AN EQUIVALENT DOMAIN INTEGRAL METHOD FOR COMPUTING CRACK-TIP INTEGRAL PARAMETERS IN NON-ELASTIC, THERMO-MECHANICAL FRACTURE

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Abstract—The crack-tip parameters, such as $J'$, $T^*$, $AT^*$ etc, which quantify the severity of the stress/strain fields near the crack-tip in elastic-plastic materials subject to thermo-mechanical loading, are often expressed as integrals over a path that is infinitesimally close to the crack-tip (front). The integrand in such integrals involves the stress-working density, stress, strain and displacement fields arbitrarily close to the crack-tip. In a numerical analysis, such data near the crack-tip are not expected to be very accurate. This paper describes simple approaches and attendant computational algorithms, wherein, the “crack-tip integral” parameters may be evaluated through “equivalent domain integrals” (EDI) alone. It is also seen that the present (EDI) approaches form the generic basis for the popular “virtual crack extension” (VCE) methods. Several examples of thermo-mechanical fracture, including: (i) thermal loading of an elastic material, (ii) arbitrary loading/unloading/reloading of an elastic-plastic material, containing a single dominant crack, are presented to illustrate the present approach and its accuracy.

1. INTRODUCTION

IT IS WELL known that in a linear elastic material containing a dominant crack, the strength of the crack-tip stress and strain field is quantified by the so-called $K$-factor. Beginning with the work of Eshelby[1], Cherepanov[2] and Rice[3], interest has been focused on certain ‘integral’-type crack-tip parameters which may quantify the severity of the crack-tip stress/strain fields. These parameters are, in general, defined, say for two-dimensional problems, as integrals over a circular path $\Gamma_c$, with a radius $\epsilon$ being ‘very small’. The integrand in these ‘crack-tip integrals’ is, in general, such that it is of $(1/\epsilon)$ type at radius $\epsilon$ from the crack-tip, which renders the integral over $\Gamma_c$ to be of a finite magnitude. This integral over $\Gamma_c$ is often sought to be represented, equivalently, as a far-field integral plus a ‘finite domain integral’ using the divergence theorem.

As discussed comprehensively, for instance, by Atluri et al.[4, 5], the aforementioned finite domain integral vanishes identically under some special circumstances, such as when (i) the material is linearly or nonlinearly elastic and appropriately homogeneous, such that if $W$ is the total stress-working density, one has $(\partial W/\partial X_i) = \sigma_{ij}(\partial e_{ij}/\partial X_i)$; (ii) the body forces due to thermal strains, electromagnetic forces, inertia, etc. are zero; (iii) for elasto-plastic materials, only conditions of monotonic and proportional loading must prevail, so that a deformation theory of plasticity (or, in essence, a nonlinear elasticity theory) is valid; and (iv) during quasi-static or dynamic crack-propagation, fully steady-state conditions must prevail everywhere in the solid, i.e. the stress/strain fields everywhere in the solid must appear invariant to an observer moving with the crack-tip. Under these special circumstances, the crack-tip integral becomes path-independent and becomes equal to the far-field contour integral. One such path-independent contour integral for two-dimensional mode-I problems is the well-known $J$, that may be attributed to the work of Eshelby[1], Cherepanov[2] and Rice[3]. For purely elastic materials, as shown in [4-6], the crack-tip integral parameter $J'$ (defined in refs [4, 5]) has the interpretation as an energy-release rate, whether or not the special conditions (i) to (iv) listed above hold (if they do not, $J'$ simply has an equivalent representation as a far-field contour integral plus a finite domain integral). On the other hand, for elasto-plastic materials, it has been shown in [4-6] that the far-field contour integral alone, as defined in [2, 3] may be interpreted as the difference in area under the load-deformation curves of two identical and identically loaded cracked bodies with slightly different crack lengths only when (i) each cracked body is loaded only quasi-statically, i.e. the kinetic energy is zero; (ii) the body forces due to, say, thermal gradients, gravity, electromagnetic, or other sources, are zero; (iii) each elastic-plastic body is loaded only monotonically and proportionally, (iv) there is no

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unloading; (v) the crack in each body remains stationary, i.e. only up to the initiation of quasi-
static crack growth.

Thus, it is apparent from the above discussion that, in general, a crack-tip integral parameter
has an equivalent representation as a far-field integral plus a finite domain integral. Also, in general,
such a crack-tip parameter may not have any physical interpretation beyond that it is simply an
integral parameter that quantifies the severity of the crack-tip fields. Such parameters have been
introduced variously, by Blackburn[7], Ainsworth, Neale and Price[8], Wilson and Yu[9],
Kishimoto, Aoki and Sakata[10], Atluri et al.[4-6, 11] and Burst et al.[12].

