New concepts in meshless methods

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SUMMARY

Meshless methods have been extensively popularized in literature in recent years, due to their flexibility in solving boundary value problems. Two kinds of truly meshless methods, the meshless local boundary integral equation (MLBIE) method and the meshless local Petrov-Galerkin (MLPG) approach, are presented and discussed. Both methods use the moving least-squares approximation to interpolate the solution variables, while the MLBIE method uses a local boundary integral equation formulation, and the MLPG employs a local symmetric weak form. The two methods are truly meshless ones as both of them do not need a 'finite element or boundary element mesh', either for purposes of interpolation of the solution variables, or for the integration of the 'energy'. All integrals can be easily evaluated over regularly shaped domains (in general, spheres in three-dimensional problems) and their boundaries. Numerical examples presented in the paper show that high rates of convergence with mesh refinement are achievable. In essence, the present meshless method based on the LSWF is found to be a simple, efficient and attractive method with a great potential in engineering applications. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: meshless methods; local boundary integral equation; local symmetric weak form; local Petrov-Galerkin formulation; moving least-squares approximation

1. INTRODUCTION

Meshless methods in computational mechanics have attracted much attention in recent decades, due to their flexibility in solving practical engineering problems, especially those problems with discontinuities or moving boundaries. The main objective of meshless methods is to get rid of or at least alleviate the difficulty of meshing and remeshing the entire structure, by only adding or deleting nodes in the entire structure.

The initial idea of meshless methods dates back to the smooth particle hydrodynamics (SPH) method for modelling astrophysical phenomena [1] while the research into meshless methods has become very active only after the publication of the Diffuse Element Method by Nayroles et al. [2]. Several meshless methods have also been reported in literature since then, such as the element-free Galerkin method [3–8], the reproducing kernel particle method [9].

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Even though no mesh is required in these methods for the interpolation of the solution variables, shadow elements are inevitable in these methods, for the integration of "energy". Therefore, even though called element-free methods, these methods are not truly meshless ones.

Recently, two truly meshless methods, the meshless local boundary integral equation (MLBIE) method, and the meshless local Petrov–Galerkin approach have been successfully developed in [10–15] for solving linear and non-linear boundary problems. Both methods use local weak forms over a local sub-domain, and shape functions from the moving least-squares (MLS) approximation. The MLBIE method employs a local unsymmetric weak form, i.e., the local boundary integral equation (LBIE), while the MLPG approach uses a local symmetric weak form. Both methods are truly meshless ones, as no 'finite element or boundary element mesh' is required in these two approaches, either for purposes of interpolation of the solution variables, or for the integration of the 'energy'. All integrals can be easily evaluated over regularly shaped domains (in general, spheres in three-dimensional problems) and their boundaries.

These two methods are also more flexible and easier in dealing with non-linear problems than the conventional FEM, EFG and BEM, as domain integrals will not cause any difficulty in implementing these two methods.

In the present paper, by 'the support of node $x_i$', we mean a sub-domain (usually taken as a circle of radius $r_i$) in which the weight function $w_j$ in the MLS approximation, associated with node $x_i$, is non-zero; by 'the domain of definition' of an MLS approximation for the trial function at any point $x$ (hereinafter simply called as the 'domain of definition of point $x$') we mean a sub-domain which covers all the nodes whose weight functions do not vanish at $x$; by 'the domain of influence of node $x_i$', we denote a sub-domain in which all the nodes have non-zero couplings with the nodal values at $x_i$ in the system stiffness matrix; and by the 'symmetric weak form' we mean the weighted residual form for the differential equation, which is integrated by parts enough times such that the differentiability requirements for the trial and test functions are the same. In our implementation, the domain of influence of a node is the union of the domains of definition of all points (in general, but the quadrature points in specific) on the local boundary (for the MLBIE method) or in the local domain (for the MLPG approach) of the node. We do not intend to mean these to be versatile definitions, but rather, explanations of our terminology.

The following discussion begins with the brief description of the moving least-squares (MLS) approximation in Section 2. The MLBIE method and its numerical implementation are presented in Section 3. The MLPG approach and its numerical implementation are presented in Section 4. Numerical examples for 2-D problems are given in Section 5. The paper ends with conclusions and discussions in Section 6.

2. THE MLS APPROXIMATION SCHEME

This section gives a brief summary of the MLS approximation. For details of the MLS approximation, see [3, 14].

Consider a sub-domain $\Omega_x$, the neighbourhood of a point $x$ and denoted as the domain of definition of the MLS approximation for the trial function at $x$, which is located in the problem domain $\Omega$. To approximate the distribution of function $u$ in $\Omega_x$, over a number of randomly located nodes $\{x_i\}, i = 1, 2, \ldots, n$, the moving least squares approximant $u^h(x)$ of $u, \forall x \in \Omega_x$, can be

defined by

\[ u^h(x) = p^T(x)a(x) \quad \forall x \in \Omega_x \]  

(1)

where \( p^T(x) = [p_1(x), p_2(x), \ldots, p_m(x)] \) is a complete monomial basis of order \( m \), and \( a(x) \) is a vector containing coefficients \( a_j(x) \), \( j = 1, 2, \ldots, m \), which are functions of the space co-ordinates \( x = [x^1, x^2, x^3]^T \).

The coefficient vector \( a(x) \) is determined by minimizing a weighted discrete \( L_2 \) norm, which can be defined as

\[ J(x) = \sum_{i=1}^{n} w_i(x)[p^T(x)a(x) - \hat{u}_i]^2 \]  

(2)

where \( w_i(x) \) is the weight function associated with the node \( i \), with \( w_i(x) > 0 \) for all \( x \) in the support of \( w_i(x) \), \( x_i \) denotes the value of \( x \) at node \( i \), \( n \) is the number of nodes in \( \Omega_x \) for which the weight functions \( w_i(x) > 0 \).

