AN EXPLICIT EXPRESSION FOR THE TANGENT-STIFFNESS
OF A FINITELY DEFORMED 3-D BEAM AND ITS USE
IN THE ANALYSIS OF SPACE FRAMES

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Abstract—Simplified procedures for finite-deformation analyses of space frames, using one beam element to model each member of the frame, are presented. Each element can undergo three-dimensional, arbitrarily large, rigid motions as well as moderately large non-rigid rotations. Each element can withstand three moments and three forces. The nonlinear bending–stretching coupling in each element is accounted for. By obtaining exact solutions to the appropriate governing differential equations, an explicit expression for the tangent-stiffness matrix of each element, valid at any stage during a wide range of finite deformations, is derived. An arc length method is used to incrementally compute the large deformation behavior of space frames. Several examples which illustrate the efficiency and simplicity of the developed procedures are presented. While the finitely deformed frame is assumed to remain elastic in the present paper, a plastic hinge method, wherein a hinge is assumed to form at an arbitrary location in the element, is presented in a companion paper.

1. INTRODUCTION

There is a renewed interest in efficient and simple analyses of three-dimensional frame structures due to their increasing viability for use as both offshore structures as well as multipurpose structures in outer space. There are plans for deploying very large structures in outer space, for a variety of reasons, such as antennae, radio telescopes, etc. While the offshore structures are in general massive, the large space structures (LSS) are necessarily of low mass and very high flexibility. A technological problem in the operation of the LSS is the need for active or passive control of transient dynamic (travelling wave type) response. Since the LSS are high flexible, large deformation behavior needs to be considered. The transient dynamic response of LSS, modeled as space frames, may be written as

\[ M\ddot{q} + D(\dot{q}, q) + S(q) = \mathbf{f} + \mathbf{Q}_e, \]  

(1.1)

where \( M \) is the mass matrix; \( D \) is the vector of nonlinear structural (or other passive) damping which may depend nonlinearly on the velocity \( \dot{q} \) as well as displacement \( q \) (depending on the joint design); \( S \) is the vector of nodal restraining forces which, for large deformations, depend nonlinearly on the nodal displacements \( q \); \( \mathbf{f} \) is the vector of control forces to be determined from a properly formulated active control algorithm; and \( \mathbf{Q}_e \) is the vector of externally applied dynamic forces, and \( \ddot{q} \) is the acceleration vector. Assuming that the control forces \( \mathbf{f} \) are determined from the control algorithm (which is a complicated problem and the object of a wide body of research in its own right), eqn (1.1) is a nonlinear initial value problem to be integrated by time-stepping algorithms. In such procedures, it is customary to write the displacement vector, \( q_{n+1} \) at time \( t_{n+1} \), as \( q_{n+1} = q_n + \Delta q \). Thus, the internal restraining nodal-force vector \( S_{n+1} \) is often written as

\[ S_{n+1} = S(q_{n+1}) = (\delta K)\Delta q + (\delta R). \]  

(1.2)

where \((\delta K)\) is the "tangent-stiffness matrix" at state \( t_n \) (accounting for geometric and material nonlinearities), and \((\delta R)\) are the internal restraining forces at \( t_n \).

In the usual finite element analysis, much effort is usually expended in evaluating \((\delta K)\). To account for large deformations and material nonlinearities, the usual procedures for analyzing space frames involve:

(i) the use of several finite elements to model each member of the space frame;
(ii) the assumption of polynomial basis functions for each component of displacement/rotation of each element; and
(iii) the numerical (quadrature) integration, over each element, of appropriate strain energy terms.

One of the aims of the present paper is to present an explicit expression for \((\delta K)\) of a three-dimensional beam element undergoing arbitrarily large rigid motion and moderately large non-rigid rotations. It is sufficient to model each member of the space frame by a single beam element of the aforementioned type. The joint design of the LSS is assumed to be such that each beam element can carry three bending moments
and three forces (axial and shear). The nonlinear bending-stretching coupling (and axial shortening of each beam element due to large rotations) in each beam element is accounted for. Under these conditions, an explicit expression for $K$ is derived, without the use of assumed polynomial basis functions for element deformation, and without the use of element-wise numerical quadrature. Analytical solutions for the appropriate differential equations are derived and used to derive explicit expressions for the stiffness coefficients. The present development for three-dimensional frame elements is an extension of that presented earlier for plane frames by Kondoh and Atluri [1].

The present paper is limited to a geometrically nonlinear quasistatic analysis of space frames. An arc-length method is used to generate the finite-deformation response solution. Several examples to illustrate the efficiency of the present approach are given. Simplified analyses accounting for material nonlinearities through a plastic-hinge method, wherein the hinge may occur at an arbitrary location along the member, are being presented in a companion paper.

The organization of the remainder of the paper is as follows. In Section 2.1, the kinematics of three-dimensional deformation of a beam element is considered. The deformation includes arbitrarily large rotations, which are characterized by finite-rotation vectors [2–4]. The governing differential equation for a three-dimensional beam undergoing large displacements and rotations are treated in 2.2. By assuming that the relative or non-rigid rotations are only moderate, those differential equations are simplified and are of the beam-column type. The axial-stretch of the beam depends on the integral over length of the squares of relative rotations. The simplified differential equations are then solved exactly; and analytical relations are derived between the “axial stretch and relative rotations” on the one hand, and the “axial force and bending moments” on the other. Using the formalism of a mixed-variational method, a closed-form (explicit) expression for the $(12 \times 12)$ tangent stiffness matrix is derived in Section 2.3.

The solution strategy is briefly discussed in Section 3; numerical examples are treated in Section 4; and Section 5 gives some concluding remarks. Two Appendices and attendant tables list the explicit expressions for the coefficients of the present three-dimensional beam tangent stiffness matrix, such that they may be directly implemented by other researchers and code developers.

### 2. DERIVATION OF AN EXPLICIT TANGENT STIFFNESS MATRIX FOR FINITE-DEFORMATION, POST-BUCKLING ANALYSIS OF SPACE FRAMES

The frame-type structures discussed herein are assumed to remain elastic, and only a conservative system of concentrated loads are assumed to act at the nodes of the frame.

#### 2.1. Three-dimensional kinematics of deformation of a member element of a space-frame

Consider a typical frame member, modeled here as a three-dimensional beam element, that spans between nodes 1 and 2 as shown in Fig. 1. The element is considered to have a uniform cross-section and to be of length $l$ before deformation. The co-ordinates $x_j$ are the local co-ordinates at the node $j$ ($j=1,2$) of an undeformed element. Likewise, $u_i$ ($i=1,2,3$) denote the displacements at the centroidal axis of the element along the coordinate directions $x_i$, respectively. Also, as shown in Fig. 1, $\theta_i$ are the angles of rotation about the axes of $x_i$. After a

![Fig. 1. Nomenclature for kinematics of deformation of a space member.](image-url)
deformation of the element, two co-ordinate systems are introduced to represent the rigid and relative (non-rigid) rotations of the element. One is the co-ordinate system \(x_i\), which is locally "tangential" and "normal" to the deformed centroidal axis; another is \(x\), which characterizes the rigid translations and rotations of the member (see Fig. 1).

Considering each rotation as a semi-tangential rotation, we can treat rotations as vectors [2-4]. Thus, the relation among the total, rigid and relative rotation vectors is given by

\[
\gamma = \beta + \alpha \quad (i = 1, 2),
\]

(2.1)

where \(\gamma\) is the total rotation vector at the node \(i\), \(\beta\) is the vector of rigid rotation of the beam as a whole, and \(\alpha\) is the relative rotation vector at the node \(i\).