Also, under circumstances discussed earlier, when the far-field integral $J$ can be related to the
difference in areas under load-deformation curves of two cracked bodies (or equivalently the rate
of change of total potential energy with respect to crack length), the so-called virtual crack-extension
(VCE) method to determine the far-field $J$ has been developed by Parks[13, 14] and Hellen[15].
For reasons discussed earlier in this paper, these methods are, strictly speaking, restricted to elastic
materials only. More recently de Lorenzi[16] and Li et al.[17] proposed methods of calculating
the far-field $J$ (which, for elastic materials, will be synonymous with a crack-tip parameter) which
are analogous to the VCE method, in that they involve only equivalent domain integrals. References
[16, 17] do not consider problems with body forces and/or non-elastic deformations and non-
proportional loading (and unloading). Miyazaki et al.[18] recently presented an invariant form
of the VCE equation, but their formulation results in a mixture of both contour and domain
integration in two-dimensional problems. The details of the numerical algorithm are not given in
[18].

This paper deals with numerical schemes and attendant algorithms for determining crack-tip
parameters under situations wherein thermal strain and non-proportional loading (and unloading)
of elastic–plastic materials are considered. In these situations, the present numerical algorithm
aims at representing the crack-tip parameter entirely by an ‘equivalent domain integral’ alone. In
that sense, the present approach is analogous to a VCE method. Specific and explicit computational
algorithms for evaluating the ‘equivalent domain integral’ (EDI), when quadratic isoparametric
finite elements are employed, are given. Several numerical examples to illustrate the validity of
the approach, are provided.

2. ‘EDI’ REPRESENTATION OF CRACK-TIP PARAMETERS IN THE PRESENCE OF
NONELASTIC STRAINS AND THERMO-MECHANICAL LOADING

In the remainder of the paper, we will consider the quasi-static thermo-mechanical loading of
solids containing only stationary cracks, i.e. only the problem of incipient crack growth. Also, we
restrict our attention to quasi-static, two-dimensional problems only, wherein the strains may be
considered to be infinitesimal except for the possibly singular strains near the crack-tip. For such
problems one may define a crack-tip parameter[4–6],

$$J_e' \equiv T^* = \int_{\Gamma_e} (W n_1 - \sigma_{ij} u_{i,1} n_j) d\Gamma$$

(1)

wherein the nomenclature is illustrated in Fig. 1, $(\ )_i$ denote $\partial(\ )/\partial X_i$, and $W$ is the total stress-
working density:

$$W = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij} = \int_0^{\epsilon_{ij}} \sigma_{ij} \frac{d\epsilon_{ij}}{dt} dt$$

(2)

The sign convention is indicated in Fig. 1. We restrict our attention to materials and loading such
that:

$$\epsilon_{ij} \equiv \epsilon_{ij}^e + \epsilon_{ij}^p + \epsilon_{ij}^\theta \equiv \epsilon_{ij}^p + \epsilon_{ij}^\theta$$

and
Equivalent domain integral method for crack-tip integral parameters

Fig. 1. Nomenclature for a plane crack.

\[
\frac{d\epsilon_{ij}}{dt} = \frac{d\epsilon_{ij}^e}{dt} + \frac{d\epsilon_{ij}^t}{dt} + \frac{d\epsilon_{ij}^p}{dt}
\]  

wherein the superscript 'e' denotes elastic, 't' denotes thermal, and 'p' denotes the 'plastic' strains. The plasticity is assumed to be of the 'rate-independent' nature. The superscript 'm' denotes the mechanical strains. Sometimes it is customary\[7,8\] to define crack-tip parameters involving mechanical strains alone, as:

\[
J_e = \int_{\Gamma_e} (W_{ep} n_i - \sigma_{ij} u_{ij} n_j) d\Gamma
\]  

where

\[
W_{ep} = \int_0^s \sigma_{ij} d\epsilon_{ij}^m
\]

\[
d\epsilon_{ij}^m = d\epsilon_{ij}^e + d\epsilon_{ij}^p.
\]

The difference between \(J_e\) and \(J_s\), as defined in eqs (1) and (4), arises mainly when the temperature field near the crack-tip may be singular as in the case of concentrated heat sources near the crack-tip.

As in [16], we introduce an arbitrary but continuous function \(S(X_1, X_2)\) such that:

\[
S = 1 \text{ on } \Gamma_e
\]

\[
S = 0 \text{ on } \Gamma_f
\]

where \(\Gamma_e\) and \(\Gamma_f\) are illustrated in Fig. 1. The use of eq. (7) allows one to rewrite eq. (1) as:

\[
J_e' = - \left[ [W n_i - \sigma_{ij} u_{ij} n_j] S d\Gamma \right]
\]

\[
\Gamma \equiv \Gamma_e - \Gamma_f + \Gamma_s
\]

with a similar representation for \(J_s\) when \(W\) in eq. (8) is replaced by \(W_{ep}\).

