A BOUNDARY/INTERIOR ELEMENT METHOD FOR QUASI-STATIC AND TRANSIENT RESPONSE ANALYSES OF SHALLOW SHELLS

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Abstract—Integral equations are derived for the representation of in-plane as well as transverse displacements of shallow shells undergoing small, quasi-static or dynamic deformations. A combined boundary/interior element method for static stress, free-vibration, and transient response analyses of shallow shells, based on these integral equations, is derived. Numerical results are presented to demonstrate not only the computational economy but also the superior accuracy realizable from the present approach, in contrast to the popular Galerkin finite element approach.

1. INTRODUCTION

For boundary-value/initial-value problems in (linear or nonlinear) solid mechanics, it is often possible to derive certain integral representations for displacements [1–3]. A key ingredient, which makes such derivations possible, is the singular-solution, in an infinite space, of the corresponding differential equation (in the fully linear case) or of the highest-order differential operator (in the nonlinear case, or even in the linear case when the full linear equation cannot be conveniently solved) for a "unit" load applied at a generic point in the infinite space. When the problem is linear, and the singular solution can be established for the complete linear differential equation of the problem, the forementioned integral representation for displacements involves only boundary integrals of the unknown trial functions and their appropriate derivatives. Such an integral representation, when discretized, leads to the so-called boundary-element method [1, 2]. Such boundary-element methods are possible, for example, in linear, isotropic, elastostatics (see, for instance, ref. [1]) and in problems of static-bending of linear elastic isotropic plates [4, 5]. On the other hand:

1. even for linear problems wherein the singular solution cannot be established for the entire linear differential equations,

2. for anisotropic materials, and

3. for problems of large deformation and material inelasticity, the integral representation, if any, for displacements would involve not only boundary integrals but also interior integrals (i.e. integrals over the domain) of the trial functions and/or their derivatives [3]. A discretization of such integral equations would lead not to a simple boundary-element method, but to a sort of hybrid boundary/interior element method [3].

The literature on the static or dynamic analysis of shells by the boundary element method is rather sparse. The primary objectives of this paper are:

1. to explore the integral equation formulations for static and dynamic analysis of shallow shells;

2. to consider the discretization of these integral equations; and

3. to explore the advantages of the present approach as compared to the usual Galerkin finite element approaches wherein higher-order continuities of trial solution for displacements, such as \( C^1 \) continuity of transverse displacement, has long plagued the successful development of shell finite elements.

As is well known, due to the curvature of the shell, the in-plane displacements and the transverse displacement in a shell are inherently coupled in the kinematics of deformation as well as in the momentum balance relations for the shell. In the present integral equation formulation, the test functions are chosen to be the singular solutions, in infinite space, of parts of the relevant momentum balance equations. However, when the dynamic motion of the shell is considered and since the in-plane displacements \( u \) are coupled to the transverse displacement \( w \) in the shell equilibrium equations, the presently considered test functions represent singular solutions to only the highest-order differential operator occurring in the elastostatic equilibrium equations of the shell. Due to this reason, the integral representations for the shell displacements involve not only boundary integrals, but also domain integrals involving the trial solutions for displacements. Thus, when discretized, the present integral equation approach leads to a hybrid boundary/domain element method. However, unlike in the Galerkin finite element method, in the present approach, the trial functions for transverse displacement need only be piecewise constant or
utmost $C^\infty$ continuous over each interior element. In this paper, several examples of quasi-static, free vibration, and transient dynamic response analysis of shallow shells, based on the developed integral equation approach, are presented. The present method for dynamic analysis of shells has some similarities to those developed in refs [5] and [6] for dynamic analysis of plates and plane, linear elastic, bodies.

The remainder of the paper is organized as follows: Section 2 deals with the statement of the boundary-value/initial-value problem; Section 3 deals with the formulation of integral equations for shell displacements and a solution strategy; Section 4 with an algebraic formulation; Section 5 with numerical examples; and Section 6 with some concluding comments.

2. THE BOUNDARY-VALUE/INITIAL-VALUE PROBLEM

Consider a shallow shell of an isotropic elastic material with the mid-surface being described by $z = z(x_1, x_2)$, $\alpha = 1, 2$ (see Fig. 1). The base-plane of the shell is defined by a domain $\Omega$ in the $x_1 x_2$ plane, and $\Omega$ is bounded by a smooth curve $\Gamma$. Using Reissner's linear theory of shallow shells [7], the pertinent equilibrium equations may be written as

$$N_{\alpha \beta, \beta} + b_\alpha = 0 \quad (\alpha, \beta = 1, 2) \quad (2.1a)$$

$$D \nabla^4 w + R_{\alpha \alpha, \alpha} - b_\alpha = f_\alpha, \quad (2.1b)$$

where $N_{\alpha \beta}$ are membrane forces; $(\cdot)_\beta = \partial(\cdot)/\partial x_\beta$; $w$ is the transverse deflection of the mid-surface of the shell; $b_\alpha (\alpha = 1, 2, 3)$ are body forces; $f_\alpha$ is the load normal to the shell mid-surface; $D = E\theta/12(1 - \nu^2)$; $\theta$ is the thickness; $E$ and $\nu$ are the elastic constants; $\nabla^4$ is the biharmonic operator in the variables $x_\alpha$; and

