0 q_1 \\
p_0 q_2
\end{pmatrix} + \begin{pmatrix}
e_{\text{p}}(1 - q_1) \\
e_{\tau}(1 - q_1) \\
3e_{\text{p}}(1 - q_2)
\end{pmatrix}
\]

Then, the plastic strain is found using the predicted stress state:

\[ \Delta \gamma = \frac{x_\mu e_{\text{p}} + \alpha p_\mu e_{\text{p}} / 3 + t_\mu e_{\tau}}{J_2^{1/2} + \sqrt{2} \tau_N + (x_\mu^2 + \alpha p_\mu^2 / 3 + t_\mu^2) / J_2^{1/2}} \]

Finally, the stress is computed according to the following formula:

\[
\begin{pmatrix}
x_N \\
t_N \\
p_N
\end{pmatrix} = \begin{pmatrix}
x_0 q_1 \\
t_0 q_1 \\
p_0 q_2
\end{pmatrix} + \begin{pmatrix}
e_{\text{p}}(1 - q_1) \\
e_{\tau}(1 - q_1) \\
3e_{\text{p}}(1 - q_2)
\end{pmatrix}
\]

Scale the components, using the factor \((\sqrt{2} \tau_N) / J_2\), such that the stress state sits exactly on the yield surface.

7. Numerical examples of elastic-plastic fracture problems

The following is a summary of the examples considered. First, an embedded crack in a small block of an elastic-plastic solid is analyzed using EPFEAM. The \( J \) integral along the crack front is obtained at different load levels along the crack front. Second, a middle crack tension specimen (also referred to as a center cracked panel, see Fig. 12) in plane stress is analyzed using the Elastic-Plastic Element Alternating Method (EPFEAM). The \( J \) integral obtained by EPFEAM agrees well with that obtained using the Electric Power Research Institute (EPRI) approach. It also shows that the HRR singularity can be captured correctly by EPFEAM. Third, the NIST 90 MSD-3 crack growth test (Fig. 16), one of the experiments carried out at National Institute of Standards and Technology (NIST), is simulated.
using EPFEAM. The $T^*$ integral versus crack growth is obtained from the numerical simulation. It is compared to the handbook solution of $J$ integral. $T^*$ integral rises before the crack initiation stage and levels out after it reaches the saturation value. The plastic zone size obtained by the EPFEAM is significantly different from the simple Irwin estimation. Fourth, an MSD problem (Fig. 19) is solved, where there are two small cracks in front of each tip of the main crack. The $T^*$ integral, which is the same as the $J$ integral for this case of monotonic loading of a stationary crack, is compared to $J$ integral for the main crack in the absence of MSD cracks to show the influence of the MSD cracks on the main crack. It is shown that MSD cracks increase the severity of possibility of fracture at the main crack significantly. Fifth, the elastic-plastic analysis of a problem with cracks emanating from the holes (Fig. 23) is carried out to show capability of the EPFEAM in dealing with complex problems. Finally, a numerical prediction is presented for the NIST 90 MSD-4 crack growth test. The $T^*$ resistance curve used in this prediction is obtained by curve fitting the $T^*$ resistance curves from the numerical simulation of NIST 90 MSD-2 and 3. The load and crack growth predicted by the alternating method agree well with the experimental results. Snapshots of stresses and plastic deformations during the crack growth are also presented to illustrate the power of the EPFEAM.

7.1. An embedded elliptical crack in an elastic-plastic solid

A block of an elastic-plastic solid with an embedded crack is analyzed. The size of the body is $4 \times 8 \times 12$. Uniform surface loads $\sigma_y$ are applied to the upper surface and the bottom along the $z$-axis. Only one-eighth of the block is analyzed because of the symmetry of the problem. The FEM mesh used in the analysis is shown in Fig. 10. The element size near the crack front is small enough to capture the plastic zone. The major semi-axis $a$ of the ellipse is on the $y$-axis. The minor semi-axis $b$ is on the $x$-axis. $a/b = 2$.