Using eqn (2.1), the total rotation vector at the node 2, \(\gamma_2\), is represented as

\[
\gamma_2 = \beta + \alpha + \alpha'.
\]

(2.2)

where

\[
\alpha' = \gamma - \gamma.
\]

(2.3)

Therefore, the relative rotation vector at the node 2 can be defined using eqns (2.1) and (2.2) as

\[
\alpha = \alpha + \alpha'.
\]

(2.4)

On the other hand, the expressions of the rotation vectors may be written, by using their components in any co-ordinate system, as follows [2-4]. Using the local co-ordinate system, the total rotation vector at the node \(i\) may be written [2-4] as

\[
\gamma = \tan \frac{\theta_i}{2} e_i, \quad (i = 1, 2)
\]

\[
\gamma = \tan \frac{\theta_i}{2} e_i, \quad (j = 1, 2, 3).
\]

(2.5)

The relative rotation vector at the node \(i\) in the co-ordinate system \(x_i\) is given by

\[
\alpha = \tan \frac{\theta_i}{2} e_i, \quad (i = 1, 2)
\]

\[
\alpha = \tan \frac{\theta_i}{2} e_i, \quad (j = 1, 2, 3).
\]

(2.6)

Substituting eqn (2.5) into eqn (2.3), the difference between the rotation vectors at nodes 1 and 2 is given by

\[
\alpha' = \left(\tan \frac{\theta_2}{2} - \tan \frac{\theta_1}{2}\right) e_i
\]

(2.7)

\[
\alpha' = \left(\tan \frac{\theta_2}{2} - \tan \frac{\theta_1}{2}\right) e_i.
\]

(2.8)

Also substituting eqns (2.6) and (2.8) into eqn (2.4), the relative rotation at node 2 is represented as

\[
\alpha = \tan \frac{\theta_2}{2} e_i
\]

(2.9)

\[
\alpha = \left(\tan \frac{\theta_2}{2} + \tan \frac{\theta_1}{2}\right) e_i.
\]

(2.10)

Furthermore, the action of a rotation \(\mathbf{R}\), which transforms a vector \(\mathbf{dX}\) to \(\mathbf{dX}'\), is represented by the relation [2-4]

\[
d\mathbf{X}' = \frac{1}{1 + \mathbf{R} \cdot \mathbf{R}} \left[(1 - \mathbf{R} \cdot \mathbf{R}) \cdot \mathbf{dX} + 2(\mathbf{R} \cdot \mathbf{dX}) \cdot \mathbf{R} + 2\mathbf{R} \times \mathbf{dX}\right].
\]

(2.11)

Using (2.11) and considering the action of the total rotation \(\gamma\) on the unit vectors \(e_i\), one obtains the following equations:

\[
T_i = \mathbf{T} e_i, \quad (i = 1, 2),
\]

(2.12)

where \(T_i\) are the vectors \(e_i\) at node \(i\), and

\[
T_i = \frac{1}{1 + \theta_i^2} \left[1 + \tan^2\left(\frac{\theta_1}{2}\right) - \tan^2\left(\frac{\theta_2}{2}\right)ight] - \tan^2\left(\frac{\theta_1}{2}\right)
\]

(2.13a)

\[
T_i = \frac{2}{1 + \theta_i^2} \left[\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} + \tan \frac{\theta_2}{2}\right]
\]

(2.13b)

\[
T_i = \frac{2}{1 + \theta_i^2} \left[\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} - \tan \frac{\theta_2}{2}\right]
\]

(2.13c)

\[
T_i = \frac{2}{1 + \theta_i^2} \left[\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} - \tan \frac{\theta_2}{2}\right]
\]

(2.13d)

\[
T_i = \frac{1}{1 + \theta_i^2} \left[1 - \tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_2}{2}\right)ight] - \tan^2\left(\frac{\theta_2}{2}\right)
\]

(2.13e)

\[
T_i = \frac{2}{1 + \theta_i^2} \left[\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} + \tan \frac{\theta_1}{2}\right]
\]

(2.13f)

\[
T_i = \frac{2}{1 + \theta_i^2} \left[\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} + \tan \frac{\theta_1}{2}\right]
\]

(2.13g)

\[
T_i = \frac{2}{1 + \theta_i^2} \left[\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2} - \tan \frac{\theta_1}{2}\right]
\]

(2.13h)
\[ \theta_1 = \tan^2 \left( \frac{\theta_1}{2} \right) + \tan^2 \left( \frac{\theta_2}{2} \right) + \tan^2 \left( \frac{\theta_3}{2} \right). \]  

\[ (2.15) \]

On the other hand, \( \mathbf{e}_s \), as a unit vector in the direction of the line joining node 1 to 2 in the deformed configuration, may be represented as

\[ \mathbf{e}_s = r \cdot \mathbf{e}_1 + s \cdot \mathbf{e}_2 + t \cdot \mathbf{e}_3. \]  

\[ (2.16) \]

where

\[ r = \frac{\bar{u}_1}{l^*}, \quad s = \frac{\bar{u}_2}{l^*}, \]  

\[ t = \frac{l + \bar{u}_1}{l^*}. \]  

\[ (2.17a, b) \]

and

\[ \bar{u}_i = u_i - \bar{u}_i. \]  

\[ (2.19) \]

Other unit vectors, \( \mathbf{e}_1, \mathbf{e}_2 \), corresponding to the coordinate system, \( \mathbf{x}_s \), may be written, using eqn (2.11) and the rotation vector, \( \mathbf{w} \), at node 1, shown in Fig. 2, as

\[ \mathbf{e}_i = \frac{1}{1 + i \mathbf{w} \cdot \mathbf{w} / l^*} \left[ (1 - l \mathbf{w} \cdot \mathbf{w}) \cdot \mathbf{e}_i^* \right. \]

\[ + 2(\mathbf{w} \cdot \mathbf{e}_i^*) \cdot \mathbf{w} + 2 \left( \mathbf{w} \times \mathbf{e}_i^* \right) \mathbf{w}, \]  

\[ (i = 1, 2), \]  

\[ (2.20) \]

where

\[ \mathbf{w} = \tan \frac{\mathbf{w} \cdot \mathbf{e}_i^* \times \mathbf{e}_i}{2 | \mathbf{e}_i^* \times \mathbf{e}_i |}. \]  

\[ (2.21) \]

From eqns (2.12)-(2.19) and eqns (2.21) and (2.22), the relative rotation vector at node 1 is represented as

\[ \mathbf{w} = h \cdot \mathbf{e}_1 + l \cdot \mathbf{e}_2 + m \cdot \mathbf{e}_3. \]  

\[ (2.23c) \]

where

\[ h = \frac{1 \mathbf{H} : t - 1 \mathbf{I} : s}{1 + \mathbf{1} \mathbf{G} : r + \mathbf{1} \mathbf{H} : s + \mathbf{1} \mathbf{I} : t} \]  

\[ (2.24a) \]

\[ l = \frac{1 \mathbf{I} : r - 1 \mathbf{G} : t}{1 + \mathbf{1} \mathbf{G} : r + \mathbf{1} \mathbf{H} : s + \mathbf{1} \mathbf{I} : t} \]  

\[ (2.24b) \]

\[ m = \frac{1 \mathbf{G} : s - 1 \mathbf{H} : r}{1 + \mathbf{1} \mathbf{G} : r + \mathbf{1} \mathbf{H} : s + \mathbf{1} \mathbf{I} : t}. \]  

\[ (2.24c) \]

Substituting eqns (2.12)-(2.15) and eqns (2.23) and (2.24) into eqn (2.20), the following equations are obtained:

\[ \mathbf{e}_1 = o \cdot \mathbf{e}_1 + p \cdot \mathbf{e}_2 + q \cdot \mathbf{e}_3, \]  

\[ (2.25) \]

\[ \mathbf{e}_2 = u \cdot \mathbf{e}_1 + v \cdot \mathbf{e}_2 + w \cdot \mathbf{e}_3, \]  

\[ (2.26) \]

where

\[ o = [C_1, A + 2h \cdot C_2 + (l \cdot C - m \cdot B)] / C_3, \]  

\[ (2.27a) \]

\[ p = [C_1, B + 2l \cdot C_2 + (m \cdot A - h \cdot C)] / C_3, \]  

\[ (2.27b) \]

\[ q = [C_1, C + 2m \cdot C_2 + 2(h \cdot B - l \cdot A)] / C_3, \]  

\[ (2.27c) \]

\[ u = [C_1, D + 2h \cdot C_3 + 2(l \cdot F - m \cdot E)] / C_3, \]  

\[ (2.27d) \]

\[ v = [C_1, E + 2l \cdot C_4 + 2(m \cdot D - h \cdot F)] / C_3, \]  

\[ (2.27e) \]

\[ w = [C_1, F + 2m \cdot C_4 + 2(l \cdot E - m \cdot D)] / C_3, \]  

\[ (2.27f) \]

\[ C_1 = 1 - h^2 - l^2 - m^2, \]  

\[ (2.28a) \]

\[ C_2 = h \cdot A + l \cdot B + m \cdot C \]  

\[ (2.28b) \]

\[ C_3 = 1 + h^2 + l^2 + m^2, \]  

\[ (2.28c) \]

\[ C_4 = h \cdot D + l \cdot E + m \cdot F. \]  

\[ (2.28d) \]

