LARGE DISPLACEMENT ANALYSIS OF PLATES
BY A STRESS-BASED FINITE
ELEMENT APPROACH

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Abstract—A mixed-hybrid incremental variational formulation, involving orthogonal rigid rotations and a symmetric stretch tensor, is proposed for finite deformation analysis of thin plates and shells. Isoparametric eight-noded elements are based upon the Kirchhoff–Love hypotheses, the assumption of plane stress, and the moderately large rotations of Von Karman plate theory. Semilinear elastic isotropic material properties are assumed, and the right polar damnation of the deformation gradient is used. The symmetrised Biot–Luré (Jaumann) stress measure gives a unique complementary energy density and a set of variational principles with a priori satisfaction of linear momentum balance, a posteriori angular momentum balance, and interelement traction reciprocity by means of Lagrange multipliers.

The incremental modified Newton-Raphson solution procedure is generated by a truncated Taylor series expansion of the functionals in a total Lagrangian formulation. The theory is applied to laterally loaded and buckled thin plates, and numerical results are compared with truncated series solutions.

INTRODUCTION

Material costs and weight reduction priorities have caused renewed interest in the load-carrying capacity of plates and shells at large deformations. At the same time, the subject has benefited from a refinement of the underlying theories of nonlinear behaviour of continua [1] and is particularly ripe for advances in discrete computational techniques. A unified treatment of the developments to date has been presented by Atluri [2] in an extensive theoretical investigation (Part I) to which this practical applications paper is the computational complement. However, the bibliographic chain for this research commences with a pioneering paper by Fraeijs de Veubeke in 1972 [3]. Most variational formulations in the area are based upon the principle of minimum potential energy, but the assumed-stress approach has appeared [4–7] in hybrid-stress and Reissner-variational formats. However, of the two most commonly used large deformation stress measures, the unsymmetrical first Piola–Kirchhoff (Piola–Lagrange) and the symmetrical second Piola–Kirchhoff (Kirchhoff–Trefitz), neither can form a convenient complementary energy functional. In the first case, Ogden [8] has shown that the inverse of the stress–deformation gradient relation is multivalued; and, in the second, displacements and stresses are inseparably interrelated. Nonetheless, the essential attributes of an effective functional, i.e. a uniquely invertible stress–strain relationship and a linear equilibrium expression suited to satisfaction by stress functions, can be obtained by employing the first Piola–Kirchhoff stress in conjunction with a rigid rotation tensor as the symmetrized Biot–Luré or Jaumann stress. This insight by Fraeijs de Veubeke has inspired a new methodology for finite deformation analysis [9–11] and underlies the theoretical derivation in [2]. The formulation, its alternatives and applications are discussed in a comprehensive paper by Atluri and Murakawa [12].

Classical first-order thin shell theory is adopted herein, with the midsurface as the reference surface of a bona fide curved two-dimensional element and the Kirchhoff–Love hypotheses defining the kinematic scheme. Although more complicated constitutive laws are available, a linear relationship is assumed between the Jaumann stress and the stretch tensor in a “semilinear” elastic isotropic homogeneous material. The theory is intended for application to problems characterized by large rotations and moderate stretching, but an especially convenient computational format results when the moderately large rotations of Von Karman thin plate theory are also considered. The added complexity of finite deformation analysis extends to the choice of co-ordinate system as well, and a total Lagrangian description with convected natural co-ordinates is selected for this work.

The right polar decomposition of the deformation gradient into an orthogonal rigid rotation tensor and a symmetrical positive definite stretch tensor is the
keystone of the analytic procedure and gives rise to a new pair of objective bending strain measures. Strain energy density also decouples into stretching and bending energies while rigorous Jaumann stress resultants and stress couples are derived in terms of the first Piola–Kirchhoff stress measure. A set of mixed-hybrid functionals is established by substituting these quantities and appropriate Legendre contact transformations into a general expression. A priori orthogonality of rotations and equilibrium of stresses is ensured through suitable functions while all other field equations are present as subsidiary equations and natural boundary conditions. In addition, the finite element discretization employs interelement boundary displacements and tractions to enforce displacement continuity and traction reciprocity in an \textit{a posteriori} weighted integral sense. The formulations are "mixed" due to the multifield use of stresses, rotations, and moments as elemental variables, in conjunction with global rotations and displacements.