The use of the divergence theorem in eq. (8) allows us, then, to give equivalent domain integral (EDI) representations for the crack-tip parameters \(J_e'\) and \(J_s\). Thus,\n
\[
J_e' = - \left[ \int_{A-A_f} \left( \frac{\partial}{\partial X_1} (WS) - \frac{\partial}{\partial X_j} \left( \sigma_{ij} \frac{\partial u_i}{\partial X_1} S \right) \right) dA \right]
\]

with a similar expression for \(J_s\) when \(W\) in eq. (9) is replaced by \(W_{ep}\). Now we assume that equilibrium conditions, in the absence of body forces and inertia, hold, as:
Further, we assume the existence of an elastic potential, $W^e$, for stress $\sigma_{ij}$, such that:

$$\sigma_{ij} = \frac{\partial W^e}{\partial \varepsilon_{ij}}$$

(11)

where

$$W^e = \int \sigma_{ij} \, d\varepsilon_{ij}.$$  

(12)

Under the assumptions of eqs (10) and (11), it can be easily shown that eq. (9) and a similar expression for $J_e$, may be rewritten as:

$$J'_e = J'(S) + J'(W)$$

(13)

$$- J'(S) + J'(W^p)$$

(14)

$$J_e = J(S) + J(W^{ep})$$

(15)

$$= J(S) + J(W^p)$$

(16)

where

$$J'(S) = - \int_{\mathcal{A} - \mathcal{A}_e} \left\{ W^e \frac{\partial S}{\partial X_j} - \sigma_{ij} \frac{\partial u_j}{\partial X_j} \right\} \, dA$$

(17)

$$J'(W) = - \int_{\mathcal{A} - \mathcal{A}_e} \left\{ \frac{\partial W}{\partial X_j} - \sigma_{ij} \frac{\partial \varepsilon_{ij}}{\partial X_j} \right\} S \, dA$$

(18)

$$J'(W^{ep}) = - \int_{\mathcal{A} - \mathcal{A}_e} \left\{ \frac{\partial W^{ep}}{\partial X_j} - \sigma_{ij} \frac{\partial \varepsilon_{ij}^{ep}}{\partial X_j} \right\} S \, dA$$

(19)

$$W^{ep} = \int \sigma_{ij} (d\varepsilon_{ij}^{ep} + d\varepsilon_{ij}^p)$$

(20)

$$d\varepsilon_{ij}^{ep} = d\varepsilon_{ij}^p + d\varepsilon_{ij}^e = d\varepsilon_{ij} - d\varepsilon_{ij}^e$$

(21)

$$J(S) = - \int_{\mathcal{A} - \mathcal{A}_e} \left\{ W^e \frac{\partial S}{\partial X_j} - \sigma_{ij} \frac{\partial u_j}{\partial X_j} \right\} \, dA$$

(22)

$$J(W^{ep}) = - \int_{\mathcal{A} - \mathcal{A}_e} \left\{ \frac{\partial W^{ep}}{\partial X_j} - \sigma_{ij} \frac{\partial \varepsilon_{ij}^{ep}}{\partial X_j} \right\} \, dA$$

(23)

$$J(W^p) = - \int_{\mathcal{A}_e} \left\{ \frac{\partial W^p}{\partial X_j} - \sigma_{ij} \frac{\partial \varepsilon_{ij}^p}{\partial X_j} \right\} S \, dA$$

(24)

$$W^p = \int \sigma_{ij} d\varepsilon_{ij}^p$$

(25)
On the other hand, instead of the above equivalent domain integral (EDI) representations, one may, upon the application of the divergence theorem, express eqs (1) and (4), respectively, as:

\[
J'_e = \int_{\Gamma} \left( W n_i - \sigma_{ij} \frac{\partial u_i}{\partial X_1} n_j \right) d\Gamma - \int_{A-\alpha_i} \left( \frac{\partial W}{\partial X_1} - \sigma_{ij} \frac{\partial e_{ij}}{\partial X_1} \right) dA
\]

(27)

and

\[
J_e = \int_{\Gamma} \left( W n_i - \sigma_{ij} \frac{\partial u_i}{\partial X_1} n_j \right) d\Gamma - \int_{A-\alpha_i} \left( \frac{\partial W}{\partial X_1} - \sigma_{ij} \frac{\partial e_{ij}}{\partial X_1} \right) dA
\]

(28)

In summary, we presented here four different ways of computing \( J'_e \), as in eqs (13), (14), (27) and (28), respectively, and four different ways of computing \( J_e \), as in (15), (16), (29) and (30), respectively. Equations (27), (28), (29) and (30) are straightforward expressions for \( J'_e \) and \( J_e \) and involve both contours as well as domain integrals. On the other hand, eqs (13), (14), (15) and (16), give equivalent domain integral representations for \( J'_e \) and \( J_e \); and, in the process, involve an arbitrary function \( S \) as defined in (7).