We denote by \( \alpha \) the relative rotation at node 1. Thus, \( \alpha \) characterizes the transformation of the coordinate system \( \mathbf{x}_s \) at node 1. From eqn (2.23), one obtains

\[ \alpha = - \mathbf{w} = -(h \cdot \mathbf{e}_1 + l \cdot \mathbf{e}_2 + m \cdot \mathbf{e}_3). \]  

\[ (2.29) \]
Also, using eqns (2.16), (2.25) and (2.26),

\[ 1\alpha = (1\alpha \cdot \hat{\epsilon}) \hat{\epsilon} + (1\alpha \cdot \hat{\epsilon}_2) \hat{\epsilon}_2 + (1\alpha \cdot \hat{\epsilon}_3) \hat{\epsilon}_3, \]  

(2.30)

Therefore, the components of the relative rotation at node 1, i.e., \( 1\alpha \), are obtained from eqns (2.6), (2.16), (2.25), (2.26) and (2.29) to (2.30), as

\[
\begin{align*}
\tan \frac{1\theta_1}{2} &= -(h \cdot o + l \cdot p + m \cdot q) \\
\tan \frac{1\theta_2}{2} &= -(h \cdot u + l \cdot v + m \cdot w) \\
\tan \frac{1\theta_3}{2} &= -(h \cdot r + l \cdot s + m \cdot t). 
\end{align*}
\]  

(2.31)

Also, the components of the relative rotation at node 2, \( 2\alpha \), are obtained from eqns (2.7)-(2.10), (2.16), (2.25) and (2.26), as

\[
\begin{align*}
\tan \frac{2\theta_1}{2} &= \tan \frac{1\theta_1}{2} + \left( \tan \frac{2\theta_1}{2} - \tan \frac{1\theta_1}{2} \right) \phi \\
&\quad + \left( \tan \frac{2\theta_2}{2} - \tan \frac{1\theta_2}{2} \right) \rho \\
&\quad + \left( \tan \frac{2\theta_3}{2} - \tan \frac{1\theta_3}{2} \right) \eta \\
\tan \frac{2\theta_2}{2} &= \tan \frac{1\theta_2}{2} + \left( \tan \frac{2\theta_1}{2} - \tan \frac{1\theta_1}{2} \right) \nu \\
&\quad + \left( \tan \frac{2\theta_2}{2} - \tan \frac{1\theta_2}{2} \right) \sigma \\
&\quad + \left( \tan \frac{2\theta_3}{2} - \tan \frac{1\theta_3}{2} \right) \tau. 
\end{align*}
\]  

(2.32)

It should be noted that the component \( 1\theta_1 \) of the relative rotation at node 1 is zero due to the rotation \( 1\omega \) being as in eqn (2.21).

Finally, the relation between the total axial stretch and displacements of the member is

\[ \delta = [\tilde{u}^2 + \tilde{v}^2 + (l + \tilde{u})]^\frac{1}{2} - l, \]  

(2.33)

where \( \delta \) is the total axial stretch, and

\[ \tilde{u}_i = u_i - 1u_i, \quad (i = 1, 2 \text{ and } 3). \]

2.2. Relations between the "strecth and relative rotations" and the "axial force and bending moments" for a frame member

In preparation for the task of deriving an explicit expression for the tangent stiffness matrix that is valid over a wide range of deformations of a frame member, in this section, certain explicit relations are derived between the kinematic variables of stretch and relative rotations, on the one hand the mechanical variables of axial force and bending moments on the other, of an individual frame member (or of a finite element if more than one finite element is contemplated for modeling an individual member). These "generalized" force–displacement relations for an individual member/element are also intended to be valid over a range of deformations that may be considered as "large".

To achieve the above purpose, a beam-column, as shown in Fig. 3, is considered. It should be noted that

Fig. 3. Sign-convention for system of generalized force on a framed member.
all of the rotations are semitangential rotations [2-4] and \( \dot{\theta}_i \) at node 1 is zero. Using the relative rotations, \( \dot{\phi}_i, \dot{\theta}_2 \) and \( \dot{\theta}_3 \), the relation between unit vectors \( e^* \) and \( \hat{e} \), at any point along the beam is written, using eqn (2.11), as

\[
e^* = S_y \cdot \hat{e}
\]

where

\[
S_y = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}
\]

\[
S_{11} = \frac{1}{1 + \beta^2} \left[ 1 + \tan^2\left(\frac{\theta_1}{2}\right) - \tan^2\left(\frac{\theta_2}{2}\right) \right] - \tan^2\left(\frac{\theta_3}{2}\right)
\]

\[
S_{12} = \frac{2}{1 + \beta^2} \left[ \tan\left(\frac{\theta_1}{2}\right) \tan\left(\frac{\theta_2}{2}\right) + \tan\left(\frac{\theta_3}{2}\right) \right]
\]

\[
S_{13} = \frac{2}{1 + \beta^2} \left[ \tan\left(\frac{\theta_1}{2}\right) - \tan\left(\frac{\theta_2}{2}\right) \right]
\]

\[
S_{21} = \frac{2}{1 + \beta^2} \left[ \tan\left(\frac{\theta_1}{2}\right) \tan\left(\frac{\theta_2}{2}\right) - \tan\left(\frac{\theta_3}{2}\right) \right]
\]

\[
S_{22} = \frac{1}{1 + \beta^2} \left[ 1 - \tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_2}{2}\right) - \tan^2\left(\frac{\theta_3}{2}\right) \right] - \tan^2\left(\frac{\theta_3}{2}\right)
\]

\[
S_{23} = \frac{2}{1 + \beta^2} \left[ \tan\left(\frac{\theta_1}{2}\right) \tan\left(\frac{\theta_3}{2}\right) - \tan\left(\frac{\theta_2}{2}\right) \right]
\]

\[
S_{31} = \frac{2}{1 + \beta^2} \left[ \tan\left(\frac{\theta_1}{2}\right) \tan\left(\frac{\theta_2}{2}\right) + \tan\left(\frac{\theta_3}{2}\right) \right]
\]

\[
S_{32} = \frac{2}{1 + \beta^2} \left[ \tan\left(\frac{\theta_2}{2}\right) \tan\left(\frac{\theta_3}{2}\right) + \tan\left(\frac{\theta_1}{2}\right) \right]
\]

\[
S_{33} = \frac{2}{1 + \beta^2} \left[ 1 - \tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_2}{2}\right) \right]
\]

\[
\theta^2 = \tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_2}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right)
\]

The curvatures along the centroidal axis of a deformed member are given by

\[
K^* = \frac{\text{de}^*}{dx^*} e^*
\]

\[
K_{12}^* = \frac{\text{de}^*}{dx^*} e^* \text{ or } -\frac{\text{de}^*}{dx^*} e^*
\]
The equation of equilibrium in the two transverse directions of the beam may be written \([5, 6]\) as

\[ -\frac{d\tilde{M}_1}{dx_1} + \dot{\tilde{Q}}_1 \cdot (e_x \cdot e_1) - \tilde{N} \cdot (e_x \cdot e_1) = 0 \]  
\[ -\frac{d\tilde{M}_2}{dx_1} + \dot{\tilde{Q}}_2 \cdot (e_x \cdot e_1) - \tilde{N} \cdot (e_x \cdot e_1) = 0. \]  