$$R_{\alpha \alpha} = -1/z_{\alpha \alpha} \quad (2.1c)$$

are the radii of curvature of the undeformed shell. Along $\Gamma$, the boundary conditions are

$$u_\alpha = \bar{u}_\alpha \quad \text{at} \quad \Gamma_\alpha; \quad N_{\alpha \beta} n_\beta = \bar{F}_\alpha \quad \text{at} \quad \Gamma_\alpha; \quad \Gamma = \Gamma_1 \cup \Gamma_2, \quad (2.2a, b)$$

where $n_\alpha$ are the direction cosines of the unit outward normal to $\Gamma$ in the base plane. The out-of-plane boundary conditions are

$$w = \bar{w} \quad \text{or} \quad V_\alpha = \bar{F}_\alpha \quad (2.3a)$$

$$\Psi_\alpha = \bar{\Psi}_\alpha \quad \text{or} \quad M_\alpha = \bar{M}_\alpha, \quad (2.3b)$$

where

$$V_\alpha = -D \frac{\partial}{\partial n} (\nabla^2 w) + \frac{\partial}{\partial t} M_\alpha$$

is the reduced Kirchhoff shear force;

$$\Psi_\alpha = \frac{\partial w}{\partial n}$$

is the rotation around the tangent to $\Gamma$;

$$M_\alpha = -D \left\{ (n_1^2 + n_2^2) \frac{\partial^2 w}{\partial x_1^2} + 2(1 - \nu) n_1 n_2 \times \frac{\partial^2 w}{\partial x_1 \partial x_2} + (n_1^2 + n_2^2) \frac{\partial^2 w}{\partial x_2^2} \right\}$$

and $n$ and $t$ are directions normal and tangential, respectively, to $\Gamma$ in the base plane.

It is well known that the equilibrium eqns (2.1) can be written more concisely in terms of a stress function (for $N_{\alpha \beta}$) and the transverse displacement $w$. However, we leave the equations in the form (2.1), which is somewhat more general, so as to treat in-plane inertia forces and to extend the development to the case of general non-shallow shells, and nonlinear kinematics, in forthcoming papers.

The in-plane strain–displacement relations are

$$\varepsilon_\alpha = \frac{1}{2} \left[ u_{\alpha, \alpha} + u_{\beta, \alpha} + \frac{2w}{R_{\alpha \alpha}} \right]$$

where $u_\alpha$ are the in-plane displacements at the shell mid-surface. The in-plane stress-resultant/strain relations are

$$N_{11} = C(\varepsilon_{11} + \nu \varepsilon_{22}); \quad N_{22} = C(\varepsilon_{22} + \nu \varepsilon_{11}); \quad N_{12} = C(1 - \nu) \varepsilon_{12}, \quad (2.4b)$$

where $C = E\theta/(1 - \nu^2)$. The moment–curvature relations are

$$M_{11} = -D(w_{11} + \nu w_{22}); \quad M_{22} = -D(w_{22} + \nu w_{11}); \quad M_{12} = -D(1 - \nu) w_{12}. \quad (2.5)$$
Finally, the initial conditions on the shell may be written as

\[ u_i(x_p, 0) = u_{i0}(x_p) \quad \text{at} \quad t = 0 \]

\[ \dot{u}_i(x_p, 0) = \dot{u}_{i0}(x_p) \quad \text{at} \quad t = 0 \]

\[ w(x_p, 0) = w_0(x_p) \]

\[ \dot{w}(x_p, 0) = \dot{w}_0(x_p) \quad \text{at} \quad t = 0, \]  

where \( \mathbf{P} \equiv d(\mathbf{P})/dt \).

### 3. INTEGRAL EQUATIONS FOR SHELL DISPLACEMENTS AND A BOUNDARY ELEMENT SOLUTION STRATEGY

In an approximate analysis of the boundary/initial-value problem described in Section 2, let \( u_i \) and \( w \) be the assumed trial solutions. We shall consider a general weighted-residual formulation, and let \( u^*_i \) and \( w^* \) be the corresponding test functions. In the familiar Galerkin finite-element method, the trial functions \( (u_i \) and \( w) \) and the test functions \( (u^*_i \) and \( w^* \) belong to the same category of function spaces. In the present formulation, however, as will be seen, the test functions \( (u^*_i \) and \( w^* \) belong to an entirely different class of function space from that of the trial functions. With this in mind the combined weak forms of the equilibrium equations and boundary conditions for the in-plane \([\text{eqns (2.1a) and (2.2a,b)}]\) and out-of-plane \([\text{eqns (2.1b) and (2.3a,b)}]\) deformations, respectively, may be written (see, for instance, ref. \[8\]) as

\[
\int_\Omega (N_{i,j,k} + b_3 u^*_j) d\Omega = \int_{\Gamma_2} (P_s - P_s) u^*_j d\Gamma \\
+ \int_{\Gamma_m} (\bar{u}_i - u_s) P^*_m (u^*_j) d\Gamma.
\]  