The remainder of the paper deals with an examination of the numerical implementation, attendant algorithms, and relative accuracies, in evaluating \( J'_e \) and \( J_e \), through the eight alternative equations discussed above.

### 3. NUMERICAL ALGORITHMS FOR EVALUATING \( J'_e \) AND \( J_e \)

We develop these algorithms, here, exclusively for use in conjunction with conventional 8-noded isoparametric quadrilateral elements to model the structure, except near the crack-tip, where collapsed quarter-point triangular elements are used.

For evaluating the domain integrals in eqs (13), (14), (15) and (16), it is convenient to use the familiar isoparametric representation:

\[
N^k (\xi, \eta) = \begin{cases} 1 & \text{if } k = 1 \ldots 8 \\
\end{cases}
\]

(31)

where \( N^k (\xi, \eta) \) are the 'shape' functions (see Fig. 2).

Consider first the evaluation of \( J' (S) \) and \( J (S) \) as in eqs (17) and (22), respectively. Under the assumptions of eqs (31) and (32), eq. (17), for instance, may be written as:

\[
J' (S) = \sum_{\text{elem}} \{ J' (S) \}_{\text{elem}}
\]

(33)

where, for each element,

\[
\{ J' (S) \} = - \int_{-1}^{+1} \int_{-1}^{+1} \left[ W \frac{\partial N^k}{\partial X_1} S^k - \sigma_{ij} \frac{\partial N^m}{\partial X_1} \frac{\partial N^k}{\partial X_j} u_i u_j S^k \right] (\det J) \, d\xi \, d\eta
\]

(34)
where \((\det J)\) is the determinant of the Jacobian of the transformation from \((X_1, X_2)\) to \((\xi, \eta)\) coordinates.

In the above, \(S^K\) are the 'nodal' values of \(S\). Since, \(S\) is an arbitrary but continuous function, numerical results will be presented in this paper for several types of the specified function \(S\). As such, it is convenient to represent (34) as:

\[
\{ J'(S) \}_{\text{elem}} = R^K S^K
\]

where,

\[
R^K = \int_{-1}^{1} \int_{-1}^{1} \left[ W \frac{\partial N^K}{\partial X_1} - \sigma_{ij} \frac{\partial N^K}{\partial X_j} \frac{\partial N^K}{\partial X_i} \right] \det J \, d\xi \, d\eta
\]

It is possible, using the properties of isoparametric finite elements, to write explicitly:

\[
\frac{\partial N^K}{\partial X_j} = \frac{1}{(\det J)} \left\{ (-1)^{j+1} \frac{\partial N^K}{\partial \eta} \frac{\partial N^K}{\partial \xi} + (-1)^{j} \frac{\partial N^K}{\partial \xi} \frac{\partial N^K}{\partial \eta} \right\} X^N_{v+1}
\]

\[
\begin{align*}
\{ K, N = 1, 8 \\
\{ j = 1, 2 
\end{align*}
\]

where \([j + 1] = \text{mod} (j+1, 2)\). Thus, \(R^K\) can be given the explicit expression,

\[
R^K = \left\{ \left( \frac{\partial N^K}{\partial \eta} \frac{\partial N^K}{\partial \xi} - \frac{\partial N^K}{\partial \xi} \frac{\partial N^K}{\partial \eta} \right) X^N_{v+1,1} \sigma_{ij} u_i^M \right\}^{(T)} \bar{w}(T)
\]

where \(K, L, M, N\) vary from 1 to 8; \(i, j = 1, 2\); \((T)\) is the integration point number; \(w^{(T)}\) is the integration weight, \(w^{(T)} = \det J^{(T)} / \det J\). Here, a \((2 \times 2)\) integration is used \(T = 1-4\).

On the other hand, the difficulty with computing the domain integrals in eqs (18), (19), (23), (24), (27), (28), (29) and (30) lies in the evaluation of the derivatives of the quantities \(W, W_p, W_p^\alpha, \epsilon_{ij}\) and \(\epsilon_{ij}^\alpha\) etc. In general these quantities \(W, W_p, W_p^\alpha, \epsilon_{ij}\) themselves are known, to a good numerical approximation, at the \((2 \times 2)\) integration points\[18\]. Consider an integral of the type:

\[
I = \int_{\text{elem}} \frac{\partial F}{\partial X_1} \, dA
\]

where the values of \(F\) are known at the integration points, as \(F^{(T)}\). The bi-linear extrapolation of \(F\) from \(F^{(T)}\) to the corner nodes, may be shown to yield\[19\]:

\[
F^N = E^{N(T)} F^{(T)}
\]

\[
\begin{align*}
E^{N(T)} &= \begin{bmatrix} A & B & C & B \\
B & A & B & C \\
C & B & A & B \\
B & C & B & A \end{bmatrix} \\
\{ N = 1, 4 \} \\
\{ T = 1, 4 \}
\end{align*}
\]
where

\[ A = 1 + \sqrt{\frac{3}{2}} ; \quad B = -\frac{1}{2} ; \quad C = 1 - \sqrt{\frac{3}{2}}. \]  