(2.42a)  

(2.42b)

Also

\[ -\frac{d\tilde{N}}{dx_1} = 0, \]  

(2.42c)

where

\[ \dot{\tilde{Q}}_1 = -\frac{1}{l + \delta} (\tilde{M}_1 - \tilde{\dot{M}}_1) \]  
\[ \dot{\tilde{Q}}_2 = -\frac{1}{l + \delta} (\tilde{M}_2 - \tilde{\dot{M}}_2). \]  

(2.43a)  

(2.43b)

Substituting eqns (2.34)–(2.36) and (2.39)–(2.41) into eqn (2.42), the following equilibrium equations are obtained:

\[ EI_1 \frac{d}{dx_1} \left[ S_{12} \left( \frac{dS_{12}}{dx_1} \cdot S_{11} + \frac{dS_{22}}{dx_1} \cdot S_{12} + \frac{dS_{32}}{dx_1} \cdot S_{13} \right) \right] \]

\[ -EI_2 \frac{d}{dx_1} \left[ S_{22} \left( \frac{dS_{12}}{dx_1} \cdot S_{11} + \frac{dS_{22}}{dx_1} \cdot S_{22} + \frac{dS_{32}}{dx_1} \cdot S_{23} \right) \right] \]

\[ + GJ \frac{d}{dx_1} \left[ S_{32} \left( \frac{dS_{12}}{dx_1} \cdot S_{11} + \frac{dS_{22}}{dx_1} \cdot S_{32} + \frac{dS_{32}}{dx_1} \cdot S_{33} \right) \right] \]

\[ + \frac{dS_{13}}{dx_1} \cdot S_{33} \right] + \dot{Q}_1 \cdot S_{31} - \tilde{N} \cdot S_{31} = 0 \]  

(2.44a)

\[ -EI_1 \frac{d}{dx_1} \left[ S_{11} \left( \frac{dS_{11}}{dx_1} \cdot S_{11} + \frac{dS_{21}}{dx_1} \cdot S_{21} + \frac{dS_{31}}{dx_1} \cdot S_{31} \right) \right] \]

\[ + EI_2 \frac{d}{dx_1} \left[ S_{21} \left( \frac{dS_{11}}{dx_1} \cdot S_{11} + \frac{dS_{21}}{dx_1} \cdot S_{22} + \frac{dS_{31}}{dx_1} \cdot S_{31} \right) \right] \]

\[ + GJ \frac{d}{dx_1} \left[ S_{31} \left( \frac{dS_{11}}{dx_1} \cdot S_{11} + \frac{dS_{21}}{dx_1} \cdot S_{22} + \frac{dS_{31}}{dx_1} \cdot S_{31} \right) \right] \]

\[ + \dot{Q}_2 \cdot S_{33} - \tilde{N} \cdot S_{32} = 0 \]  

(2.44b)

and

\[ -EI_1 \frac{d}{dx_1} \left[ S_{13} \left( \frac{dS_{13}}{dx_1} \cdot S_{11} + \frac{dS_{23}}{dx_1} \cdot S_{22} + \frac{dS_{33}}{dx_1} \cdot S_{33} \right) \right] \]

\[ + EI_2 \frac{d}{dx_1} \left[ S_{23} \left( \frac{dS_{13}}{dx_1} \cdot S_{11} + \frac{dS_{23}}{dx_1} \cdot S_{22} + \frac{dS_{33}}{dx_1} \cdot S_{33} \right) \right] \]

Also, the boundary conditions are given by

\[ EI_1 \frac{d^2\theta_i}{dx_1^2} \bigg|_{x=0} = \dot{\tilde{M}}_1, \]

(2.48a, b)

\[ -EI_1 \frac{d^2\theta_i}{dx_1^2} \bigg|_{x=L} = \dot{\tilde{M}}_2, \]  

(2.48c, d)

\[ \theta_3 \bigg|_{x=0} = 0, \]  

(2.48e, f)
\[ G_\alpha \frac{d\theta_i}{dx^2} \bigg|_{x = \xi} = \gamma \tilde{M}_3, \quad (2.48c, f) \]

The solutions of eqns (2.47b) and (2.48c, d) are given by:

\[ \delta_i = -\gamma m_2 \left\{ \frac{1}{f^2} - \frac{1}{f} \sin f \cdot x^* \frac{x^*}{l} - \frac{1}{f} \cot f \cdot \cos f \cdot x^* \frac{x^*}{l} \right\} \]

\[ -\gamma m_1 \left\{ \frac{1}{f} + \frac{1}{f} \cdot \cos f \cdot \cos f \cdot x^* \frac{x^*}{l} \right\}, \quad (2.55) \]

where

\[ f = \sqrt{-n_2}, \quad (2.56) \]

(4) For \( n_2 > 0 \)

\[ \delta_i = -\gamma m_1 \cdot \frac{-1}{g^2} \cdot \sinh \frac{g \cdot x^*}{l} + \frac{1}{g} \cdot \coth g \cdot \cos \frac{g \cdot x^*}{l} \]

\[ + \gamma m_2 \left\{ \frac{1}{g^2} - \frac{1}{g} \cdot \coth g \cdot \cosh \frac{g \cdot x^*}{l} \right\}, \quad (2.57) \]

The non-dimensional axial forces and bending moments, denoted \( n_1, n_2, m_1, \) and \( m_2, \) may be defined, respectively, through the relations

\[ n_1 = \frac{Nl^2}{EI}, \quad m_1 = \frac{M_1 l}{EI}, \quad (2.50a, b) \]

\[ n_2 = \frac{Nl^2}{EI}, \quad m_2 = \frac{\tilde{M}_2 l}{EI}, \quad (2.50c, d) \]

The solutions of eqns (2.47a) and (2.48a, b) are given by:

(1) For \( n_1 < 0 \)

\[ \theta_2 = \gamma m_1 \cdot \left[ \frac{1}{d^2} - \frac{1}{d} \cdot \sin d \cdot x^* \frac{x^*}{l} - \frac{1}{d} \cdot \cot d \cdot \cos d \cdot x^* \frac{x^*}{l} \right] \]

\[ + \gamma m_2 \cdot \left[ -\frac{1}{d^2} + \frac{1}{d} \cdot \cosec d \cdot \cos d \cdot x^* \frac{x^*}{l} \right], \quad (2.51) \]

where

\[ d = \sqrt{-n_1}, \quad (2.52) \]

(2) For \( n_1 > 0 \)

\[ \theta_2 = \gamma m_1 \cdot \left[ \frac{1}{d^2} - \frac{1}{d} \cdot \sinh e \cdot x^* \frac{x^*}{l} + \frac{1}{e} \cdot \coth e \cdot \cosh e \cdot x^* \frac{x^*}{l} \right] \]

\[ + \gamma m_2 \cdot \left[ -\frac{1}{d^2} + \frac{1}{e} \cdot \cosech e \cdot \cosh e \cdot x^* \frac{x^*}{l} \right], \quad (2.53) \]

where

\[ e = \sqrt{n_1}, \quad (2.54) \]

Equations (2.51)-(2.58) lead to the following relations between the relative rotations, \( \theta_1, \gamma \theta_1, \gamma \theta_2, \) and \( \gamma \theta_2, \) at the ends of the member and the corresponding bending moments, \( m_1, \gamma m_1, m_2, \) and \( \gamma m_2. \)

(1) For \( n_1 < 0 \)

\[ \gamma \theta_2 = \gamma m_1 \cdot \left[ \frac{1}{d^2} - \frac{1}{d} \cdot \cot d \frac{x^*}{l} \right] + \gamma m_2 \cdot \left[ -\frac{1}{d^2} + \frac{1}{d} \cdot \cosec d \frac{x^*}{l} \right] \]

\[ (2.59a) \]