(3.1)

and

\[
\int_\Omega (D \nabla^2 w - R_{i,j,k}) u^* d\Omega = \int_{\Gamma_2} (\bar{v}_i - V_s) w^* d\Gamma \\
+ \int_{\Gamma_m} (\bar{M}_i - M_s) \Psi^* d\Gamma \\
+ \int_{\Gamma_m} (\Psi_s - \Psi^*) M^* d\Gamma \\
+ \int_{\Gamma_m} (w - \bar{w}) V^* d\Gamma.
\]  

(3.2)

To make a specific choice for the test functions that results in convenient integral representations for the shell-displacements \( u_i \) and \( w \), we rewrite the in-plane equilibrium equations in a slightly different form, as follows. From the relations between \( (N_{i,j}) \) and \( (u_i, w) \) as given in eqns \( (2.4a,b) \), we may write

\[ N_{i,j} = N^*_{i,j} + C \kappa_{i,j} w, \]  

where

\[ N^*_i = C(u_{i1} + w_{i2}); \]

\[ N^*_{i2} = C(u_{i2} + w_{i1}); \]

\[ N^*_{i3} = \frac{1}{2} C(1 - \nu) (u_{i1} + u_{i2}) \]

or

\[ N^*_{i,j} = C_p \kappa_{i,j} u_{i,j}. \]  

(3.4)

and

\[ \kappa_{11} = \frac{1}{R_{11}} + \frac{v}{R_{22}}; \]

\[ \kappa_{22} = \frac{1}{R_{22}} + \frac{v}{R_{11}}; \]

\[ \kappa_{12} = \frac{1 - v}{R_{12}}. \]  

(3.5)

Use of (3.3) in (3.1) results in

\[ \int_\Omega [N^*_{i,j} + C(\kappa_{i,j} w)] u^*_j d\Omega \\
= \int_{\Gamma_2} (P_s - P_s) u^*_j d\Gamma \\
+ \int_{\Gamma_m} (\bar{u}_i - u_s) P^*_m (u^*_j) d\Gamma. \]  

(3.6)

Use of the Divergence theorem in eqn (3.6) results in

\[ \int_\Omega N^*_{i,j} u^*_j d\Omega - \int_\Omega N^*_{i,j} u_{i,j} d\Omega \\
+ \int_\Omega C(\kappa_{i,j} w) u^*_j d\Omega + \int_\Omega b_i u^*_j d\Omega \\
= \int_{\Gamma_2} (P_s - P_s) u^*_j d\Gamma \\
+ \int_{\Gamma_m} (\bar{u}_i - u_s) P^*_m (u^*_j) d\Gamma. \]  

(3.7)

Since the material is linear elastic and isotropic, we have

\[ N^*_{i,j} u^*_j = C_p \kappa_{i,j} u_{i,j} \]  

(3.8)
where the definitions of \( N_{\text{,}} \) are apparent. We now introduce the additional notations:

\[
P_{\text{,}} = N_{\text{,}} n_{\text{,}}; \quad P_{\text{,}} = N_{\text{,}} n_{\text{,}}
\]

or

\[
P_{\text{,}} = P_{\text{,}} + C_{\text{,}} u_{\text{,}} n_{\text{,}}.
\]

Using (3.8, 3.9a, 3.9b) in (3.7) and applying the Divergence theorem, it is easy to obtain:

\[
\int_{0}^{\partial} [N_{\text{,}} u_{\text{,}}]_{\partial} u_{\text{,}} d\Omega + \int_{0}^{\partial} h_{\text{,}} u_{\text{,}} d\Omega
\]

\[
+ \int_{f} \hat{P}_{\text{,}} u_{\text{,}} d\Gamma - \int_{f} P_{\text{,}} \hat{u}_{\text{,}} d\Gamma
\]

\[
- \int_{\Omega} C_{\text{,}} w_{\text{,}} u_{\text{,}} d\Omega = 0, \quad (3.10a)
\]

where

\[
\hat{P}_{\text{,}} = \hat{P}_{\text{,}} \text{ at } \Gamma_{\text{,}}; \quad \text{and} \quad \hat{P}_{\text{,}} = P_{\text{,}} \text{ at } \Gamma_{\text{,}} \quad (3.10b)
\]

and

\[
\hat{u}_{\text{,}} = \hat{u}_{\text{,}} \text{ at } \Gamma_{\text{,}}; \quad \text{and} \quad \hat{u}_{\text{,}} = u_{\text{,}} \text{ at } \Gamma_{\text{,}} \quad (3.10c)
\]