(42)

Now, one may interpolate \( F \) over the element in terms of \( F' \), and evaluate \( \frac{\partial F}{\partial X_1} \) at the center of the element \( (\eta = \xi = 0) \), from the relation:

\[
\frac{\partial F}{\partial X_1} = \frac{\partial L^N}{\partial X_1} E^{N(T)} F(T)
\]

(43)

where \( L^N \) are bi-linear shape functions:

\[
L^N = \frac{1}{4} (1 + \xi \xi^N) (1 + \eta \eta^N).
\]

(44)

Use of (43) in (39), and a one-point integration, leads to:

\[
I = \frac{1}{4 (\det J)} (\eta^K \xi^N - \xi^K \eta^N) X_2^K E^{N(T)} \Delta
\]

(45)

where \( \Delta \) is the area of the bi-linear element defined by the corner nodes; and \( \xi^K, \eta^K \) are the nodal values of the coordinates \( (\xi, \eta) \). Use of (41), and the explicit expression for \( \Delta \), results in:

\[
I = \frac{(\sqrt{3})}{2} \left\{ \left(F^{(1)} - F^{(3)}\right) \left(X_2^1 - X_2^3\right) + \left(F^{(4)} - F^{(2)}\right) \left(X_2^4 - X_2^2\right) \right\}.
\]

(46)

Equation (46) is a surprisingly simple result. It is worth remembering, however, that superscripts imply node numbering; and superscripts in parentheses imply numbering of integration points (see Fig. 2).

The results in eq. (46) now enable us to give the following explicit expression for the domain integral such as \( J'(W) \) in eq. (18):

\[
J'(W) = -\frac{(\sqrt{3})}{2} \left\{ W^{(1)} - W^{(3)} \right\} \left(X_2^1 - X_2^3\right) + \left(W^{(4)} - W^{(2)}\right) \left(X_2^4 - X_2^2\right)
\]

(47)

\[
- \sigma^0 \left[ \left( \sigma^{(1)} - \sigma^{(3)} \right) \left(X_2^1 - X_2^3\right) + \left( \sigma^{(4)} - \sigma^{(2)} \right) \left(X_2^4 - X_2^2\right) \right] S^0
\]

where \( \sigma^0 \) and \( S^0 \) are the average values of \( \sigma^{(i)} \) and that of \( S^K \) at the center of the element, respectively.

4. NUMERICAL EXAMPLES

4.1 Finite element models and choices for function \( S \)

The solutions were carried out using two finite element models, as in Figs 3 and 4. In each case, degenerate, 'quarter-point' singular sector-shaped elements are used immediately surrounding

Fig. 2. An eight-noded isoparametric element.
the crack-tip. The first mesh consists of 48 elements and 165 nodes, while the second mesh consists of 84 elements and 277 nodes. The radius of the crack-tip quarter-point elements is 2.5% of the crack length.

Several convenient choices for the function $S$ of eq. (7), which is required to be continuous, can be made. The most natural choice is to consider $S$ to be linear between $r_c$ and $r_f$ as:

$$S = \left(\frac{r_f - r}{r_f - r_c}\right)$$

(48)
as shown in Figs 5 and 6. Here \( r_i \) is the inner radius of the integrated area, \( r_f \) the outer radius and \( r \) the radius of the position of the node in question. A schematic representation of \( S \) for several areas of integration, with outer radius \( r_1; r_2; r_3; r_4 \) respectively, is given in Fig. 6. Another choice for the function \( S \) is the ‘saw-tooth’ function as shown in Fig. 7. The input data for defining such an \( S \) in a computer program may be considerably simpler in some instances. It is interesting to note that when an algorithm such as the one in eq. (47) is used to evaluate the domain integrals in eqs (18), (19), (23) and (24), with \( S \) as defined in Fig. 7, the resulting integral is one-half of the corresponding domain-integrals in eqs (27), (28), (29) and (30) respectively.

4.2 A thermally loaded elastic plate with an edge crack

A schematic definition of the problem is given in Fig. 8. Three different temperature fields of the types:

\[
T(X_i) = \left( \frac{2X_1}{W} \right) T_0; \quad (49a)
\]

\[
T(X_i) = -\left( \frac{X_1 - (W/2)}{W} \right)^2 T_0; \quad (49b)
\]

and

\[
T(X_i) = \frac{T_0}{2} - \left[ \frac{X_1 - (W/2)}{W} \right]^2 T_0; \quad (49c)
\]

respectively, are considered. As will be seen momentarily, the third example of eq. (49c) is one that is created to test the numerical accuracies of various procedures employed in this work.