We used the linear isotropic hardening material model with the Mises yield criterion for this example. The yield surface is

$$\tau_y(\xi) = \tau_0(1 + k\xi)$$  \hspace{1cm} (152)

where $\xi$ is the plastic strain, $\tau_y$ is the equivalent stress on the yield surface. The equivalent stress is

$$\tau_y^2 = \sigma' : \sigma'$$  \hspace{1cm} (153)

where $\sigma'$ is the deviatoric stress. The following material constants are used. The Young's modulus is $E = 1$. The hardening parameter is $k = 0.1$. The equivalent stress at initial yielding is $\tau_0 = 1.2$. The Poisson's ratio is $\nu = 0.3$.

![Fig. 10. FEM mesh for the uncracked block.](image-url)
The *Equivalent Domain Integral* (EDI) method [25] is used to evaluate the $J$ integrals along the crack front. Fig. 11 shows the $J_i$ along the crack front at different surface loads $\sigma_i$. The load increases from $\sigma_i = 0.1$ to $\sigma_i = 1.1$. The load step is $\Delta \sigma_i = 0.1$. The $J_i$ has smallest values at the intersections between the crack front and the major-axis. The largest values are at the intersections between the crack front and the minor-axis.

### 7.2. A middle crack tension problem

The middle crack tension problem is shown in (Fig. 12). A power law hardening is used in order to compare the solution by EPRI [1]. The following strain–stress curve in the uniaxial tension test is used in the EPFEAM.

$$
\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left( \frac{\sigma}{\sigma_0} - 1 \right)^n
$$

where $\varepsilon_0$ and $\sigma_0$ are the strain and stress at the initial yielding, $n$ is the hardening exponent, and $\alpha$ is the hardening parameter. $\sigma = 47$ ksi, $n = 13$ and $\alpha = 1$ are taken in this numerical example. Young's modulus is $E = 10,500$ ksi. Poisson's ratio is $\nu = 0.33$.

Fig. 13 shows the $J$ integrals obtained by the EPFEAM, EPRI, Linear Elastic Fracture Mechanics (LEFM) and the effective crack length approaches. It shows that the EPFEAM solution agrees well with the EPRI solution, differs significantly with those obtained by LEFM and the effective crack length approach when the applied load is high.

The $J$ is the sum of the fully plastic value $J_{pl}$ and the effective elastic value $J_{el}$ in the EPRI approach.

![Graph showing $J_i$ along the crack front at different load levels.](image)
The crack load increases from intersections between the crack front and the specimen boundary. Elastic-plastic analysis is used in order to determine the load and the effective crack length from the J integral. The Fracture Mechanics solutions used in this example agree well with the experimental results for the effective crack length

\[ \frac{J}{J_{el}} = \alpha \epsilon_0 \sigma_0 \frac{a(W-a)}{W} h_i(a/W, n)(P/P*)^{n-1} \]

where \( h_i = 2.65 \) for this problem [1].
In the LEFM approach, the $J$ is estimated by the following formula

$$K = \sigma \sqrt{\pi a} \sqrt{\text{sec}(\frac{\pi a}{2W})} \left[ 1 - 0.025 \left( \frac{a}{W} \right)^2 + 0.06 \left( \frac{a}{W} \right)^4 \right]$$

(155)

and

$$J = \frac{K^2}{E}$$

(156)

where $2a$ is the crack length and $2W$ is the width of the panel.

The effective crack size is the following:

$$a_{\text{eff}} = a + \frac{1}{1 + (P/P_0)^2} \left( \frac{K}{\sigma_0} \right)^2$$

(157)

where $K$ is the stress intensity factor obtained by using the crack size $a$ in Eq. (155), $P$ is the applied load, and $P_0 = 2\sigma_0(W - a)t$ is the limit load. $t$ is the thickness. Observe that the HRR field dominates over a significant length here.

The effective elastic value $J_{\text{eff}}$ is obtained by using the effective crack length $a_{\text{eff}}$ in Eqs (155) and (156).