Here it should be noted that \( \hat{u}_i \), \( i = 1, 2, \ldots, n \), in equation (2) are the fictitious nodal values, and not the nodal values of the unknown trial function \( u^h(x) \) in general (see Figure 1 for a simple one-dimensional case for the distinction between \( u_i \) and \( \hat{u}_i \)).

Solving for \( a(x) \) by minimizing \( J \) in equation (2) and substituting it into equation (1) give a relation which may be written as the form of an interpolation function similar to that used in the FEM, as

\[ u^h(x) = \sum_{i=1}^{n} \phi_i(x) \hat{u}_i, \quad u^h(x) = u_i \neq \hat{u}_i, \quad x \in \Omega_x \]  

(3)

where

\[ \phi_i(x) = \sum_{j=1}^{m} p_j(x)[A^{-1}(x)B(x)]_{ij} \]  

(4)

with the matrices \( A(x) \) and \( B(x) \) being defined by

\[ A(x) = \sum_{i=1}^{n} w_i(x)p(x_i)p^T(x_i) \]  

(5)

\[ B(x) = [w_1(x)p(x_1), w_2(x)p(x_2), \ldots, w_n(x)p(x_n)] \]  

(6)

The MLS approximation is well defined only when the matrix \( A \) in equation (5) is non-singular.
\( \phi_i(\mathbf{x}) \) is usually called the shape function of the MLS approximation corresponding to nodal point \( \mathbf{x}_i \). From equations (4) and (6), it may be seen that \( \phi_i(\mathbf{x}) = 0 \) when \( w_i(\mathbf{x}) = 0 \). The fact that \( \phi_i(\mathbf{x}) \) vanishes, for \( \mathbf{x} \) not in the support of nodal point \( \mathbf{x}_i \) preserves the local character of the moving least squares approximation.

In implementing the MLS approximation for the present local symmetric weak form, the basis functions and weight functions should be chosen at first. Both Gaussian and spline weight functions with compact supports can be considered in the present work. The Gaussian weight function corresponding to node \( i \) may be written as

\[
w_i(\mathbf{x}) = \begin{cases} 
\exp\left[-(d_i/c_i)^2\right] - \exp\left[-(r_i/c_i)^2\right], & 0 \leq d_i \leq r_i \\
0, & d_i > r_i
\end{cases}
\]

(7)

where \( d_i = |\mathbf{x} - \mathbf{x}_i| \) is the distance from node \( \mathbf{x}_i \) to point \( \mathbf{x} \), \( c_i \) is a constant controlling the shape of the weight function \( w_i \) and therefore the relative weights and \( r_i \) is the size of the support for the weight function \( w_i \) and determines the support of node \( \mathbf{x}_i \). In the present computation, \( k = 1 \) was chosen.

A spline weight function is defined as

\[
w_i(\mathbf{x}) = \begin{cases} 
1 - 6 \left( \frac{d_i}{r_i} \right)^2 + 8 \left( \frac{d_i}{r_i} \right)^3 - 3 \left( \frac{d_i}{r_i} \right)^4, & 0 \leq d_i \leq r_i \\
0, & d_i \geq r_i
\end{cases}
\]

(8)

The size of support, \( r_i \), of the weight function \( w_i \) associated with node \( i \) should be chosen such that \( r_i \) should be large enough to have sufficient number of nodes covered in the domain of definition of every sample point \( n \geq m \) to ensure the regularity of \( A \). A very small \( r_i \) may result in a relatively large numerical error in using Gauss numerical quadrature to calculate the entries in the system matrix. On the other hand, \( r_i \) should also be small enough to maintain the local character of the MLS approximation.

A generalization of the MLS interpolation scheme, suitable for 4th order problems of beam and plate bending is given in Reference [14].

3. THE MESHLESS LOCAL BOUNDARY INTEGRAL EQUATION (MLBIE) METHOD

In this section, we use the following potential problem to illustrate the development of the MLBIE approach:

\[
\nabla^2 u(\mathbf{x}) = p(\mathbf{x}), \quad \mathbf{x} \in \Omega
\]

(9)

where \( p \) is a given source function and the domain \( \Omega \) is enclosed by \( \Gamma = \Gamma_u \cup \Gamma_q \), with boundary conditions:

\[
u = \bar{u} \quad \text{on} \quad \Gamma_u
\]

(10a)

\[
\frac{\partial u}{\partial n} = q = \bar{q} \quad \text{on} \quad \Gamma_q
\]

(10b)

where \( \bar{u} \) and \( \bar{q} \) are the prescribed potential and normal flux, respectively, on the essential boundary \( \Gamma_u \) and on the flux boundary \( \Gamma_q \), and \( n \) is the outward normal direction to the boundary \( \Gamma \).
In the present local boundary integral equation, a local weak form over a local sub-domain, \(\Omega_0(\in \Omega)\), is employed. The local sub-domain \(\Omega_0\) is conveniently taken to be a sphere (in 3-D, or a circle in 2-D) centred at a point \(x\) in question. The local weak form of the differential equation (9) and the boundary conditions (10), over a local sub-domain \(\Omega_0\), can be written as

\[
\int_{\Omega_0} u^*(x, y) \left[ \nabla^2 u(x) - \rho(x) \right] \, d\Omega = 0
\]  

(11)

where \(u\) is the trial function and \(u^*\) is the test function which is chosen to be the solution, in infinite space, of the following equation

\[
\nabla^2 u^*(x, y) + \delta(x, y) = 0
\]  

(12)

with \(\delta(x, y)\) being the Dirac delta function.