Figure 9 shows the results for \( J_e \) for the thermal load case of eq. (49a) when the coarse mesh is used. Results for the normalized value of \( J_e \) as computed by using the alternate expressions of eqs (16) and (30), respectively, are shown in Fig. 9. The broken lines in Fig. 9 show, respectively, the first term, i.e. \( J(S) \) of eq. (16), and the first term, i.e. contour integral in eq. (30), respectively. The effect of mesh refinement for the same problem is shown in Fig. 10. The solid lines in Fig. 10 show results for \( J_e \) as computed by using eqs (16) [with two different types of function \( S \)] and (30), respectively. Once again, the broken lines indicate the ‘first terms’ as defined earlier. Figures 9 and 10 indicate the relative insensitivity of the obtained results to mesh refinement as well as to the type of function \( S \) employed.
The results in the sets of Figs 11 and 12, and 13 and 14 are for the cases of thermal loading of eqs (49b) and (49c), respectively; and the nomenclature in these sets of Figures is entirely analogous to that in the set of Figs 9 and 10.

From Figs 13 and 14 it is seen that, for the thermal loading of eq. (49c), the numerical value of $J_e$ as computed by eq. (30) is nearly path-independent except for large values of $r$. It is also seen that for this case, the first term of eq. (30), i.e. the contour integral, tends to be very large. Thus, while $J$ of eq. (30) is in itself a very small number as shown in Figs (13) and (14), it is the result of an algebraic sum of two large numbers, which usually results in a loss of accuracy.

It is noted that the results in Figs 9-17 are normalized with respect to the quantity $(\alpha T^2 EI)$, where $\alpha$ is the coefficient of thermal expansion, $E$ the Young's modulus, and $l$ is the crack length.
Fig. 10. Variation of \( J \), with path-radius, for the case of linear temperature distribution (84 Elements).

Fig. 11. Variation of \( J \), with path-radius, for the case of parabolic temperature distribution (48 Elements).

Fig. 12. Variation of \( J \), with path-radius, for the case of parabolic temperature distribution (84 Elements).
Fig. 13. Variation of $J_x$ with path-radius, for the case of combination of linear and parabolic temperature distribution (48 Elements).

Fig. 14. Variation of $J_x$ with path-radius, for the case of combination of linear and parabolic temperature distribution (84 Elements).

Fig. 15. Variation of $J_x$ and $J_y$ with path-radius for the case of linear temperature distribution.
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Figure 15 shows the results for normalized values of $J_e$ and $J'_e$ for the thermal loading case of eq. (49a), with a mesh of 84 elements being used. The solid lines in Fig. 15 show, the results for $J_e$ as computed by using eqs (15) and (29), and those for $J'_e$ as computed by using eqs (13) and (27), respectively. As may be expected for the present case of non-singular temperature field, the values of $J_e$ and $J'_e$ are nearly identical. The broken lines in Fig. 15 indicate the first terms in eqs (13) and (15) [involving a domain-integral with the presence of $S$], and in eqs (27) and (29) [the 'contour' integrals], respectively. Figure 15 clearly shows that when eqs (27) and (29) are used, $J_e$ and $J'_e$ are the result of algebraically adding two large numbers which results in a loss of accuracy. On the other hand when eqs (13) and (15) are used, as seen from Fig. 15, each of the domain-integrals in eqs (13) and (15) is smaller in comparison to the respective integrals in eqs (27) and (29). Thus, algebraically adding the two integrals in the eqs (13) and (15) results in a better accuracy for $J'_e$ and $J_e$, respectively, as compared to the procedures based on eqs (27) and (29). Results for the thermal loading cases of eqs (49b) and (49c), while using a mesh of 84 elements, are shown respectively in Figs 16 and 17, wherein the nomenclature is identical to that in Fig. 15. The results in Fig. 17 can be particularly seen to illustrate the numerical stability and accuracy in computing $J'_e$ or $J_e$, using the equivalent domain-integrals representations of eqs (13) and (15) respectively, as opposed to computing $J'_e$ and $J_e$ based on eqs (27) and (29) respectively.

Tables 1, 2 and 3 provide details of numerical values pertaining to the results presented in Figs (15), (16) and (17), respectively. In those tables the standard deviation (s.d.) of the respective value of $J_e$ and $J'_e$, as evaluated over a number of paths $n$ (in the present case $n = 9$), is defined as:

![Fig. 16. Variation of $J_e$ and $J'_e$ with path-radius for the case of quadratic temperature distribution.](image)

![Fig. 17. Variation of $J_e$ and $J'_e$ with path-radius for the case of a combination of linear and parabolic temperature distributions.](image)
### Table 1