Fig. 14 shows the HRR slope captured by the EPFEAM. The EPFEAM utilises only the elastic analytical solution, where the singularity of the stress is $1/\sqrt{r}$. But the EPFEAM is able to capture the HRR type of singularity through the iterations of plastic corrections.

Fig. 15 shows the plastic zone size obtained by the EPFEAM and Irwin corrections. The first order Irwin approximation, denoted by Irwin-1.00 in the figure, is calculated using

$$\frac{1}{2\pi} \left( \frac{K}{\sigma_0} \right)^2$$

(158)

7.3. A simulation

The crack in the simulated panel is made by the crack tip field. In unloading, the crack tip field may not be used.

The $T^*$-integrated crack tip field and unloading, the plastic zone size obtained by the Irwin approximation may be found.

Fig. 14: The HRR slope captured by the EPFEAM.

Fig. 15: Plastic zone sizes obtained by the EPFEAM and Irwin corrections.

Footnote 4: Note that Eq. (157) is the EPRI modification to the Irwin approach to account for strain hardening. The normal plastic zone estimation of Irwin is shown in Eq. (158).
The simple force balance for the elastic perfect plastic material leads to a second order Irwin approximation. Denoted Irwin-2.00 in the figure, the second order Irwin approximation is calculated using

\[
\frac{1}{\pi} \left( \frac{K}{\alpha_0} \right)^2
\]

(159)

It is noticed that the second-order approximation is twice as large as the first-order approximation. The EPFEAM solution shows that an intermediate one, which is 1.65 times that of the first-order approximation, gives the best estimation when the load level is low (less than half of the limiting load in this case). However, the error of the Irwin type of approximation becomes greater when the load is high.

7.3. A simulation of a crack growth test in an elastic-plastic sheet

The crack growth test 90 MSD-3, carried out at National Institute of Standards and Technology, is simulated numerically using the EPFEAM. The test configuration is shown in Fig. 16. The cracked panel is made of AL 2024-T3.

The \( T^* \)-integral parameter is a versatile fracture parameter [2]. It can be used to characterize the crack tip fields in an elastic-plastic material or an elastoviscoplastic material [5, 6]. In the absence of unloading, the \( T^* \) reduced to the well-known \( J \). In the elastic-plastic crack growth analysis, the unloading effect is significant when the amount of crack growth is not small. Therefore, \( J \)-integral can not be used. But \( T^* \) is still a valid fracture parameter. A recent summary about \( T^* \) integral parameters may be found in [56].

![Fig. 16. The cracked panel for NIST 90 MSD-3.](image-url)
Fig. 17. $T^*$ evaluated at contours of different sizes, and $J$.

Fig. 18. The plastic zone sizes obtained by the EPFEAM and Irwin corrections for NIST 90 MSD-3.

Fig. 17 shows the $T^*$ evaluated at contours of different sizes. The numbers in the legend indicate the radii of the contours, i.e., 'ALT-0.087' indicates that the contour is of radius 0.087 in. The curve labeled 'LEFM' is the $J$ evaluated using Eqs. (155) and (156). It is seen that $T^*$ increases and becomes close to $J$ as the contour size increases. $T^*$ increases at the crack initiation stage and levels out after it reaches the saturation value for small contours.

Fig. 18 shows that the first-order Irwin approximation under-estimates the plastic zone size in front of
the crack tip, while the second-order approximation over-estimates the size. The intermediate one, which is 1.65 times that of the first-order approximation, gives the best answer for this particular test. But it still remains to be investigated whether the Irwin type of approximation is able to capture the plastic zone size accurately for general conditions.

7.4. A problem of multiple site damage (MSD)

The MSD cracks are shown in Fig. 19. There are two small MSD cracks in front of each tip of the main crack. The far field stress increases monotonically. Crack growth is not considered. Fig. 21 shows the $T^*$ integral, which is the same as the $J$ integral in this case, at different load levels for each crack tip (see Fig. 19 for the locations of the crack tips). When the load is small, the $T^*$ values at the MSD cracks are much smaller than those at the main crack tips, but they become significant as the load increases. Fig. 22 shows the $T^*$ value at the main crack tip with/without MSD cracks. The points labeled 'Tip
A-1' are evaluated at a contour of radius 0.0873 in. The curve labeled 'Tip A-2' is evaluated at a contour of radius 0.2825 in. These two agree well with each other indicating the path independent nature of $T^*$ here. The curve labeled 'Tip AA' is the one evaluated for the same main crack tip in the absence of MSD cracks. It is seen that the influence of the MSD cracks becomes greater as the load increases.