(2) For \( n_1 > 0 \)

\[ \gamma \theta_2 = \gamma m_1 \cdot \left[ \frac{1}{d^2} - \frac{1}{d} \cdot \cosech e \frac{x^*}{l} + \frac{1}{e} \cdot \coth e \cdot \cosh e \frac{x^*}{l} \right] \]

\[ + \gamma m_2 \cdot \left[ -\frac{1}{d^2} + \frac{1}{e} \cdot \cosech e \cdot \cosh e \frac{x^*}{l} \right], \quad (2.53) \]

where

\[ e = \sqrt{n_1}, \quad (2.54) \]

(2.59b)

*Similar solutions for planar deformation of a beam-column were given earlier in [5, 6].*
(3) For \( n_2 < 0 \)

\[
\frac{1}{\gamma_1} = -m_1 \left[ -\frac{1}{f^2} + \cot \frac{f}{g} \right]
\]

\[
\frac{1}{\gamma_1} = -m_2 \left[ -\frac{1}{f^2} + \cosec \frac{f}{g} \right]
\]

\[
\frac{1}{\gamma_2} = -m_1 \left[ -\frac{1}{f^2} + \cosec \frac{f}{g} \right]
\]

\[
\frac{1}{\gamma_2} = -m_2 \left[ -\frac{1}{f^2} + \cot \frac{f}{g} \right]
\]

(2.61a)

(2.61b)

(4) For \( n_2 > 0 \)

\[
\frac{1}{\gamma_1} = -m_1 \left[ \frac{1}{f^2} + \coth \frac{g}{f} \right]
\]

\[
\frac{1}{\gamma_1} = -m_2 \left[ \frac{1}{g^2} + \cosech \frac{g}{f} \right]
\]

\[
\frac{1}{\gamma_2} = -m_1 \left[ \frac{1}{g^2} + \cosech \frac{g}{f} \right]
\]

\[
\frac{1}{\gamma_2} = -m_2 \left[ \frac{1}{f^2} + \coth \frac{g}{f} \right]
\]

(2.62a)

(2.62b)

The set of eqns (2.59)–(2.63) may be written in a more convenient form by decomposing the kinematic and mechanical variables of the beam into “symmetric” and “antisymmetric” parts, as

\[
\frac{1}{\gamma_1} = \frac{1}{2} (\gamma_1 + \gamma_2), \quad \frac{1}{\gamma_2} = \frac{1}{2} (\gamma_1 - \gamma_2)
\]

also

\[
\gamma_1 = m_{n1} - m_n, \quad \gamma_2 = m_{n1} + m_n
\]

\[
\gamma_1 = m_{n2}, \quad \gamma_2 = -m_{n2}, \quad (i = 1, 2)
\]

(2.64a)

(2.64b)

(2.65a)

(2.65b)

where the superscripts \( a \) and \( s \) refer to “antisymmetric” and “symmetric” parts, respectively.

Therefore, in terms of the variables \( *n_1, *n_2, *m_1 \) and \( *m_2 \), eqns (2.59)–(2.62) may be written as

\[
\frac{1}{\gamma_1} = *n_{12} *m_1, \quad \frac{1}{\gamma_1} = *n_{12} *m_2
\]

\[
\frac{1}{\gamma_2} = *n_{12} *m_1, \quad \frac{1}{\gamma_2} = *n_{12} *m_2
\]

\[
\gamma_1 = 2n_1, \quad \gamma_2 = -2n_2
\]

(2.66a)

(2.66b)

(2.67a)

(2.67b)

Also, using eqns (2.49) and (2.51)–(2.58), the following expressions concerning the total axial stretch, \( \delta \), are obtained as:

(1) For \( n_1 < 0 \)

\[
\delta = \frac{2}{f} \left\{ \frac{1}{2(1-n)^2} \frac{\cosec^2 (\sqrt{1-n})}{4(1-n)} - \frac{\cot (\sqrt{1-n})}{4(1-n) \sqrt{1-n}} (m_1^2 + m_2^2) + \left( \frac{1}{2} \frac{\cosec (\sqrt{1-n})}{4(1-n) \sqrt{1-n}} + \frac{N}{EA} \right) \right\}
\]

(2.63a)

(2) For \( n_1 > 0 \)

\[
\delta = \frac{2}{f} \left\{ \frac{1}{2n^2} \frac{\cosech^2 (\sqrt{n})}{4n} - \frac{\coth (\sqrt{n})}{4n \sqrt{n}} (m_1^2 + m_2^2) + \left( \frac{1}{2} \frac{\coth (\sqrt{n}) \cosech (\sqrt{n})}{4n \sqrt{n}} \right) \right\}
\]

(2.63b)

(3) For \( n_2 < 0 \)

\[
\delta = \frac{2}{f} \left\{ \frac{1}{2n^2} \frac{\cosech^2 (\sqrt{n})}{4n} - \frac{\coth (\sqrt{n})}{4n \sqrt{n}} (m_1^2 + m_2^2) + \left( \frac{1}{2} \frac{\coth (\sqrt{n}) \cosech (\sqrt{n})}{4n \sqrt{n}} \right) \right\}
\]

(2.63c)

(2.63d)

(4) For \( n_2 > 0 \)

\[
\delta = \frac{2}{f} \left\{ \frac{1}{2n^2} \frac{\cosech^2 (\sqrt{n})}{4n} - \frac{\coth (\sqrt{n})}{4n \sqrt{n}} (m_1^2 + m_2^2) + \left( \frac{1}{2} \frac{\coth (\sqrt{n}) \cosech (\sqrt{n})}{4n \sqrt{n}} \right) \right\}
\]

(2.63e)

Also, in terms of the new variables, eqns (2.63a, b)
may be rewritten in a unified form as follows:

\[
\frac{d^2 h_1}{dn_1} = \frac{1}{(-n_1)^2} - \frac{1}{4(-n_1)\sqrt{-n_1}} \cdot \cot \sqrt{-n_1} - \frac{1}{8(-n_1)} \cdot \csc(e \sqrt{-n_1})^2 \quad (2.73a)
\]

\[
\frac{d^2 h_1}{dn_1} = \frac{1}{4(-n_1)\sqrt{-n_1}} \cdot \tan \sqrt{-n_1} - \frac{1}{8(-n_1)} \cdot \sec(e \sqrt{-n_1})^2 \quad (2.73b)
\]

\[
\frac{d^2 h_2}{dn_2} = -\frac{1}{(-n_2)^2} + \frac{1}{4(-n_2)\sqrt{-n_2}} \cdot \cot \sqrt{-n_2} + \frac{1}{8(-n_2)} \cdot \csc(e \sqrt{-n_2})^2 \quad (2.75a)
\]

\[
\frac{d^2 h_2}{dn_2} = \frac{1}{4(-n_2)\sqrt{-n_2}} \cdot \tan \sqrt{-n_2} + \frac{1}{8(-n_2)} \cdot \sec(e \sqrt{-n_2})^2 \quad (2.75b)
\]

\[
\frac{d^2 h_3}{dn_3} = \frac{1}{n_3^2} + \frac{1}{4n_3\sqrt{n_3}} \cdot \cot \sqrt{n_3} + \frac{1}{8n_3} \cdot \csc(e \sqrt{n_3})^2 \quad (2.76a)
\]

Equations (2.66), (2.67) and (2.72) are the sought-after relations between the generalized displacements and forces at the nodes of an individual frame member, for the range of deformations considered. In connection with eqns (2.66), (2.67) and (2.72), it is worthwhile to recall that:

1. \( \delta \) is in the direction of the straight line connecting the nodes of the frame member after its deformation.

2. The parameters \( \delta, \theta, \Delta \) and \( \phi \) are calculated from eqs (2.31)-(2.33), which are valid in the presence of an arbitrarily large rigid motion (translations and rotations) of the individual member.

Thus, while the local stretch (pure strain) and relative rotation (non-rigid) of a differential element of an individual frame-member may be small, the individual member as a whole (and as a part of the overall frame) may undergo arbitrarily large rigid motion. Hence, the generalized force-displacement relations embodied in eqns (2.66), (2.76) and (2.72) remain valid in the presence of an arbitrarily large rigid motions of the individual member of the frame. Also, it is important to note that the present relations for each element account, as in the Von Karman plate theory, the non-linear coupling between the bending and stretching deformations, as seen from eqns (2.66), (2.67) and (2.72).