These functionals are nonlinear expressions, but solutions can be conveniently found through a linearized incremental procedure involving modified Newton–Raphson iterations on a truncated Taylor series expansion. Derivations of the theoretical expressions appear in Part I, but a summary is included here for completeness. Numerical results from selected laterally and inplane-loaded plate examples illustrate the capabilities of this finite deformation analysis vs conventional series solutions.

**THEORY**

The kinematic scheme for large deformation problems requires a distinction between deformed (\(B\)) and undeformed (\(\mathcal{B}\)) configurations (Fig. 1). Let \(x'\) and \(y'\) be fixed right-handed Cartesian co-ordinates in the undeformed and deformed bodies, respectively, and \(\xi'\) a natural convected curvilinear co-ordinate system. The reference midsurface \(S\) of the undeformed thin shell is defined by the two co-ordinates \(\xi''\), while a generic point \(P_0\) on \(S\) has a position vector \(\mathbf{R}_0\) and the thickness co-ordinate \(\xi^3\) lies along the unit normal \(N\) to \(S\) at \(P_0\). Any arbitrary point \(P\) in the undeformed shell domain has the position vector

\[
\mathbf{R} = \mathbf{R}_0 + N \xi^3. \tag{1}
\]

Covariant base vectors on \(S\) are defined by

\[
A_\xi = \frac{\partial \mathbf{R}_0}{\partial \xi'}, \quad N = \frac{1}{2} \epsilon_{xy} A_x \times A_y; \quad A_3 = N, \tag{2}
\]

where \(|N| = 1\). Similarly, the base vectors at \(P\) are

\[
G_\xi = \frac{\partial \mathbf{R}}{\partial \xi'} = (\xi_0 - \xi^3 B) \cdot A_\xi; \quad G_3 = N \tag{3}
\]

with a symmetric undeformed curvature \(B\) thus:

\[
B = -\left( \frac{\partial N}{\partial \xi''} \right) A'' = B'' A, A^\ast. \tag{4}
\]

Deformation uniquely maps surface \(S\) and points \(P_0\), \(P\) on to deformed surface \(s\) and points \(p_0\), \(p\) which, owing to the convenience of convected co-ordinates, are still defined by \(\xi'\). Midsurface displacement \(P_0 p\) is \(u\), but overall displacements \(u_t = P p\) are defined by means of Love's first approximation which states that material fibers originally straight and normal to \(S\) are

![Fig. 1. Kinematic scheme for large deformation analysis.](image)
mapped, without stretching, onto fibers straight and normal to $s$. Hence, the position vectors of $p_0$ and $p$, respectively, are

$$r_0 = R_0 + u$$  \hspace{1cm} (5)
$$r - r_0 + n \xi^3 = R_0 + u + n \xi^3$$  \hspace{1cm} (6)

where $n$ is the unit normal to $s$, and the total displacement of $P$ is:

$$u = r - R = u + (n - N) \xi^3.$$  \hspace{1cm} (7)

Deformed base vectors $a_b$ and $g_b$ at $p_0$ and $p$ and a symmetrical deformed curvature tensor $\mathbf{R}$ of $s$ can be defined in direct analogy with $A_s$, $G_s$, and $\mathbf{B}$ [2].