Now, we choose \( u_{\text{,}} \) to be the "singular solution" of the equation

\[
[N_{\text{,}} u_{\text{,}}]_{\partial} + \delta(x_{\text{,}} - \xi) \delta_{\text{,}} e_{\text{,}} = 0, \quad (3.11)
\]

where \( \delta(x_{\text{,}} - \xi) \) is the Dirac delta function at \( x_{\text{,}} = \xi \); \( \delta_{\text{,}} \) is the Kronecker delta; and \( e_{\text{,}} \) denotes that the direction of the application of the point load is along the \( x_{\text{,}} \) direction. The "singular solution" of (3.11) will be denoted as \( u_{\text{,}} \); where \( u_{\text{,}} \) is the displacement along the \( x_{\text{,}} \) direction in a plane infinite body at any point \( x_{\text{,}} \), due to a unit load along the \( x_{\text{,}} \) direction, applied at the location \( x_{\text{,}} = \xi_{\text{,}} \). Likewise, \( P_{\text{,}} \) will be considered to be the traction along the \( x_{\text{,}} \) direction on an oriented surface at \( x_{\text{,}} \), with a unit normal \( n_{\text{,}} \), due to a unit load along \( x_{\text{,}} \) at the location \( x_{\text{,}} \). These solutions are well known [9] and may be written as

\[
u_{\text{,}} = \frac{1}{8\pi G} \left[ (v - 3) + \frac{\partial p}{\partial x_{\text{,}}} - \frac{\partial p}{\partial x_{\text{,}}} \right] \quad (3.12a)
\]

and

\[
P_{\text{,}} = \frac{1}{4\pi \rho p} \left( 1 - v \right) \frac{\partial p}{\partial n_{\text{,}}} - \frac{\partial p}{\partial n_{\text{,}}} \quad (3.12b)
\]

where \( \rho = |x_{\text{,}} - \xi_{\text{,}}| \) is the radius vector from \( x_{\text{,}} \) to \( \xi_{\text{,}} \) and \( G = E/2(1 + v) \).

Due to the property of integrals involving Dirac functions, we have

\[
\int_{0}^{\partial} \delta(x_{\text{,}} - \xi) \delta_{\text{,}} u_{\text{,}} d\Omega = - u_{\text{,}}(\xi_{\text{,}}). \quad (3.13)
\]

Using eqns (3.12) and (3.13) in (3.10a), we have

\[
u_{\text{,}}(\xi) = \int_{0}^{\partial} h_{\text{,}} u_{\text{,}} d\Omega + \int_{f} \hat{P}_{\text{,}} u_{\text{,}} d\Gamma
\]

\[
- \int_{f} \hat{u}_{\text{,}} P_{\text{,}} d\Gamma
\]

\[
- \int_{\Omega} C_{\text{,}} w_{\text{,}} u_{\text{,}} d\Omega = 0. \quad (3.14)
\]

It can be shown that while the coefficient \( \gamma \) in the left-hand side of (3.14) is the unity when \( \xi_{\text{,}} \) is in the interior of \( \Omega \), the value of \( \gamma \) is \( (0.5) \) when \( \xi_{\text{,}} \) falls on the "smooth" boundary \( \Gamma \) [1]. Equation (3.14) is the sought-after integral equation for \( u_{\text{,}} \) in a shallow shell.

We now choose the test function \( w_{\text{,}}(x_{\text{,}}) \) to be the "singular solution" in an infinite plate corresponding to a unit point load at the location \( \xi_{\text{,}} \). Thus, \( w_{\text{,}} \) corresponds to the solution of the linear equation

\[
D \nabla^{4} w_{\text{,}} = \delta(x_{\text{,}} - \xi) \quad (3.15)
\]

in an infinite domain in the base-plane of the shallow shell. It is well known [9] that the solution for \( w_{\text{,}} \) is given by

\[
w_{\text{,}}(x_{\text{,}}, \xi) = \frac{1}{8\pi} \rho^{3} \ln \rho \quad (3.16)
\]

where \( \rho = |x_{\text{,}} - \xi_{\text{,}}| \).

Using eqns (3.16) and (3.3) in eqn (3.2) and employing repeated integrations by part in the resulting equation, one easily obtains the integral equation

\[
\gamma D w_{\text{,}}(\xi) = \int_{f} \hat{P}_{\text{,}} w_{\text{,}}(x_{\text{,}}, \xi) d\Gamma
\]

\[
- \int_{f} \hat{u}_{\text{,}} M_{\text{,}} w_{\text{,}}(x_{\text{,}}, \xi) d\Gamma
\]

\[
+ \int_{f} \hat{u}_{\text{,}} M_{\text{,}} w_{\text{,}}(x_{\text{,}}, \xi) d\Gamma
\]

\[- \int_{f} \hat{u}_{\text{,}} V_{\text{,}} w_{\text{,}}(x_{\text{,}}, \xi) d\Gamma
\]
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\[ - \int_\Omega \left[ \frac{N_{xx}^d}{R_{sp}} + C \frac{N_{zz}^d}{R_{sp}} w - b_1 - f_2 \right] \times (x, \nu) w^*(x, \xi, \eta) d\Omega \]

\[ + \sum_i \left\{ \left[ M_i \right] w^* - \left[ ^*M_i \right] w \right\}, \quad (3.17) \]

where

\[ V_n = - D \frac{\partial}{\partial n} (\nabla^2 w) + \frac{\partial}{\partial \nu} M_{ij} \]

\[ M_{ij} = M_{i1} n_1^2 + 2 M_{i2} n_1 n_2 + M_{i3} n_2^2 \]

\[ M_i = (M_{i2} - M_{i1}) n_1 n_2 + M_{i3} (n_1^2 - n_2^2) \]

\[ \gamma = 1 \text{ for } \xi \in \gamma ; \]

\[ \gamma = \frac{1}{2} \text{ for } \xi \in \Gamma_{(\text{smooth})} \]

In eqn (3.17) the terms with the superposed symbol \( ^* \) should be taken to imply the respective prescribed values, if any, at \( \Gamma \); otherwise, they are to be treated as the unknown solution variables. Also, the symbol \( \left\{ \right\} \) denotes the jump in the quantity \( \left( \right) \) at a corner at \( \Gamma \), in the direction of the increasing arc length along \( \Gamma \); and the summation \((1 \text{ to } k)\) extends to all the \( k \) such corners.