Using \((\nabla^2 u)u^* = (u, u^*), i - (uu^*), i + uu^*, i\) and the divergence theorem twice, in equation (11) yields

\[
u(y) = \int_{\partial\Omega_0} u^*(x, y) \frac{\partial u(x)}{\partial n} \, d\Gamma - \int_{\partial\Gamma} u(x) \frac{\partial u^*(x, y)}{\partial n} \, d\Gamma
\]

\[\]

\[= \int_{\Omega_0} u^*(x, y) \rho(x) \, d\Omega\]

(13)

where \(\partial\Omega_0\) is the boundary of the local domain \(\Omega_0\).

It should be noted that equation (13) holds irrespective of the size and shape of \(\partial\Omega_0\). We can deliberately choose a simple regular shape for \(\Omega_0\) and thus for \(\partial\Omega_0\). The most regular shape of a sub-domain should be an \(n\)-dimensional sphere for a boundary-value problem defined on an \(n\)-dimensional space. Thus, an \(n\)-dimensional sphere (or a part of an \(n\)-dimensional sphere for a boundary node), is chosen in our development (see Figure 2).

The above equation can be further simplified by using a modified test function \(\tilde{u}^*(x, y)\), such that it vanishes over \(L_s\), where \(L_s\) is a part of \(\partial\Omega_0\), on which no boundary conditions are specified (see Figure 2). This can be easily accomplished by introducing a ‘companion solution’, \(\tilde{u}\), which satisfies

\[
\begin{align*}
\nabla^2 \tilde{u} &= 0 & \text{in } \Omega_0' \\cap L_s \\
\tilde{u} &= u^*(x, y) & \text{on } \partial\Omega_0'
\end{align*}
\]  

(14)

where \(\Omega_0' \supseteq \Omega_0\), such that \(\Omega_0' = \Omega_0\) for an interior source point \(y\); and \(\Omega_0'\) is the extended whole sphere which encloses \(\partial\Omega_0\), a part of the sphere, for a boundary source point \(y\) (see Figure 2).

Using the modified test function \(\tilde{u}^*(x, y) = u^* - \tilde{u}\) in equation (13) and rearranging equation (13), we obtain the following local boundary integral equation (LBIE):

\[
\alpha(y)u(y) = -\int_{\partial\Omega_0} u(x) \frac{\partial \tilde{u}^*(x, y)}{\partial n} \, d\Gamma + \int_{\Gamma_s} \frac{\partial u(x)}{\partial n} \tilde{u}^*(x, y) \, d\Gamma
\]

\[\]

\[= -\int_{\partial\Omega_0} \tilde{u}^*(x, y) \rho(x) \, d\Omega\]

(15)

where \(\Gamma_s = \partial\Omega_0 \cap \Gamma_s\), i.e. \(\Gamma_s\) is a part of \(\partial\Omega_0\), over which boundary conditions are specified.
The domain of definition of the MSL approximation for the trial function at point \( x \)

Figure 2. The local domains, the support of nodes, the domain of definition of a point, and the domain of influence of a node: (1) the domain of definition of any point \( x \) is the domain which covers all the nodes whose weight functions do not vanish at \( x \). (2) the domain of influence for source point \( y \) is the union of all \( \Omega_k \), \( \forall x \in \partial \Omega_k \) (for the MLHIE method) or \( \forall x \in \Omega_k \) (for MLPG approach)

Upon solving for the modified test function, we can solve the problem by using a numerical discretization technique. For the 2-D harmonic operator, the modified test function \( \tilde{u}^* \) can be easily solved as

\[
\tilde{u}^* = u^* - \tilde{u} = \frac{1}{2\pi} \ln \frac{r_0}{r}
\]

(16)
since \( u^* = -1/(2\pi) \ln r \) and therefore \( \tilde{u} = -1/(2\pi) \ln r_0 \), where \( r = |x - y| \) denotes the distance from the source point to the generic point under consideration, and \( r_0 \) is the radius of the local sub-domain, \( \Omega_k \).

Substituting equation (3) into equation (15), imposing boundary conditions on the right-hand side for node \( i \), and carrying out the integrals, we obtain the following linear equations:

\[
a_i u_i = \sum_j K_{ij}^* \tilde{u}_j + f_i^* \quad i = 1, 2, \ldots, N
\]

(17)

where \( N \) is the total number of nodes in the entire domain \( \Omega \).

\[
K_{ij}^* = \int_{\Gamma_{i\alpha}} \tilde{u}^*(x, y_i) \frac{\partial \phi_j(x)}{\partial n} d\Gamma - \int_{\Gamma_{i\beta}} \phi_j(x) \frac{\partial \tilde{u}^*(x, y_i)}{\partial n} d\Gamma
- \int_{\Gamma_{i\mu}} \phi_j(x) \frac{\partial \tilde{u}^*(x, y_i)}{\partial n} d\Gamma + \int_{\Omega_k} \omega^* \phi_j(x) \tilde{u}^*(x, y_i) d\Omega
\]

(18a)

and

\[
f_i^* = \int_{\Gamma_{i\alpha}} \tilde{u}^*(x, y_i) q d\Gamma - \int_{\Gamma_{i\beta}} \frac{\partial \tilde{u}^*(x, y_i)}{\partial n} d\Gamma - \int_{\Gamma_{i\mu}} \frac{\partial \tilde{u}^*(x, y_i)}{\partial n} d\Gamma - \int_{\Omega_k} \tilde{u}^*(x, y_i) p(x) d\Omega
\]

(18b)
in which $\Gamma_{{aq}}$ and $\Gamma_{{q}}$ are the flux and essential boundary sections, respectively, of $\Gamma$, with $\Gamma = \Gamma_{{aq}} \cup \Gamma_{{u}}$, and $\tilde{u}$ and $\tilde{q}$ are the prescribed values of $u$ and $\partial u/\partial n$ on $\Gamma_{{u}}$ and $\Gamma_{{aq}}$, respectively. For those interior nodes located inside the domain $\Omega$, $L_{{i}} = \nabla \Omega_{{i}}$, and the boundary integrals involving $\Gamma_{{u}}$ and $\Gamma_{{aq}}$ vanish in equations (18a) and (18b).