2.3 Tangent stiffness matrix of a space frame member/element

Recall that, for the most part of the previous subsection, each member of the frame is treated as a beam column; but in extreme cases, i.e. of "pathological" deformations, it may be modeled by two or three elements at most.

Now we consider the strain energy due to axial stretch of the member. Since the total axial stretch, \( \delta \), is related in a highly non-linear fashion to the axial force, \( N \), as well as the bending moments, \( \delta_m \) and \( \delta_m \) (i = 1, 2), from eqn (2.72), the inversion of this relation in an explicit form, which expresses the axial force \( N \) as a function of \( \delta \), appears impossible. With a view towards carrying out this inversion of the \( \delta \) vs \( N \) relation incrementally, the strain energy due to stretching, which is denoted as \( \pi_\delta \), needs to be expressed in a "mixed" form using the well-known concept of a Legendre contact transformation [7] as

\[
\pi_\delta = N \cdot \delta - \frac{1}{2} \cdot \frac{N^2}{2EA}. \quad (2.77)
\]
On the other hand, the strain energy due to bending is introduced as follows. The "flexibility" coefficients, \( h_i \) and \( h_i \) (i = 1, 2), are highly non-linear functions of the axial force in eqns (2.66)–(2.71). However, unless the flexibility coefficients are equal to zero, one may invert eqns (2.66) and (2.67) to write the "force-displacement" relations as

\[
\begin{align*}
\sigma m_1 &= h_1, \quad \tau m_1 = -h_1, \\
\sigma m_2 &= h_2, \quad \tau m_2 = -h_2.
\end{align*}
\]  

(2.78a, b)

Using the definition of non-dimensional moments as in eqn (2.50), one may express the strain energy due to bending, which is denoted as \( \pi_B \), as

\[
\pi_B = \frac{E I_1}{2} \left[ \frac{\theta_1^2}{h_1} + \frac{\theta_2^2}{h_2} \right] + \frac{E I_2}{2} \left[ \frac{\theta_3^2}{h_3} + \frac{\theta_4^2}{h_4} \right].
\]

(2.80)

However, when in the limit as \( \bar{N} \) tends to \(( -4\pi^2E/12) \), as explained in [1], \( h_i \) (i = 1, 2) tend to zero; thus, the inversions of eqns (2.66) and (2.67) to obtain eqns (2.78) and (2.79) are not meaningful. In such a case, one may use a mixed form for the bending energy of the symmetric mode, treating both \( \tau m \) and \( \theta i \) (i = 1, 2) as variables, as

\[
\begin{align*}
\pi_B &= \frac{E I_1}{2} \left[ \frac{\theta_1^2}{h_1} + \frac{\theta_2^2}{h_2} \right] + \frac{E I_2}{2} \left[ \frac{\theta_3^2}{h_3} + \frac{\theta_4^2}{h_4} \right],
\end{align*}
\]

(2.80a)

\[
\pi_B = \frac{E I_2}{2} \left[ \frac{\theta_3^2}{h_3} + \frac{\theta_4^2}{h_4} \right].
\]

(2.80b)

However, as explained in [1], without loss of generality for a practical frame-structure, we may consider the strain energy in the form of eqn (2.80). It should be noted that in the view of the dependence of \( h_i \) and \( h_i \) on \( n_i \) (i = 1, 2) as in eqns (2.68)–(2.71), there is coupling between "bending" and "stretching" variables.

The strain energy due to torsion, which is denoted as \( \pi_T \), may be written as

\[
\pi_T = \frac{G J}{2 l} \cdot \theta_3. \tag{2.81}
\]

The internal energy in the member due to combined bending, stretching, and torsion is represented as

\[
\begin{align*}
\pi &= \frac{E I_1}{2 l} \left[ \frac{\theta_1^2}{h_1} + \frac{\theta_2^2}{h_2} \right] + \frac{E I_2}{2 l} \left[ \frac{\theta_3^2}{h_3} + \frac{\theta_4^2}{h_4} \right] + \frac{G J}{2 l} \cdot \theta_3 + \dot{N} \cdot \delta - \frac{l \cdot \dot{N}^2}{2 E A}.
\end{align*}
\]

(2.82)

The condition of vanishing of the first variation of \( \pi \), which is denoted here as \( \pi^* \), in eqn (2.82) due to a variation in \( \dot{N} \), which is denoted here as \( \dot{N}^* \), is given by

\[
\frac{\pi^*}{l} = 0 = -\frac{1}{2} \left[ \frac{\theta_1^2}{h_1} \cdot \frac{d^2 h_1}{d n_1^2} + \frac{\theta_2^2}{h_2} \cdot \frac{d^2 h_2}{d n_2^2} \right] \cdot \dot{N}^* + \frac{\theta_3^2}{h_3} \cdot \frac{d^2 h_3}{d n_3^2} + \frac{\theta_4^2}{h_4} \cdot \frac{d^2 h_4}{d n_4^2} \cdot \dot{N}^* + \frac{\delta}{l \cdot E A} \cdot \dot{N}^* \cdot \dot{N}^*.
\]

(2.83)

Equation (2.83) leads clearly to the relation between \( \delta \) and the generalized forces as given in eqn (2.72).

The reason for using the "mixed" form for the stretching energy in eqn (2.77) is now clear from the above result. By using a similar mixed form for the increment of stretching energy, the incremental axial stretch vs incremental generalized force relation can be derived in a manner analogous to that used in obtaining eqn (2.83) from eqn (2.82). This incremental relation, which is, by definition, piecewise linear, may easily be inverted, as demonstrated in the following. Also, it is shown in the following that eqn (2.82) forms the basis for generating an explicit form for the "tangent-stiffness" of the member.

The increment of the internal energy of the member, which is denoted as \( \Delta \pi \), involving terms up to second order in the "incremental" variables, \( \Delta \theta_i \), \( \Delta \theta_i \), \( \Delta \dot{N} \), and \( \Delta \delta \) can be seen from eqn (2.82) as

\[
\Delta \pi = \frac{E I_1}{l} \left[ \frac{\theta_1^2}{h_1} \cdot \Delta \theta_i + \Delta \theta_i \cdot \Delta \theta_i \cdot \frac{1}{h_1} \right] + \frac{E I_2}{2 l} \left[ \frac{\theta_3^2}{h_3} \cdot \Delta \theta_i + \Delta \theta_i \cdot \Delta \theta_i \cdot \frac{1}{h_3} \right] + \frac{G J}{2 l} \cdot \Delta \theta_i \cdot \Delta \dot{N} + \Delta \theta_i \cdot \Delta \delta - \frac{l \cdot \Delta N^2}{2 E A}.
\]

(2.84)

In the above equation, it should be recalled that \( h_i \) and \( h_i \) (i = 1, 2) are functions of \( \ddot{N} \).
\[ \Delta \phi \] may be expressed in terms of \( \phi_i \), and \( \phi_j \) \( (i = 1, 2, 3; j = 1, 2) \) and/or their increments. Henceforth, we used the notation for the vector \( d'' \) that

\[ d'' = [u_1; i_1; u_2; i_2; u_3; i_3; \phi_1; i_1; \phi_2; i_2; \phi_3; i_3] \]  

(2.85)
as shown in Fig. 1.

In terms of the increment \( \Delta d'' \), eqn (2.84) may be written as

\[ \Delta \pi = \frac{1}{2} \Delta d'' \cdot A_{ad} \cdot \Delta d'' + \Delta N \cdot A_{ad} \cdot \Delta d'' \]

\[ + A_{am} \cdot \Delta d'' + B \cdot \Delta d'' \]  

(2.86)
The details of \( A_{ad}, A_{am}, B \), and \( B \) are as shown in Appendix A.