Using eqn (3.4) and the Divergence theorem, it is easy to see that

\[ \int_\Omega \frac{N_{xx}^d}{R_{sp}} (x, \nu) w^*(x, \xi, \eta) d\Omega = \int_\Gamma C \kappa_{sp} n_2 \psi^*(x, \xi, \eta) d\Gamma \]

\[ - \int_\Omega \left( C \kappa_{sp} w^*(x, \xi, \eta) \right)_\nu \psi(x, \xi, \eta) d\Omega. \quad (3.18) \]

Use of (3.18) in (3.17) results in the final integral equation for \( w \) as follows:

\[ \gamma D \frac{\partial w}{\partial \eta}(\xi, \eta) = \int_\Gamma \hat{V}_n (x, \nu) w^*(x, \xi, \eta) d\Gamma \]

\[ - \int_\Gamma \hat{M}_n (x, \nu) \psi^*(x, \xi, \eta) d\Gamma \]

\[ + \int_\Gamma \hat{\psi}_n (x, \nu) M_{n1}^*(x, \xi, \eta) d\Gamma \]

\[ - \int_\Omega w (x, \nu) V^*_n (x, \xi, \eta) d\Omega \]

\[ - \int_\Gamma C \kappa_{sp} n_2 \hat{u}_n (x, \nu) w^*(x, \xi, \eta) d\Gamma \]

\[ + \int_\Omega \left( C \kappa_{sp} w^*(x, \xi, \eta) \right)_\nu \hat{u}_n (x, \nu) d\Omega \]

\[ - \int_\Gamma C \kappa_{sp} \hat{u}_n (x, \nu) \psi^*(x, \xi, \eta) d\Gamma \]

\[ + \int_\Omega \left( C \kappa_{sp} w^*(x, \xi, \eta) \right)_\nu \hat{u}_n (x, \nu) d\Omega \]

\[ - \int_\Gamma C \kappa_{sp} w^*(x, \xi, \eta) \psi^*(x, \xi, \eta) d\Gamma \]

\[ + \sum_i \left\{ \left[ M_i \right] w^* - \left[ ^*M_i \right] w \right\}. \quad (3.19) \]

Since \( \partial w / \partial \eta \) is also an independent variable at \( \Gamma \) in the present boundary-value problem of the shallow shell, an integral relation for \( \partial w / \partial \eta \), likewise, be derived as

\[ \gamma D \frac{\partial w}{\partial \eta}(\xi, \eta) = \int_\Gamma \hat{V}_n (x, \nu) \frac{\partial w^*}{\partial \eta}(x, \xi, \eta) d\Gamma \]

\[ - \int_\Gamma \hat{M}_n (x, \nu) \frac{\partial \psi^*}{\partial \eta}(x, \xi, \eta) d\Gamma \]

\[ + \int_\Gamma \hat{\psi}_n (x, \nu) \frac{\partial M_{n1}^*}{\partial \eta}(x, \xi, \eta) d\Gamma \]

\[ - \int_\Gamma \hat{\omega}_n (x, \nu) \frac{\partial \omega^*}{\partial \eta}(x, \xi, \eta) d\Gamma \]

\[ - \int_\Gamma C \kappa_{sp} n_2 \hat{u}_n (x, \nu) \frac{\partial w^*}{\partial \eta}(x, \xi, \eta) d\Gamma \]

\[ + \int_\Omega \left( C \kappa_{sp} w^*(x, \xi, \eta) \right)_\nu \hat{u}_n (x, \nu) d\Omega \]

\[ - \int_\Gamma C \kappa_{sp} \hat{u}_n (x, \nu) \frac{\partial \psi^*}{\partial \eta}(x, \xi, \eta) d\Gamma \]

\[ + \int_\Omega \left( C \kappa_{sp} w^*(x, \xi, \eta) \right)_\nu \hat{u}_n (x, \nu) d\Omega \]

\[ + \sum_i \left\{ \left[ M_i \right] \frac{\partial w^*}{\partial \eta} - \left[ ^*M_i \right] w \right\}. \quad (3.20) \]

Remarks

In summary, eqns (3.14), (3.19) and (3.20) represent the complete set of integral equations for \( u_\nu, w, \) and \( \partial w / \partial \eta \). An examination of eqns (3.14), (3.19), and (3.20) reveals the following features:

(1) For given body forces \( b_\nu \), the integral relation for \( u_\nu \) [eqn (3.14)] involves the trial functions \( u_\nu \) only at the boundary \( \Gamma \). On the other hand, due to the curvature induced coupling of the trial functions \( (u_\nu, w) \) in the shallow-shell problem, the integral relation for \( w \) contains a domain-integral (over \( \Omega \)) involving the trial function for \( w \).
If the body forces $b_i$ include in-plane inertia forces ($\rho u_i$), then the integral relation for $u_i$ involves a domain integral (over $\Omega$) of $\bar{u}_i$ as well.