\( T = T(X), \) 84 ELTS

<table>
<thead>
<tr>
<th>( N )</th>
<th>( R/L )</th>
<th>( J )</th>
<th>( J' )</th>
<th>( J'' )</th>
<th>( J')</th>
<th>( J'' )</th>
<th>( J')</th>
<th>( J'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1.340</td>
<td>1.338</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
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<td>1.323</td>
<td>1.319</td>
<td>1.311</td>
<td>1.304</td>
<td>1.342</td>
<td>1.341</td>
<td>1.348</td>
</tr>
<tr>
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<td>1.236</td>
<td>1.218</td>
<td>1.343</td>
<td>1.342</td>
<td>1.344</td>
</tr>
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<td>1.228</td>
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<td>1.341</td>
<td>1.342</td>
<td>1.342</td>
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<td>1.341</td>
<td>1.341</td>
<td>1.341</td>
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<td>1.088</td>
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<td>1.339</td>
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<td>1.033</td>
<td>0.957</td>
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<td>1.334</td>
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<td>1.118</td>
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<td>1.716</td>
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<td>1.316</td>
<td>1.285</td>
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</tbody>
</table>

Standard deviation × 100: 0.629 0.809 2.08 2.765

\( J \) from displacement field: 1.368

### Table 2

\( T = T(X^2), \) 84 ELTS

<table>
<thead>
<tr>
<th>( N )</th>
<th>( R/L )</th>
<th>( J )</th>
<th>( J' )</th>
<th>( J'' )</th>
<th>( J')</th>
<th>( J'' )</th>
<th>( J')</th>
<th>( J'' )</th>
</tr>
</thead>
<tbody>
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<td>2.021</td>
<td>2.034</td>
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<td></td>
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<tr>
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<td>0.071</td>
<td>2.012</td>
<td>2.037</td>
<td>2.025</td>
<td>2.039</td>
<td>2.034</td>
<td>2.047</td>
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<td>1.967</td>
<td>2.012</td>
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<td>2.396</td>
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<td>2.025</td>
<td>2.035</td>
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<tr>
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<td>0.339</td>
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<td>2.036</td>
<td>2.026</td>
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</tr>
<tr>
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<td>1.735</td>
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<td>2.024</td>
<td>2.034</td>
<td>2.023</td>
<td>2.026</td>
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<tr>
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<td>1.656</td>
<td>2.418</td>
<td>2.024</td>
<td>2.031</td>
<td>2.022</td>
<td>2.018</td>
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</tr>
<tr>
<td>8</td>
<td>0.800</td>
<td>1.592</td>
<td>2.987</td>
<td>2.023</td>
<td>2.027</td>
<td>2.021</td>
<td>1.948</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1.273</td>
<td>1.485</td>
<td>5.430</td>
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<td>2.001</td>
<td>2.016</td>
<td>1.910</td>
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</tbody>
</table>

Standard deviation × 100: 0.173 1.186 0.485 4.649

\( J \) from displacement field: 2.047

### Table 3

\( T = T(X^2) + TO/2, \) 84 ELTS, \( J \times 100 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( R/L )</th>
<th>( J )</th>
<th>( J' )</th>
<th>( J'' )</th>
<th>( J')</th>
<th>( J'' )</th>
<th>( J')</th>
<th>( J'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>1.569</td>
<td>1.413</td>
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<td>1.407</td>
<td>0.903</td>
<td>1.263</td>
<td>0.424</td>
<td>1.570</td>
<td>1.415</td>
<td>1.578</td>
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<tr>
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<td>1.136</td>
<td>0.180</td>
<td>0.660</td>
<td>2.264</td>
<td>1.572</td>
<td>1.452</td>
<td>1.569</td>
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<tr>
<td>4</td>
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<td>-7.614</td>
<td>1.569</td>
<td>1.487</td>
<td>1.554</td>
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<tr>
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<td>0.339</td>
<td>0.386</td>
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<td>-0.521</td>
<td>-16.157</td>
<td>1.560</td>
<td>1.508</td>
<td>1.531</td>
</tr>
<tr>
<td>6</td>
<td>0.471</td>
<td>0.206</td>
<td>-10.386</td>
<td>0.012</td>
<td>-27.252</td>
<td>1.547</td>
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</tr>
<tr>
<td>7</td>
<td>0.625</td>
<td>0.475</td>
<td>-16.041</td>
<td>2.280</td>
<td>-37.859</td>
<td>1.533</td>
<td>1.421</td>
<td>1.502</td>
</tr>
<tr>
<td>8</td>
<td>0.800</td>
<td>1.433</td>
<td>-21.266</td>
<td>7.592</td>
<td>-43.916</td>
<td>1.527</td>
<td>1.238</td>
<td>2.455</td>
</tr>
</tbody>
</table>

Standard deviation × 100: 1.715 8.553 32.148 64.234

\( J \) from displacement field: 1.527

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is shown for $J_c$ and $J_i$ as evaluated from eqs (15, 29) and (13, 27), respectively. Once again, these tables indicate the relative stability and accuracy of numerical results for $J_c$ or $J_i$ obtained from the EDI representations of eqs (13) and (15).