From equation (18a), it is seen that no derivatives of the shape functions are needed in constructing the stiffness matrix for the interior nodes and for those boundary nodes with no essential-boundary-condition-prescribed sections on their local boundaries. This is attractive in engineering applications as the calculation of derivatives of shape functions from the MLS approximation is quite costly.

Here, it should be noted that since the unknown variable $u$ at the source point $y$ (or more precisely, the nodal value of $u^h(x)$ itself) appears on the left-hand side of equation (17), it is very convenient to impose the essential boundary conditions if any, at the global boundary $\Gamma$. Upon imposing the essential boundary condition for $u_{{i}}$ on the left-hand side of equation (17) for those nodes where $u$ is specified; or using equation (3) to represent $u$ for those nodes with $u$ unknown, and rearranging equation (17), we have the following linear system of $\tilde{u}$:

$$K \cdot \tilde{u} = f$$

(19)

4. THE MESHLESS LOCAL PETROV–GALERKIN (MLPG) APPROACH

The MLPG approach is first proposed in [12, 13], for solving linear and non-linear potential problems. Unlike in the MLBIE method which uses a local \textit{unsymmetric} weak form, i.e. the LBIE, the MLPG approach uses a local \textit{symmetric} weak form over a local sub-domain $\Omega_{{i}}$ as a formulation and uses the MLS approximation to develop a truly meshless method. As the MLS approximation is used to construct shape functions, the essential boundary conditions in the MLPG approach is enforced, \textit{a posteriori}, by a penalty formulation.$^1$

In the present paper, we use the following problem in linear elasticity to illustrate the formulation:

$$\sigma_{{ij,j}} + b_{{j}} = 0 \quad \text{in } \Omega$$

(20)

where $\Omega$ is the global domain bounded by $\Gamma$, $\sigma_{{ij}}$ is the stress tensor, which corresponds to the displacement field $u_{{i}}$, $b_{{j}}$ is the body force, and $(\ )_{{,j}}$ denotes $\partial ( ) / \partial x^j$. The corresponding boundary conditions are given as follows:

$$u_{{i}} = \tilde{u}_{{i}} \quad \text{on } \Gamma_{{u}}$$

(21a)

$$t_{{i}} \equiv \sigma_{{ij}n_{{j}}} = \tilde{t}_{{i}} \quad \text{on } \Gamma_{{t}}$$

(21b)

where $\tilde{u}_{{i}}$ and $\tilde{t}_{{i}}$ are the prescribed displacements and tractions, respectively, on the displacement boundary $\Gamma_{{u}}$ and on the traction boundary $\Gamma_{{t}}$, and $n_{{i}}$ is the unit outward normal to the boundary $\Gamma$.

$^1$A simple and elementary way to satisfy the essential b.c exactly, \textit{a priori}, in a meshless method, is presented in Reference [15].
A generalized local weak form of the differential equation (20) and the boundary conditions (21), over a local sub-domain \( \Omega_s \subset \Omega \), can be written as

\[
\int_{\Omega_s} (\sigma_{ij,j} + b_i) v_i \, d\Omega - \alpha \int_{\Gamma_m} (u_l - \bar{u}_l) v_i \, d\Gamma = 0
\]  

(22)

where \( u_l \) and \( v_i \) are the trial and test functions, respectively, \( \Gamma_m \) is a part of the boundary \( \partial \Omega_s \) of \( \Omega_s \), over which the essential boundary conditions are specified, and \( \alpha \gg 1 \) is a penalty parameter used to impose the essential boundary conditions.

Using \( \sigma_{ij,j} v_i = (\sigma_{ij} v_i)_{,j} - \sigma_{ij,j} v_i \), and the divergence theorem, in equation (22) yields the following expression:

\[
\int_{\Omega_s} \sigma_{ij} p_j v_i \, d\Omega - \int_{\Gamma_m} (\sigma_{ij} v_{i,j} - b_i v_i) \, d\Omega - \alpha \int_{\Gamma_m} (u_l - \bar{u}_l) v_i \, d\Gamma = 0
\]  

(23)

in which \( \partial \Omega_s \) is the boundary of the sub-domain \( \Omega_s \) and \( n_i \) is the outward unit normal to the boundary \( \partial \Omega_s \).

Imposing the natural boundary condition, \( t_i = \sigma_{ij} n_j = \bar{t}_i \) on \( \Gamma_m \) in equation (23) we obtain

\[
\int_{\Omega_s} t_i v_i \, d\Omega + \int_{\Gamma_m} t_i v_i \, d\Gamma + \int_{\Gamma_m} \bar{t}_i v_i \, d\Gamma - \int_{\Omega_s} (\sigma_{ij} v_{i,j} - b_i v_i) \, d\Omega - \alpha \int_{\Gamma_m} (u_l - \bar{u}_l) v_i \, d\Gamma = 0
\]  

(24)

in which \( \Gamma_m \) is a part of \( \partial \Omega_s \), over which the natural boundary condition, \( t_i = \bar{t}_i \), is specified.