By setting to zero the variation of \( \Delta \pi \) in eqn (2.86) with respect to \( \Delta N \), one obtains the following relation as

\[ A_{am} \cdot \Delta d'' + B = -A_{ad} \cdot \Delta N. \]  

(2.87)
Thus, the above equation is the incremental counterpart of \( \delta \) vs the generalized force relation obtained in eqn (2.83). Unlike the non-linear relation in eqn (2.83), the piecewise linear relation, eqn (2.87), can be inverted to express \( \Delta N \) in terms of the generalized displacements as

\[ \Delta N = \frac{1}{A_{am}} [A_{ad} \cdot \Delta d'' + B]. \]  

(2.88)
Substituting eqn (2.88) into eqn (2.86), one obtains the internal energy expression as

\[ \Delta \pi = \frac{1}{2} \Delta d'' \cdot K'' \cdot \Delta d'' + \Delta d'' \cdot R'' - \frac{B^2}{2A_{am}}, \]  

(2.89)
where \( K'' \) is the tangent stiffness matrix of member/element,

\[ = A_{ad} - \frac{1}{A_{am}} \cdot A_{am} \cdot A_{ad}, \]  

(2.90)
and \( R'' \) is the internal generalized force vector for member/element,

\[ = B - \frac{B}{A_{am}} \cdot A_{ad}. \]  

(2.91)
Recall that the tangent stiffness matrix and the internal force vector are written in the member co-ordinate system as shown in Fig. 1. Thus, it is necessary to transform \( d'' \) from a member co-ordinate system to a global co-ordinate system.

It should be emphasized once again that the tangent stiffness matrix \( K'' \) of eqn (2.90) is given an explicit expression, as in Appendix A; and likewise, the internal generalized force vector \( R'' \) is also given explicitly. No member-wise numerical integrations are involved. During the course of deformation of the frame, once the nodal displacements of the frame at stage \( C_N \) are known, the tangent stiffness of each of the members and hence of the frame structure, which governs the deformation of the frame from stage \( C_N \) to an incrementally close neighboring stage \( C_{N+1} \), can be easily evaluated from eqn (2.90). This distinguishing feature of the present formulation renders the large deformation analysis of framed structures much more computationally inexpensive than the standard incremental (updated or total Lagrangean) finite element formulations reported in current literature [8]. Numerical examples illustrating this are given later.

3. SOLUTION STRATEGY

Although a number of solution procedures are available for non-linear structural analyses, a reliable approach to trace the structural response near limit points, and in a post-buckled range, is the arc-length method which was proposed by Ricks [9] and Wempner [10] and modified by Crisfield [11, 12] and Ramm [13]. This method is the incremental/iterative procedure which represents a generalization of the displacement control approach. The arc-length method, in which the Euclidean norm of the increment in the displacement and load space is adopted as the prescribed increment, allows one to trace the equilibrium path beyond limit points such as in snap-through and snap-back phenomena.

A full description of the arc-length method, as presently adopted, is given in Ref. [14] and is not repeated here.

4. NUMERICAL EXAMPLES

Several numerical examples are considered in this section, to demonstrate the validity of the present study.

The first example is that of the so-called Williams’ toggle frame, which was first treated by Williams [15] and later analyzed by Wood and Zienkiewicz [16] and Karamanlidis et al. [17]. A schematic of the structure is shown in Fig. 5. The structure has a semispan of

![Fig. 5. Schematic diagram of Williams’ toggle frame.](image-url)
Tangent-stiffness of a finitely deformed 3-D beam

12.943 (in.), a raise of 0.386 (in.), and is composed of two identical members, each with a rectangular cross section of width 0.753 (in.), depth of 0.243 (in.), and $E = 1.03 \times 10^7$ (psi). Each member of the frame is modeled by a single element of the type derived in this paper. Figure 6 shows the presently computed relation between the external load $P$ and the conjugate displacement $\delta$, and also that between $P$ and the horizontal reaction ($R$) at the fixed end. Also, shown in Fig. 6 are the comparison experimental results of Williams [15] as well as the numerical solutions obtained by Wood and Zienkiewicz [16]. Excellent agreement between all the three sets of results may be noted. However, the efficiency of the present method is clearly borne out by the facts that: (a) the present solution uses one element to model each member, while Ref. [16] uses five elements to model each member; and (b) no numerical integrations are used, in the present, to derive the tangent stiffness of the element during each step of deformation, since an explicit expression for such is given.

Prior to consideration of space frames, we consider the case of large-deformation bending response of a single member, through the example of a cantilever beam subject to a transverse load at the tip, as shown in Fig. 7. It is seen that the present results, using just two elements, agree excellently with those of Bathe and Bolourchi [18]. The relative rotation at tip, as computed from the present procedure, is shown in Fig. 8 and is found to agree excellently with an independent analytical solution.

We now consider the example of a space frame, whose geometry and pertinent material properties are shown in Fig. 9.

The results for the case of axial loading are shown in Fig. 10. In this case, to trigger global buckling, a loading imperfection of magnitude ($P/1000$) is considered in the transverse direction (where $4P$ is the axial load) as shown in the inset of Fig. 10. Also shown in Fig. 10 is the comparison response of the structure when modeled as a space truss with and without local buckling [19]. An examination of Fig. 10 shows that the response of the space frame under an axial load system indicated in Fig. 10 is nearly the same as that predicted when a space-truss-type model is employed and when the local (member) buckling is accounted for. (Note that both the responses, i.e. those predicted by a space-frame modeling as well

![Image of Figure 6: Variations of load-point displacement and support reaction with load, for Williams' toggle in the post-buckling range.](image-url)
Fig. 7. Deflections of a cantilever under a concentrated load.

Fig. 8. Rotations of a cantilever under a concentrated load.
Tangent-stiffness of a finitely deformed 3-D beam

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Longerons</th>
<th>Diagonal Batters</th>
<th>Short Longerons</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_A$</td>
<td>$7.08 \times 10^6$</td>
<td>$2.70 \times 10^5$</td>
<td>$1.65 \times 10^6$</td>
</tr>
<tr>
<td>$E_I$</td>
<td>$2.16 \times 10^6$</td>
<td>$6.43 \times 10^4$</td>
<td>$2.20 \times 10^5$</td>
</tr>
<tr>
<td>$G_J$</td>
<td>$1.03 \times 10^5$</td>
<td>$4.77 \times 10^4$</td>
<td>$1.40 \times 10^5$</td>
</tr>
</tbody>
</table>

Fig. 9. Schematic of a 12-bay space frame.

Fig. 10. Deflections at free end under axial loads.
as a space-truss modeling with member buckling, are considerably more flexible than that predicted by a space-truss modeling without local buckling being considered.) This points to the potential use of space-truss-type modeling with local buckling being accounted for.

The results for the case of transverse (bending) loading are shown in Fig. 11, when the structure is modeled as a space frame. Also included in Fig. 11 are the comparison results [19], when the structure was modeled as a space truss when local buckling was suppressed or accounted for. Figure 11 reveals that the bending response of the structure, when modeled as a space frame, is nearly similar to that of a space truss when local (individual) buckling is properly accounted for.

5. CLOSURE

In this paper, simple and effective procedures of explicitly determining the tangent stiffness matrix, and an arc-length method, have been presented for analyzing the large deformation and post-buckling response of (three-dimensional) space frames. Certain salient features of the present methodology are indicated below.

(1) An explicit expression (i.e. requiring no further element-numerical integration) is given for the "tangent-stiffness" matrix of an individual element (which may then be assembled in the usual fashion to form the "tangent-stiffness matrix" of the frame structure). The formulation that is employed accounts for (a) arbitrarily large rigid rotations and translations of the individual element, (b) the non-linear coupling between the bending and axial stretching motions of the element. Each element can withstand bending moments, a twisting moment, transverse shear forces, and an axial force.

(2) The presently proposed simplified methodology has excellent accuracy in that only one element may be sufficient, in most cases (of practical interest in the behavior of structural frames), to model each member of the frame structure. Inasmuch as the relative (non-rigid) rotation of a differential segment of the present element is restricted to be small, a single element alone is not enough to model the post-buckling response of an entire beam column undergoing excessively large deformations as in an elastica. However, when considered as a part of a practical frame structure, the situation of each member of the frame undergoing abnormally large deformations, as in an elastica, represents a pathological case.