(3) Again, due to the curvature-induced coupling of the in-plane and out-of-plane displacements in the presently considered shallow shell, the integral equations for $w$ and $\partial w/\partial n$, eqns (3.19) and (3.20) respectively, contain domain-integrals (over $\Omega$) involving trial functions for both $w$ and $u_i$. Also, in a transient dynamic analysis, the term $\ddot{w}$ appears inside a domain-integral.

(4) For reasons (1) to (3) above, unlike the classical homogeneous isotropic elasto-statics [1] wherein a discretization of the relevant integral equations requires a use of basis functions for the displacements at the boundary alone, the present shallow-shell formulation requires the assumption of basis functions for the trial solutions $u_i$ and $w$, at the boundary $\Gamma$ as well as in the interior $\Omega$. Thus, the present solution methodology may, strictly speaking, be classified as hybrid boundary-element/interior (finite) element method based on a direct discretization of integral equations.

(5) Unlike in the homogeneous isotropic elasto-statics [1], the present integral equations are no longer boundary-integral equations alone.

(6) Suppose now that in eqns (3.14), (3.19), and (3.20) we let $\zeta_i$ tend to a point on the boundary, i.e. $\zeta_i \in \Gamma$. Thus, we obtain three integral relations for the boundary values of $u_i$, $w$, and $\partial w/\partial n$. An examination of these relations reveals that, in order to discretize these equations, one needs to assume only very simple trial functions $u_i$, $w$, $\partial w/\partial n$ not only at the boundary but also in the interior of $\Omega$. For instance, $\Omega$ may be discretized into a number of finite elements and $\Gamma$ into a number of boundary elements. As $\zeta_i$ tends to $\Gamma$, the integral relations (3.14), (3.19), and (3.20) clearly show that $w$ and $u_i$ need only be piecewise constant functions in each finite element in $\Omega$, in any discretization process. Finally, one may need only consider $C^0$ continuous functions for $w$ and $u_i$ in each element (to extend the formulations, later, to finite deformation cases).

(7) In contrast, it is recalled that in the Galerkin finite element method, $u_i$ need be $C^0$ continuous and $w$ be $C^1$ continuous in each element. The difficulties with such a finite element approach are too well documented in literature to warrant further comment here.

(8) At each point on the boundary, two of the in-plane variables $u_i (i = 1, 2)$, $P_r (i = 1, 2)$ are specified; and the other two are unknown. Likewise, two of the out-of-plane variables, $V_o$, $M_z$, $\psi_n$, and $w$ are specified; and the other two are unknown. At each point in $\Omega$, as seen from eqns (3.14), (3.19), and (3.20), the three displacements, $u_i$ and $w$, are unknown. Thus, if eqns (3.14), (3.19), and (3.20) are discretized, through finite as well as boundary elements and the nodal values of the variables are appropriately understood to be either specified or unknown, one would easily obtain exactly as many equations as the number of unknowns, so that the problem may be considered as well posed.

4. ALGEBRAIC FORMULATION

We discretize the boundary into $M$ segments $\Gamma_m$, $m = 1, \ldots, M$, such that the corner points, if any, on $\Gamma$ coincide with the nodes of $\Gamma_m$. The domain $\Omega$ is discretized into $L$ finite elements $\Omega_l$, $l = 1, \ldots, L$. Altogether, one has $N$ nodes, of which $N_\Gamma$ are on $\Gamma$ and $N_\Omega$ inside $\Omega$. Of course, the discretization is such that each $\Gamma_m$ is a boundary segment of some $\Omega_l$. At each nodal point on the boundary, $u_i$, $w$, $\partial w/\partial n$, $P_r$, $V_o$, and $M_z$ are treated as nodal variables; and at each point in the interior of $\Omega$, only $u_i$ and $w$ are treated as nodal variables. While a $C^0$ continuity is not necessary in the present formulation for $u_i$ and $w$ inside $\Omega$, such a continuity is used in the present, for algebraic convenience (as well as for reasons of extending the present formulation to a finite deformation case).

Let the numerals 1, 2, 3, 4, 5 denote respectively: the vectors $u_i$ (values of in-plane displacements at nodes on $\Gamma$); $u_i$ (values of in-plane displacements at nodes in $\Omega$), $w_1$ (values of out-of-plane displacements $w$ as well as rotations $\partial w/\partial n$ at nodes on $\Gamma$); $w_0$ (values of $w$ at nodes in $\Omega$); and $P_r$ (values of in-plane tractions $N_{\alpha} N_{\beta}$, as well as out-of-plane generalized forces, viz. the reduced Kirchhoff shear and bending moments, at nodes on $\Gamma$).