In the following development, the Petrov–Galerkin method is used. Unlike in the conventional Galerkin method in which the trial and test functions are chosen from the same space, the Petrov–Galerkin method uses the trial and the test functions from different spaces. In particular, the test functions need not vanish on the boundary where the essential boundary conditions are specified. In the present work, the trial functions \( u_l \) are approximated by the MLS approximation, while the test functions \( v_i \) will be chosen from known functions.

As the test functions are chosen from known functions, the above equation can be further simplified, by deliberately selecting the test functions \( v_i \) such that they vanish over \( L_n \), the circle (for an internal node) or the circular arc (for a node on the global boundary \( \Gamma \)). This can be easily accomplished by using the weight function in the MLS approximation as also a test function, with the radius \( r_l \) of the support of the weight function being replaced by the radius \( r_0 \) of the local domain \( \Omega_s \), such that the test function vanishes on \( L_n \).

Using such test functions and rearranging equation (24), we obtain the following local symmetric weak form (LSWF) in linear elasticity, as

\[
\int_{\Omega_s} \sigma_{ij} v_{i,j} \, d\Omega + \alpha \int_{\Gamma_m} u_l v_i \, d\Gamma - \int_{\Gamma_m} \bar{t}_i v_i \, d\Gamma = \int_{\Gamma_m} \bar{u}_i v_i \, d\Gamma + \int_{\Omega_s} b_i v_i \, d\Omega
\]  

(25)

As the test functions \( v_i \) is known, the discretization of equation (25) for one local domain will only yield one linear algebraic equation. In order to obtain two independent linear equations, we can apply two (for 2-D problems) or three (for 3-D problems) independent sets of test functions in equation (25), as has been done in the boundary integral equation method, to give the following local symmetric weak form (LSWF):

\[
\int_{\Omega_s} \sigma_{ij} v_{i,j} \, d\Omega + \alpha \int_{\Gamma_m} u_l v_i \, d\Gamma - \int_{\Gamma_m} \bar{t}_i v_i \, d\Gamma = \int_{\Gamma_m} \bar{u}_i v_i \, d\Gamma + \int_{\Omega_s} b_i v_i \, d\Omega
\]  

(26)

where \( v_{ij} \) denotes the \( r \)th component of the test function in the \( k \)th set.
For brevity, equation (26) can also be written in a matrix form as

$$\int_{\Omega_i} \varepsilon \sigma \, d\Omega + \int_{\Gamma_{w}} v u \, d\Gamma - \int_{\Gamma_{w}} v t \, d\Gamma = \int_{\Gamma_{w}} \varepsilon \frac{d\Omega}{d\Gamma} + \int_{\Omega_i} \varepsilon \sigma \, d\Omega + \int_{\Omega_i} v b \, d\Omega$$

(27)

where $\varepsilon$, denotes the strain matrix from the test functions, and $\sigma$ denotes the stress vector from the trial functions. For 2-D elasticity

$$\sigma = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

(28)

in which the superscript $(i)$ denotes the $i$th set of test functions, and

$$v = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{21} \\ v_{22} \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(29)

It should be noted that the two sets of test functions in $v$ must be independent. In the present work, $v$ is chosen as

$$v_{ij} = v \delta_{ij} \quad \text{or} \quad v = vI$$

(30)

where $\delta_{ij}$ is the Kronecker delta, $I$ is the identity matrix, and $v$ is taken as the weight function in the MLS approximation, with the support $r_i$ being replaced by $r_0$, the size of the local sub-domain.

Substitution of equation (3) into the LSWF (27) for all nodes leads to the following discretized system of linear equations:

$$\sum_{j=1}^{N} K_{ij} \tilde{u}_j = f_i, \quad i = 1, 2, \ldots, N$$

(31)

where $N$ is the total number of nodes,

$$K_{ij} = \int_{\Omega_i} \varepsilon(x, x_i) DB_j \, d\Omega + \int_{\Gamma_{w}} v(x, x_i) S \phi_j \, d\Gamma - \int_{\Gamma_{w}} v(x, x_i) NDSB_j \, d\Gamma$$

(32)

and

$$f_i = \int_{\Gamma_{w}} v(x, x_i) \tilde{I} \, d\Gamma + \int_{\Gamma_{w}} v(x, x_i) S \tilde{u} \, d\Gamma + \int_{\Omega_i} v(x, x_i) b \, d\Omega$$

(33)

with $v(x, x_i)$ being the value of the test function matrix, corresponding to node $i$, evaluated at the point $x$.

$$N = \begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix}$$

(34)

$$B_j = \begin{bmatrix} \phi_{j,1} & 0 \\ 0 & \phi_{j,2} \\ \phi_{j,2} & \phi_{j,1} \end{bmatrix}$$

(35)
D is the stress–strain matrix:

\[
    D = \frac{E}{1 - \nu^2} \begin{bmatrix}
        1 & \nu & 0 \\
        \nu & 1 & 0 \\
        0 & 0 & (1 - \nu)/2
    \end{bmatrix}
\]

for plane stress \((36)\)

and \(S\) denotes the essential boundary condition index, defined as

\[
    S = \begin{bmatrix}
        S_1 \\
        0 \\
        0 & S_2
    \end{bmatrix}
\]

with

\[
    S_i = \begin{cases} 
        1 & \text{if } u_i \text{ is prescribed on } \Gamma_u \\
        0 & \text{if } u_i \text{ is not prescribed on } \Gamma_u 
    \end{cases} \quad (i = 1, 2) \quad (38)
\]

It can be easily seen that the system stiffness matrix in the present method is banded but unsymmetric. The locations of the non-zero entries in the system 'stiffness' matrix depend upon the nodes located inside the domain of influence of the node.