![Figure 24](image)

Fig. 24. $T^*$ at the crack tips at the edge of the hole.
7.5. Cracks emanating from holes

The cracks and holes are shown in Fig. 23. The FEM mesh is only for the sheet with the holes. The cracks are modeled using the analytical solution. Since the piecewise linear basis functions are used in the analytical solution, it is very easy to construct analytical solutions where the crack surface traction is zero at the segment within the hole. The $T^*$ values are found in Fig. 24.

7.6. A prediction of a crack growth in an elastic-plastic sheet with multiple site damage

A numerical prediction is carried out for the NIST 90 MSD-4 crack growth test.

The configuration of the test panel is similar to that in Fig. 19. The main crack length is $2a = 14$ in. The sizes of the MSD cracks are the same, $2a_{\text{msd}} = 0.4$ in. The distance between the main crack and the first MSD crack is 7.5 in. The distance between adjacent MSD cracks is 1.0 in. There are three small MSD cracks in front of each main crack tip.

The $T^*$ resistance curve used in the prediction is obtained by curve fitting the $T^*$ resistance curves from the numerical simulation of NIST 90 MSD-2 and 3.

$$ T^* = 0.706 - 0.0398 \Delta a - \frac{0.0697}{0.175 + \Delta a} $$

The unit of $T^*$ is (Klb/in) and the unit of $\Delta a$ is in.

The load and crack growth predicted by the alternating method are shown in Fig. 25. They agree well with the experimental results. Snapshots of stresses and plastic deformations in the vicinities of the crack tips during the crack growth are shown in Figs. 26–29. A detailed discussion of these results may be found in [55–57].
(a) $a = 7.2$ in

Figure 26. Equivalent plastic strain contour plot ($\varepsilon_p$) at different crack growth locations.

(b) $a = 8.0$ in
(c) \( a = 9.0 \text{ in} \)

(d) \( a = 9.9 \text{ in} \)

Fig. 26 (Contd.)
Fig. 27. Stress contour plot ($\sigma_x$) at different crack growth locations.
(c) $a = 9.0 \text{ in}$

(d) $a = 9.9 \text{ in}$

Fig. 27 (Contd.)
Fig. 28. Stress contour plot ($\sigma_{xx}$) at different crack growth locations.
(c) $a = 9.0 \text{ in}$

(d) $a = 9.9 \text{ in}$

Fig. 28 (Contd.)
Fig. 29. Stress contour plot ($\sigma_z$) at different crack growth locations.
(c) $a = 9.0$ in

(d) $a = 9.9$ in

Fig. 29 (Contd.)
8. Conclusion

The finite element alternating method is an iterative approximation method based on the superposition principle. It can be extended to perform elastoplastic analysis and crack growth analysis, when it is used together with the initial stress method. In this paper we summarized the recent theoretical/algorithic developments in the Finite Element Alternating Method, including: (1) the alternating method and its convergence for mixed boundary value problems, with the presence of both traction and displacement boundary conditions; (2) a non-iterative method to construct solutions for multiple arbitrarily located embedded cracks using the solution for a single embedded crack in an infinite body; (3) the analysis of elastic-plastic fracture mechanics problems; (4) the analysis of crack-growth in plane fracture situations; and (5) an efficient and accurate algorithm for the evaluation of elastoplastic stress state in a cracked structure, based on the generalized mid-point radial return for 3D constitutive laws and the stress subspace method for the plane stress analysis, and a study of link-up of multiple cracks in a wide-spread fatigue damage situation in an aircraft panel. Some numerical examples are also given.

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References