Applying the above discussed interpolations, for the various variables in $\Omega$ and at $\Gamma$, in eqs (3.14), (3.19), and (3.20), it is easy to generate the algebraic equations:

\begin{equation}
G_{ii} u_i + G_{i1} w_1 + G_{i4} w_0 + G_{i5} P_r = F_i \quad (4.1)
\end{equation}

\begin{equation}
G_{1i} u_i - 1 u_0 + G_{2i} w_1 + G_{24} w_0 + G_{25} P_r = F_2 \quad (4.2)
\end{equation}

\begin{equation}
G_{3i} u_i + G_{31} u_0 + G_{32} w_1 + G_{34} w_0 + G_{35} P_r = F_3 \quad (4.3)
\end{equation}

\begin{equation}
G_{4i} u_i + G_{41} u_0 + G_{42} w_1 + G_{44} w_0 + G_{45} P_r = F_4 \quad (4.4)
\end{equation}

The notation in eqns (4.1)-(4.4) is as follows:

(1) the numerals 1, 2, 3, 4, 5 should be identified as explained before;

(2) the matrices are defined such that, for instance, $G_{ii}$ implies that it has as many rows as the dimension of the (2) vector, i.e. $u_i$, and has as many columns as the (1) vector, i.e. $u_i$;

(3) likewise, for instance, $F_i$ is a vector whose dimension is the same as the (3) vector, i.e. $w_i$.

Equation (4.2) can be trivially solved for the (2) vector, i.e. $u_i$, as

\begin{equation}
\mathbf{u}_0 = G_{21} u_1 + G_{22} w_1 + G_{24} w_0 + G_{25} P_r - F_2. \quad (4.5)
\end{equation}

Use of (4.5) in (4.1), (4.3), and (4.4) results in equations in the only unknown variables, $u_i$, $w_1$, $w_0$, and $P_r$, thereby reducing the dimensionality of the algebraic equations yet to be solved.
For dynamic response problems, one may consider the body forces \( \delta_r \) and \( \delta_t \) to include the inertia forces \( -\rho \ddot{u} \) and \( -\rho \ddot{w} \) respectively. Thus, in eqns (4.1)–(4.4), for dynamic response analyses, one may redefine \( F_1 \ldots F_4 \) as

\[
F_1 = F_1^* - M_{11} \ddot{u} \quad F_2^* - M_{12} \ddot{u} \\
F_3 = F_3^* - M_{13} \ddot{w} \quad F_4^* - M_{14} \ddot{w}_n.
\]

(4.6a-d)

It should be noted that, in the present approach, the mass matrices, say for instance \( M_{12} \), are obtained from integrals of the type

\[
\int_0^1 \rho (x_s) w_t^n(x_s, y_s) \, dx_s.
\]

(4.6c)

Equations (4.6a)–(4.6d) are now used in eqns (4.1)–(4.4), and the resulting equations are easily rearranged to read as

\[
\begin{bmatrix}
M_{11} & G_{11} & G_{13} \\
M_{12} & G_{12} & G_{14} \\
M_{13} & G_{13} & G_{15}
\end{bmatrix}
\begin{bmatrix}
\ddot{u} \\
\ddot{w} \\
\ddot{w}_n
\end{bmatrix}
+
\begin{bmatrix}
G_{11} & G_{31} & G_{33} \\
G_{12} & G_{32} & G_{34} \\
G_{13} & G_{33} & G_{35}
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
v_n
\end{bmatrix}
=
\begin{bmatrix}
F_1^* \\
F_2^* \\
F_3^* \\
F_4^*
\end{bmatrix}
\]

(4.7)

Recall that the number (1) denotes \( u_r \) and (3) denotes \( w_r \) (including boundary rotations) and (5) denotes \( P_r \) (boundary nodal tractions). It is clear that, at the boundary nodes, the number of nodal generalized displacements is exactly the same as that of the generalized nodal tractions. Thus, the matrix

\[
\begin{bmatrix}
G_{15} \\
G_{35}
\end{bmatrix}
\]

(4.9)

is a square matrix. From (4.7), one may thus eliminate the unknown nodal generalized forces and write an equation of the type

\[
P_r = \begin{bmatrix}
G_{11} & G_{13} & G_{15} \\
G_{12} & G_{14} & G_{15} \\
G_{13} & G_{15} & G_{15}
\end{bmatrix}^{-1}
\begin{bmatrix}
F_1^* \\
F_2^* \\
F_3^*
\end{bmatrix}
-
\begin{bmatrix}
G_{11} & G_{31} & G_{33} \\
G_{12} & G_{32} & G_{34} \\
G_{13} & G_{33} & G_{35}
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
v_n
\end{bmatrix}
-
\begin{bmatrix}
0 & G_{41} & G_{43} \\
0 & G_{42} & G_{44} \\
0 & G_{43} & G_{45}
\end{bmatrix}
\begin{bmatrix}
u \\
w \\
v_n
\end{bmatrix}.
\]

(4.10)

When eqn (4.10) is used in (4.8), the resulting equations may be rearranged in the standard form, as

\[
Gq + Mq = F.
\]

(4.11)

where now, \( q \) stands for the vector of generalized nodal displacements including both the in-plane and transverse type. The transient response, or modal analysis, based on (4.11) is standard, and further details are omitted. It should be pointed out that a modal analysis of free vibrations of flat plates, similar to the one detailed above, has been presented in ref. [10].