A comprehensive discussion of the MLPG method, which may also lead to a symmetric stiffness, and may use other meshless interpolations such as partition of unity and Shepard functions, may be found in Reference [15].

5. NUMERICAL EXAMPLES

In this section, some numerical results are presented to illustrate the implementation and convergence of the present MLBIE and MLPG approaches. For the purpose of error estimation and convergence studies, the displacement and energy norms, \(\|u\|\) and \(\|\varepsilon\|\) (for the MLPG approach), and the Sobolev norms \(\|u\|_k\), \(k = 0, 1\) (for the MLBIE method) are calculated. These norms are defined as

\[
    \|u\| = \left( \int_{\Omega} u^T \cdot u \, d\Omega \right)^{1/2} \quad (39a)
\]

\[
    \|\varepsilon\| = \left( \frac{1}{2} \int_{\Omega} \varepsilon^T \cdot D \cdot \varepsilon \, d\Omega \right)^{1/2} \quad (39b)
\]

\[
    \|u\|_0 = \left( \int_{\Omega} u^2 \, d\Omega \right)^{1/2} \quad (39c)
\]

and

\[
    \|u\|_1 = \left( \int_{\Omega} u^2 + (|\nabla u|)^2 \, d\Omega \right)^{1/2} \quad (39d)
\]

The relative errors for \(\|u\|\), \(\|\varepsilon\|\) and \(\|u\|_k\) are defined as

\[
    r_u = \frac{\|u_{\text{num}} - u_{\text{exact}}\|}{\|u_{\text{exact}}\|} \quad (40a)
\]

\[
    r_\varepsilon = \frac{\|\varepsilon_{\text{num}} - \varepsilon_{\text{exact}}\|}{\|\varepsilon_{\text{exact}}\|} \quad (40b)
\]
Figure 3. Relative errors and convergence rates for the Dirichlet problem of Poisson’s equation: (a) for norm $\| \cdot \|_{0}$; and (b) for norm $\| \cdot \|_{1}$

and

$$r_k = \frac{\| u^\text{num} - u^\text{exact} \|_k}{\| u^\text{exact} \|_k}, \quad k = 0, 1 \quad (40c)$$

5.1. Examples for the MLBIE approach

5.1.1. Poisson’s equation. The results from the present MLBIE formulation are studied for the Poisson’s equation with a given source function $\rho = x^1 + x^2$ in the $2 \times 2$ domain, for which the exact solution is taken to be

$$u = -\frac{x^2}{6}[(x^1)^3 + (x^2)^3] + 3(x^1)^2x^2 + 3x^1(x^2)^2 \quad (41)$$

A Dirichlet problem is solved, for which the essential boundary condition is imposed on all sides, and a mixed problem, for which the essential boundary condition is imposed on top and bottom sides and the flux boundary condition is prescribed on left and right sides of the domain. The MLS approximation with both linear and quadratic bases as well as Gaussian and spline weight functions are employed in the computation. The size of support for both weight functions are taken to be 2, and the parameter $c_i$ for Gauss weight function is $r_i/4$.

Regular meshes of $9(3 \times 3)$, $36(6 \times 6)$ and $64(8 \times 8)$ nodes are used to study the convergence of the method. The local boundary integrals on $\partial \Omega_i$ are evaluated by using 20 Gauss points on each local boundary. The size (radius) of the local boundary for each node is taken as 0.005 in the computation.

The convergence with mesh refinement of the present method is studied for this problem. The results of relative errors and convergence for the $\| \cdot \|_{0}$ and $\| \cdot \|_{1}$ norms are shown in Figure 3 for the Dirichlet problem and in Figure 4 for the mixed problem, respectively. These figures show that the present meshless method works quite well for the Poisson’s equation.

In this example, it can be seen that the quadratic basis yields somewhat of a better result than the linear basis while both bases possess high accuracy. Also, the Gauss weight function works better than the spline weight function. We should keep in mind that the appropriate parameters $c_i$ in equation (7) need to be determined for all nodes for the Gauss weight function. The values of these parameters will affect the numerical results considerably. With inappropriate $c_i$ used in

the Gaussian weight function, the results may become very unsatisfactory. The optimal choice of these parameters is still an open research topic.

5.2. Potential flow

Consider the problem of a potential flow around a cylinder of radius $a$ in an infinite domain, shown in Figure 5. $u$ represents the stream function.

Due to symmetry, here only a part, $0 \leq x^1 \leq 4$ and $0 \leq x^2 \leq 2$, of the upper left quadrant of the field is modelled as shown in Figure 5. The exact solution for this problem is given by

$$ u = x^2 \left[ 1 - \frac{x^1}{(x^2)^2 + (x^1 - L)^2} \right] $$

Figure 5 shows the prescribed $u$ and $\partial u/\partial n$ values along all boundaries. The essential boundary condition on the left and top edges is imposed according to the exact solution (42).

The initial mesh of 24 nodes is considered as shown in Figure 6(a). Subsequently, the number of nodes is increased to 47 in Figure 6(b) and 74 in Figure 6(c) to study the convergence of the present meshless method.

Both linear and quadratic bases as well as Gaussian and spline weight functions are considered. We use \( c_i = l_i \) and \( r_i = 4c_i \) in the calculation, where \( l_i \) is defined as

\[
    l_i = \max_{j \in S_i} \| x_j - x_i \|
\]

where \( S_i \) is the minimum set of neighbouring nodes of \( x_i \) which construct a polygon surrounding \( x_i \).