Also shown in Tables 1-3 are the values of $J$ as computed from the expression $J = K_i^2/(\text{plane stress})$. In the present computations,

$$K_i = \frac{E u_r}{4} \frac{2\pi}{r}$$

where $u_r$ is the displacement, normal to the crack face, of the quarter-point node of the crack-tip element, and $r$ is the distance of the node from the crack-tip.

4.3 Elastic-plastic plate, with a central crack, and subject to be a history of loading/unloading

This problem, with its geometry and material properties, is identical to that solved in ref. [18]. The material is of an elastic, linear strain-hardening type; with $E$ (Young's Mod.) = $2.06 \times 10^5$ MPa, $\nu$ (Poisson's ratio) = 0.3, $\sigma_y$ (yield stress) = 480 MPa, and $H'$ (strain-hardening modulus) = $E/100$. The centrally cracked elastic-plastic plate is subject to a loading/unloading cycle. The plate is monotonically loaded from a far-field tension of $\sigma = 0$ to $\sigma = 0.7 \sigma_y$, and then unloaded to $\sigma = 0$.

At each load level, the crack-tip parameter $J_c$ is computed alternatively, from either the set of eqs (13, 14) or from the set of equations (27, 28).

Figure 18 shows the values of $J_c$, at the far-field load value of $\sigma = 0$ after a total unloading as computed from using the coarse mesh of 48 elements. When eqs (13, 14) are used, $J_c$ is calculated by using various values of domain area ($A - A_e$), which are signified by the radius of the outer-edge of the area, $(R/L)$. When eqs (27, 28) are used to compute $J_c$, $(R/L)$ indicates the 'Radius' of the far-field path in eqs (27, 28). Figure 18 shows that $J_c$ as computed from eqs (13, 14) displays a better 'path-independency' than that computed from eqs (27, 28). The broken lines in Fig. 18 show the first term in the expression for $J_c$ as in the sets of equations (13, 14) and (27, 28),

$$\text{s.d.} = \left\{ \frac{\sum f_i^2 - n f_m}{n} \right\}^{1/2}$$

$$f_m = \frac{1}{n} \sum_{i=1}^{n} f_i$$
Elastic - Plastic Problem, $J_e^{(KN/M)}$ after unloading ($\sigma = 0$)

84 ELTS

---

**Fig. 19.** Variation of $J_e$ with path-radius, after a total unloading in an elastic-plastic problem (84 Elements).

---

Elastic - Plastic Problem

---

**Fig. 20.** Variation of $J_e$ with applied load, during a loading/unloading cycle in an elastic-plastic problem.

---

respectively. It is seen that the first term in the expression for $J_e$ as in eqs (27, 28), is the usual definition for the "$J$ contour integral" as is widely used, and attributed to Rice[3]. That the usual $J$ contour integral of Rice, totally loses its 'path-independent' property after unloading is evident from Fig. 18.

Figure 19 shows the results that are of analogous nature to those in Fig. 18, except the ones in Fig. 19 are for the 'fine mesh' with 84 elements. Figure 19 shows that the mesh refinement improves the 'path-independent' character of $J_e$ as computed from eqs (13, 14) or (27, 28). However, the mesh refinement has no bearing on the wide path-dependence of the $J$ contour integral alone, i.e. the first term of eqs (27, 28).

Figure 20 shows the variation of the path-independent crack-tip parameter $J_e$ [as averaged over the values for various paths, as illustrated, for instance, for $\sigma = 0$ in Fig. 19] as evaluated from either eqs (13, 14) or (27, 28), for various values of $\sigma$, in the loading/unloading history. Also shown in Fig. 20 are the comparison results of Aoki et al.[21], and Yagawa et al.[18] using both a 'virtual crack-extension' type of approach, as well as the contour integral plus domain integral type of representation as in the present eqs (27, 28), for $J$. It is found that in all loading cases,
the present EDI representation of eqs (13, 14), along with the numerical algorithms presented in Section 3 of this paper, lead to the most efficient and numerically stable computation of the path-independent crack-tip integral parameter $J_e$.

**Closure**

In this work, equivalent domain integral EDI representations for crack-tip parameters such as $J_L$, $J_e$ are given. Specific computational algorithms, in the context of the use of 8-noded isoparametric finite elements to solve plane problems of elastic–plastic thermo-mechanical loading, are given, to evaluate the domain integrals in the EDI representations eqs (13, 14) and (15, 16) for $J_e$ and $J_{ct}$ respectively. These algorithms have been shown to be simple and efficient in some examples of thermoelastic crack problems, as well as elastic–plastic cracked solids that are subject to loading/unloading. In each case, it has been found that the present EDI representations yield the most accurate, stable, and path-independent numerical values for the crack-tip parameters $J_L$ and $J_e$. Thus, the present EDI representations and attendant numerical algorithms may be recommended for use in general purpose computer programs to evaluate the crack-tip integral parameters such as $J_L$ or $J_e$.

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**REFERENCES**


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