5. NUMERICAL EXAMPLES

Here we present numerical results for the problem of a shallow spherical cap as shown in Fig. 2. for the present case,

\[
\frac{1}{R_{11}} = \frac{1}{R_{22}} = \frac{1}{R} = \frac{1}{R_{12}} = 0;
\]

(5.1)

In the present series of computations, basis functions for each of the trial functions are assumed as follows:

1. over each boundary element at \( \Gamma \), the boundary variables \( u_r, w, \partial w/\partial n, P_r, M_r, \) and \( V_r \) were interpolated linearly, and
2. over each interior finite element, \( u, w \) were interpolated bilinearly such that each of these functions is \( C^0 \) continuous at the element boundaries. The boundary and interior finite elements are shown in Fig. 3. Results were obtained for four different meshes, as shown in Fig. 4. The three-noded elements, each with a node at the center of the shell, are generated from the four-noded element shown in Fig. 3 by collapsing two of the nodes together. The boundary conditions at \( \Gamma \) are: (1) \( w = 0, \partial w/\partial n = 0 \) and (2) the shell is free to move in the in-plane directions (in-plane traction-free).

The deflection at the center of the shell (with the geometry: \( R = 100, r_g = 5.0, \) due to an applied concentrated load at the center of the shell, is shown in Fig. 5. It is seen that the numerical results for all the meshes shown in Fig. 4 agree excellently with the
Fig. 4. Different meshes used in the analyses of a shallow spherical cap.

Fig. 5. Transverse displacement along a radial line in the base plane.

exact solution [11]. Note that the total number of degrees of freedom for each of the meshes is: 3 (Mesh 1); 27 (Mesh 2); 99 (Mesh 3); 183 (Mesh 4). The shell, in this example, is very shallow and approaches the geometry of a circular plate.

To examine the accuracy of the presently employed simple theory of shallow shells, problems of shells with various values of $r_0/R$ ($r_0/R = 1$ denotes a hemispherical shell) were analyzed under a uniformly applied transverse pressure loading (see Fig. 6). From the results for the crown deflection plotted in Fig. 6, it may be seen that for $(r_0/R) \leq 0.3$, the shallow-shell theory is a very good approximation (i.e. errors of less than 10%) to the deep-shell theory.

The stresses in the present integral equation approach may be computed in several alternate ways. The first approach is to use eqns (3.14), (3.19), and (3.20) for $u_1$, $w$, and $\partial u_1 / \partial n$, respectively, and analytically derive integral equations for the in-plane stress resultants $N_{ij}$, bending moments $M_{ij}$, and the reduced Kirchhoff shear, directly. However, this procedure involves rather tedious algebra. The second, and simpler, approach is to numerically differentiate the computed displacement field to derive the stress field. Both approaches are used in the present work. For the case of $(r_0/R) = (1/20)$, the obtained numerical results for $M_{ij}$ and $M_{ij}$ are compared with the analytical results in Fig. 7. At the boundary, the exact solution for $M_{ij}$ is 3.09 (KN m), whereas that obtained through the direct integral representation is 3.08 (KN m) and that obtained through numerical differentiation of the displacement field is 3.24 (KN m).

Next, the eigenvalues of free vibration of the shell were computed for various values of $(r_0/R)$. The
Fig. 6. Crown deflection for shells of various depth ratios.

Fig. 7. Results for radial and tangential bending moments.

Fig. 8. Time variation of crown deflection due to a concentrated pulse at the crown.
results for the first six eigenmodes are shown in Table 1, for Mesh 2 (27 d.o.f) and Mesh 3 (81 d.o.f). For the first mode, even Mesh 2 gives an eigenvalue of acceptable accuracy, for all the (r/R) cases, as compared to the analytical solution of [12]. From a comparison of the results for the six eigenvalues from the two meshes, it may be seen that even the 27 d.o.f. mesh gives results of acceptable accuracy for all the six modes. Also, Table 1 indicates the increased stiffening effect in the shell as the depth of the shell [(r/R)] increases.

The transient dynamic response of the shell subjected to a time-varying concentrated load at the center of the shell is analyzed for different types of time variation of the load. These results, computed from using the Mesh 3, are shown in Figs 8 and 9.
with the time variation of the applied load being depicted in the inset of each of these figures. These results indicate that the frequency content of the response of the shell is, of course, altered by the nature of the applied load.

6. CONCLUSION

A simple “boundary-element/interior-element” method, based on an integral equation formulation for static and dynamic response analysis of shallow shells, is presented. Unlike in the Galerkin finite element method, the trial functions for displacements are not required to be C¹ continuous in the present approach. Thus, the present approach is much simpler than the Galerkin FEM and yields results of acceptable accuracy with a discrete model of much smaller number of degrees of freedom.

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