The convergence for the Sobolev norms \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \) is shown in Figure 7. The mesh size \( h \) in this problem is defined as the average mesh sizes on the bottom edge. From Figure 7, we notice that the quadratic basis is surprisingly not as good as the linear basis in this problem for both Gaussian and spline weight functions.

The streamlines from the exact solution and from the numerical solution of the present meshless formulation with a linear basis and Gaussian weight functions for the cases of 47 and 74 nodes are also shown in Figure 8. It can be seen that the streamlines are well approximated by the present method with 74 nodes as compared to the closed-form solution.

5.3. Examples for the MLPG approach

In the computation, we find that the spline weight function works not as good as the Gaussian weight function, therefore, only the Gaussian weight function is considered. Also, in the test function \( v, r_0/c_i \) is taken as 1.
5.3.1. Cantilever beam. The behaviour of the present MLPG formulation is studied in the cantilever beam problem (see Figure 9), for which the following exact solution is given [15] as

\[
\begin{align*}
    u_1 &= -\frac{P}{6EI} \left( x^2 - \frac{D}{2} \right) \left[ 3x^1(2L - x^1) + (2 + \nu)x^2(x^2 - D) \right] \\
    u_2 &= \frac{P}{6EI} \left[ (x^1)^2(3L - x^1) + 3\nu(L - x^1) \left( x^2 - \frac{D}{2} \right)^2 + \frac{4 + 5\nu}{4} D^2x^1 \right]
\end{align*}
\]  

(44a) (44b)

where

\[ I = \frac{D^3}{12} \]

The stress corresponding to equations (44a) and (44b) are

\[
\begin{align*}
    \sigma_{11} &= -\frac{P}{I} (L - x^1) \left( x^2 - \frac{D}{2} \right) \\
    \sigma_{22} &= 0 \\
    \sigma_{12} &= -\frac{Px^2}{2I} (x^2 - D)
\end{align*}
\]  

(45a) (45b) (45c)

The problem is solved for the plane stress case with \( P = 1, \ E = 1, \ D = 1, \ L = 8 \) and the penalty parameter \( a = 10^6 \). Both \( \nu = 0.25 \) and \( \nu = 0.4999 \) are considered, to study the locking property of the present MLPG approach. Regular meshes of 52 \((13 \times 4)\), 72 \((18 \times 4)\) and 92 \((23 \times 4)\) nodes are used.

In all cases, \( r_i = 6h \) and \( c_i = r_i/4 \) are used, with \( h \) being the mesh size. In the formulation, the size of each local sub-domain should be big enough such that the union of all local sub-domains
covers as much as possible of the global domain. In this example, the size (radius) of the local sub-domain of each internal node is taken as $h_2$, the mesh size in the $x^2$ direction, and that of each boundary node is taken as $D/2$. In the computation, nine Gauss points are used on each section of $\Gamma_1$, and $6 \times 9$ points are used in each local domain $\Omega_2$ for numerical quadratures.

The convergence with mesh refinement of the present MLPG formulation is studied for this problem. The results of relative errors and convergence rates are shown in Figures 10 and 11 for displacements and strain energy, respectively. The mesh size $h$ in these figures is defined as the distance in $x^1$ direction between two neighbouring nodes with same $x^2$ co-ordinate. It can be seen that the convergence rates for displacements and strain energy of the present formulation exceed the corresponding convergence rates of the displacement-based FEM method, which are 2 and 1 for a linear basis, and 3 and 2 for a quadratic basis [16, 17].

Figures 10 and 11 also show that the present MLPG approach does not exhibit any volumetric locking, even though no modification is made for nearly incompressible materials with $v = 0.4999$ in the plane stress case.

Figure 12 shows the normal stress $\sigma_{11}$ and the shear stress $\sigma_{12}$ for $v = 0.25$ at $x^1 = L/2 = 4$ for the present MLPG formulation with 52 and 72 nodes. The shear stress and normal stress are almost as same as the exact solutions for 72 nodes, which are generally hard to obtain for the standard
Figure 12. (a) The normal stress $\sigma_{11}$; and (b) the shear stress $\sigma_{12}$ at $x^1=L/2$ for the cantilever beam problem

FEM. It is noted that the results for the present formulation satisfy the traction-free boundary conditions at $x^2 = 0$ and $x^2 = D = 1$ almost exactly.

5.3.2. Infinite plate with a circular hole. We consider an infinite plate with a central hole: $(x^1)^2 + (x^2)^2 \leq a^2$ of radius $a$. The plate is subjected to a uniform tension, $\sigma = 1$, in the $x^1$ direction at infinity. The exact solution for stresses is

$$
\sigma_{11} = \sigma \left[ 1 - \frac{a^2}{r^2} \left( \frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3a^4}{2r^4} \cos 4\theta \right] \tag{46a}
$$

$$
\sigma_{12} = \sigma \left[ -\frac{a^2}{r^2} \left( \frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3a^4}{2r^4} \sin 4\theta \right] \tag{46b}
$$

$$
\sigma_{22} = \sigma \left[ -\frac{a^2}{r^2} \left( \frac{1}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3a^4}{2r^4} \cos 4\theta \right] \tag{46c}
$$

where $(r, \theta)$ are the polar co-ordinates and $\theta$ is measured from the positive $x^1$-axis counterclockwise. The corresponding displacements, in plane strain case, are given by

$$
u_1 = \frac{1 + \nu}{E} \sigma \left( \frac{1}{1 + \nu} r \cos \theta + \frac{2}{1 + \nu} \frac{a^2}{r} \cos \theta + \frac{1}{2} \frac{a^2}{r^3} \cos 3\theta - \frac{1}{2r^3} \cos 3\theta \right) \tag{47a}
$$

$$
u_2 = \frac{1 + \nu}{E} \sigma \left( -\frac{r}{1 + \nu} \sin \theta - \frac{1}{1 + \nu} \frac{a^2}{r} \sin \theta + \frac{1}{2} \frac{a^2}{r^3} \sin 3\theta - \frac{1}{2r^3} \sin 3\theta \right) \tag{47b}
$$