(3) Because of (1) and (2), the present method is by far the most computationally inexpensive method to analyze three-dimensional (space) frame structures and, therefore, is of considerable potential applicability in analyzing large practical space-structures.

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REFERENCES

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**APPENDIX A**

Representations of matrices forming tangent stiffness matrix of a frame member

The vectors for representing eqn (7.86), herein, are defined as

\[ M' = \begin{bmatrix} i_1 M_1 & i_2 M_2 & i_3 M_3 & i_4 M_4 \end{bmatrix} \]

\[ T' = \begin{bmatrix} i_1 T_1 & i_2 T_2 & i_3 T_3 & i_4 T_4 \end{bmatrix} \]

\[ C' = \begin{bmatrix} E I_1 & E I_1 & E I_2 & E I_2 \end{bmatrix} \]

\[ \delta (A' J') \]

\[ V = \begin{bmatrix} -1 & 1 \end{bmatrix} \]

\[ E = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

\[ D' = \begin{bmatrix} \frac{d}{d \theta} (\frac{1}{\theta h_1}) & \frac{d}{d \theta} (\frac{1}{\theta h_2}) & \frac{d}{d \theta} (\frac{1}{\theta h_3}) & \frac{d}{d \theta} (\frac{1}{\theta h_4}) \end{bmatrix} \]

\[ = (B J') \]

\[ J = \text{unit vector (3 x 1)} \]

1. \( A_{ul} \)

\[ A_{ul} \] is represented as a (12 x 12) matrix as shown in Table A.1, which the components \( F_{ul}, G_{ul}, \) and \( H_{ul} \) are given by

\[ F_{ul} = M_1 \frac{\partial \bar{T}_1}{\partial h_1} + \frac{\partial \bar{T}_1}{\partial h_2} A_{ul} \]

\[ G_{ul} = M_1 \frac{\partial \bar{T}_1}{\partial \theta} \frac{\partial \theta}{\partial \theta} A_{ul} \]

\[ H_{ul} = M_1 \frac{\partial \bar{T}_1}{\partial \theta} \frac{\partial \theta}{\partial \theta} A_{ul} \]

where \( i, j = 1, 2, 3; m, n = 1, 2. \)

2. \( A_{ul} \)

\[ A_{ul} \] is represented as a (12 x 1) vector as shown in Table A.2, which the components \( L_1 \) and \( N_1 \) are given by

\[ L_1 = T_1 B_{ul} \frac{\partial \bar{T}_1}{\partial \theta} + \frac{\partial \bar{T}_1}{\partial \theta} \]

\[ N_1 = T_1 B_{ul} \frac{\partial \bar{T}_1}{\partial \theta} \]

where \( i = 1, 2, 3; j = 1, 2. \)

3. \( A_{ul} \)

\[ A_{ul} \] is a scalar factor as follows:

\[ A_{ul} = \frac{1}{2 E I_1} \begin{bmatrix} \alpha \frac{\partial \bar{T}_1}{\partial h_1} + \beta \frac{\partial \bar{T}_1}{\partial h_2} \end{bmatrix} + \frac{1}{2 E I_2} \begin{bmatrix} \alpha \frac{\partial \bar{T}_1}{\partial h_1} + \beta \frac{\partial \bar{T}_1}{\partial h_2} \end{bmatrix} \]

\[ \times \begin{bmatrix} \alpha \frac{\partial \bar{T}_1}{\partial h_1} + \beta \frac{\partial \bar{T}_1}{\partial h_2} \end{bmatrix} \]

\[ - \frac{1}{E A} \]

4. \( B_{ul} \)

\[ B_{ul} \] is represented as a (12 x 1) vector as shown in Table A.3, which the components \( R \) and \( S \) are given by

\[ R = M_1 \frac{\partial \bar{T}_1}{\partial \theta} + N \]

\[ S = M_1 \frac{\partial \bar{T}_1}{\partial \theta} \]

where \( i = 1, 2, 3; j = 1, 2. \)

5. \( B_{ul} \)

\[ B_{ul} \] is a scalar factor as follows:

\[ B_{ul} = \frac{1}{2} \begin{bmatrix} \alpha \frac{\partial \bar{T}_1}{\partial h_1} + \beta \frac{\partial \bar{T}_1}{\partial h_2} + \gamma \frac{\partial \bar{T}_1}{\partial h_3} + \delta \frac{\partial \bar{T}_1}{\partial h_4} \end{bmatrix} + \frac{1}{E A} \]

\[ - \frac{1}{E A} \]
APPENDIX B

Approximations of relation between total and relative rotations of a frame member

It is necessary that eqns (2.31) and (2.32) are approximated to form the tangent stiffness matrix for frame-type elements because eqns (2.31) and (2.32) have high order terms and are too complicated to formulate. To keep the formulations simple and yet to achieve the intended purpose, the following approximations to eqns (2.24), (2.27) and (2.28) are made and used in eqns (2.31) and (2.32) to obtain

\[ \tan \frac{\theta_1}{2} = -(h \cdot 1A + l \cdot 1B + m \cdot 1C) \]  
\[ \text{(B.1)} \]

\[ \tan \frac{\theta_2}{2} = -(h \cdot 1D + l \cdot 1E + m \cdot 1F) \]  
\[ \text{(B.2)} \]

\[ \tan \frac{\theta_3}{2} = -(h \cdot r + l \cdot s + m \cdot t) \]  
\[ \text{(B.3)} \]

\[ \tan \frac{\gamma_1}{2} = \tan \frac{\theta_1}{2} + \left( \tan \frac{\theta_2}{2} - \tan \frac{\theta_1}{2} \right) \cdot 1A \]
\[ + \left( \tan \frac{\theta_3}{2} - \tan \frac{\theta_2}{2} \right) \cdot 1B \]
\[ + \left( \tan \frac{\theta_3}{2} - \tan \frac{\theta_1}{2} \right) \cdot 1C \]  
\[ \text{(B.4)} \]

\[ \tan \frac{\gamma_2}{2} = \tan \frac{\theta_1}{2} + \left( \tan \frac{\theta_2}{2} - \tan \frac{\theta_1}{2} \right) \cdot 1D \]
\[ + \left( \tan \frac{\theta_2}{2} - \tan \frac{\theta_3}{2} \right) \cdot 1E \]
\[ + \left( \tan \frac{\theta_3}{2} - \tan \frac{\theta_1}{2} \right) \cdot 1F \]  
\[ \text{(B.5)} \]

\[ \tan \frac{\gamma_3}{2} = \tan \frac{\theta_1}{2} + \left( \tan \frac{\theta_3}{2} - \tan \frac{\theta_1}{2} \right) \cdot r \]  

Table A1. Matrix of $A_{\omega}$

Table A2. Vector of $A_{\omega}$

Table A3. Vector of $B_{\omega}$
Substituting eqns (2.14), (2.15) and (2.24) into eqns (B.1) to (B.3), one obtains the following equations as

\[
\begin{align*}
\tan \frac{\theta_1}{2} &= \frac{1}{1 + \varepsilon} \left\{ 2r \left( \tan \frac{\theta_1}{2} - \tan \frac{\theta_2}{2} - \tan \frac{\theta_3}{2} \right) \right. \\
&\left. + s \left( 1 - \tan^2 \frac{\theta_1}{2} + \tan^2 \frac{\theta_2}{2} - \tan^2 \frac{\theta_3}{2} \right) \right\} \\
&\left. + 2t \left( \tan \frac{\theta_2}{2} \tan \frac{\theta_3}{2} + \tan \frac{\theta_1}{2} \right) \right\} \\
\tan \frac{\theta_2}{2} &= \frac{1}{1 + \varepsilon} \left\{ r \left( 1 + \tan^2 \frac{\theta_2}{2} - \tan^2 \frac{\theta_1}{2} - \tan^2 \frac{\theta_3}{2} \right) \right. \\
&\left. + 2s \left( \tan \frac{\theta_1}{2} \tan \frac{\theta_3}{2} + \tan \frac{\theta_2}{2} \right) \right\}
\end{align*}
\]