Due to symmetry, only a part, $0 \leq x^1 \leq 4$ and $0 \leq x^2 \leq 4$, of the upper right quadrant of the plate is modelled as shown in Figure 13. Symmetry conditions are imposed on the left and bottom edges, i.e. $u_1 = 0, u_2 = 0$ is prescribed on the left edge and $u_2 = 0, u_1 = 0$ on the bottom edge, and the inner boundary at $a = 1$ is traction free. The traction boundary conditions given by the exact solution (46a)–(46c) are imposed on the right $(x^1 = 4)$ and top $(x^2 = 4)$ edges (see Figure 13). In the computation, a plane stress case with $E = 1.0$ and $v = 0.25$ is considered.

The initial mesh of 54 nodes, with six nodes in the $r$ direction and nine nodes in the $\theta$ direction, is considered as shown in Figure 14(a). Subsequently, the number of nodes is increased to 99(11 x 9)
and 135(15 × 9) to study the convergence of the present MLPG approach. The nodes are arranged regularly in the \( \theta \) direction and irregularly in the \( r \) direction.

Both linear and quadratic bases are considered. We use \( c_i = l_i \) and \( r_j = 4c_i \) in the calculation, where \( l_i \) is defined as the \( j \)th smallest distance between node \( i \) and the other nodes. The size of a local sub-domain is chosen as the minimum distance between the node in question and the other nodes. In the computation, 11 Gauss points are used on each section of \( \Gamma_i \), and \( 6 \times 9 \) points are used in each local domain \( \Omega_2 \) for numerical quadratures.

The convergence for displacements and strain energy is shown in Figure 15. The mesh size \( h \) in this problem is defined as the largest distance between neighboring nodes which form a quadrilateral. For the present MLPG formulation, even though the convergence rates from the quadratic basis are higher than those from the linear basis, the convergence rates for displacements and strain energy with the linear basis exceed those of the conventional FEM, which are 2 and 1, respectively [16, 17], while the convergence rates with the quadratic basis are not as good as those of the FEM, which are 3 and 2, respectively. We note that the convergence rates for the present formulation depend on the constants \( c_i \) and \( r_j \) in the weight functions. Optimal values of \( c_i \) and \( r_j \) should be explored in order to obtain better results.
Figure 15. Relative errors and convergence rates for the problem of a plate with a circular hole: (a) for the displacement norm; and (b) for the energy norm.

Figure 16. Normal stress $\sigma_{11}$ at $x^1 = 0$ for the problem of plate with a hole.

The stress $\sigma_{11}$ at $x^1 = 0$ obtained by the present MLPG method for linear elasticity with a quadratic basis is also depicted in Figure 16. It can be seen that the steep stress $\sigma_{11}$ is well approximated by the present method when 135 nodes are used.

6. CONCLUSIONS AND DISCUSSIONS

Two truly meshless methods, the MLBIE method based on a local boundary integral equation and the MLPG approach based on the local symmetric weak form, are presented for solving boundary-value problems, with the shape functions being from the MLS approximation. Convergence studies in the numerical examples show that these two methods possess excellent rates of convergence for both the unknown variables (displacements) and there derivatives (strain energy). Only a simple numerical manipulation is needed for calculating the derivatives of the unknown variables as the original approximated trial solution is smooth enough to yield reasonably accurate results for derivatives. The numerical results show that using both linear and quadratic bases in the MLS approximation can give quite accurate numerical results.

As is the case with other meshless methods based on a global Galerkin method, the present two methods also possess the following advantages over the finite element method.

(a) The present methods are considerably more accurate for computing the values of the unknown variables (displacements) and their derivatives (strains) than the finite element method.

(b) No smoothing technique is required to compute the derivatives, as the original result is smooth enough.

(c) No element connectivity is needed; and only randomly distributed nodes and the corresponding sub-domains are constructed in the global domain.

Compared with the other meshless techniques, discussed in literature, based on a global Galerkin formulation (for instance, the EFG method), the present approaches are found to have the following advantages.

(i) Absolutely no elements are needed in the present two formulations, either for interpolation purposes or for integration purposes, while shadow elements are required to evaluate volume integrals, in the so-called meshless methods based on global weak forms.

(ii) No special integration scheme is needed to evaluate the volume and boundary integrals. The integrals in the present methods can be easily evaluated over regularly shaped sub-domains and their boundaries. The local boundary in general is the surface of a 'sphere' centred at the node in question. This flexibility in choosing the size and the shape of the local sub-domain will lead to more convenient formulations in dealing with the non-linear problems.

Even though meshless methods have been reported in literature for about 20 years, meshless methods based on local weak forms are very new. There are many aspects in these methods which need to be explored, such as the selection of the parameters in the MLS approximation, the selection of the sizes of the local sub-domains, the selection of test functions in the MLPG method, etc. However, due to their flexibility in solving practical problems, especially those problems with discontinuities and moving boundaries, the meshless methods based on local weak forms have shown a great promise in engineering applications.

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