The deformation gradient tensor $\mathbf{F}$ relates an undeformed differential vector $dR$ at $p$ to its image $d\mathbf{r}$ at $p$ in the deformed domain as follows:

$$d\mathbf{r} = F \cdot dR = d\mathbf{R} \cdot F^T.$$  \hspace{1cm} (8)

The general form of $\mathbf{F}$ is

$$F = (\nabla u_p)^T$$  \hspace{1cm} (9)

$$= \frac{\partial v^i}{\partial x^j} a_j$$  \hspace{1cm} (10)

$$= g_{a} G + n N.$$  \hspace{1cm} (11)

Provided $\mathbf{F}$ is nonsingular, it can be decomposed into an orthogonal rigid rotation tensor $\mathbf{B}$ and symmetrical positive definite right and left stretch tensors $\mathbf{U}$ and $\mathbf{Y}$, respectively, thus:

$$\mathbf{E} = \mathbf{B} \cdot \mathbf{U} = \mathbf{Y} \cdot \mathbf{B}.$$  \hspace{1cm} (12)

Properties of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{U}$, and $\mathbf{Y}$ are presented in [2]. Confining attention to the right polar decomposition, $\mathbf{U}$ is sometimes written as $(\mathbf{I} + \mathbf{b})$, where $\mathbf{b}$ is the engineering strain tensor [9], and, for unit normals $N$ and $n$ in a Kirchhoff-Love formulation:

$$n = F \cdot N = B \cdot U \cdot N = B \cdot N.$$  \hspace{1cm} (13)

Through use of a set of identities [2], the deformed basis $g_b$ can be represented in two ways:

$$g_a = A_a + u_a + (B \cdot N)_a \xi^3$$  \hspace{1cm} (14)
$$g_a = (U - \xi^3 \mathbf{b}) \cdot a_a,$$  \hspace{1cm} (15)

and accordingly the following forms of $\mathbf{E}$ are available:

$$\mathbf{E} = (B \cdot U_0 - \xi^3 \mathbf{b} \cdot B \cdot U_0) \cdot A_a G_a^* + (B \cdot N N)$$  \hspace{1cm} (16)

$$= (A_a + u_a + (B \cdot N)_a \xi^3) G^* + (B \cdot N) N.$$  \hspace{1cm} (17)

Midplane stretching $U_b$ has already been introduced, but two new objective derived bending strains $\bar{b}^*$ and $\bar{b}^*$, will also be relevant to future developments.

Bending strains $b$ and $\bar{b}^*$ are symmetrical while $\bar{b}^*$, in general, is not.

A number of alternate finite deformation stress and strain measures can be defined by application of Nansen's Law, which relates corresponding oriented differential areas in the undeformed and deformed configurations. These definitions are motivated by consideration of the differential traction $df$ acting upon deformed oriented surface $n \mathbf{d}a$ at $p$ in $b$ (Fig. 1).

$$df = (d \mathbf{a} \mathbf{n}) \cdot \tau$$  \hspace{1cm} (20)

$$= (d \mathbf{A} \mathbf{N}) \cdot \mathbf{g} \cdot \mathbf{E}^T$$  \hspace{1cm} (21)

$$= (d \mathbf{A} \mathbf{N}) \cdot \mathbf{L}$$  \hspace{1cm} (22)

Clearly, $\mathbf{E}$ is true (Cauchy) stress; but, since it is defined in the unknown deformed domain, it cannot be calculated directly. The most commonly used finite deformation stress measure is the symmetrical second Piola-Kirchhoff ($\mathbf{g}$). However, the nominal undeformed stress $\mathbf{L}$ (unsymmetrical first Piola-Kirchhoff), the generally unsymmetrical Biot-Lurie stress $\mathbf{L}^*$, and the symmetrized Biot-Lurie (Jaumann) stress $\mathbf{L}$ prove to be more useful, where

$$\mathbf{L} = J^{-1} (E^{-1} \cdot \tau)$$  \hspace{1cm} (23)

$$\mathbf{L}^* = J E^{-1} \cdot \tau \cdot \mathbf{B} = \mathbf{L} \cdot \mathbf{B}$$  \hspace{1cm} (24)

$$\mathbf{L} = \mathbf{L}^* + \mathbf{L}^* / 2.$$  \hspace{1cm} (25)

Stress and moment resultants for plates and shells can now be established in terms of these measures in familiar fashion. Considering the deformed differential element of Fig. 2, the traction on a strip of height $d \xi^3$, unit curvilinear length ($d t^2 = 1$), and distance $\xi^3$ above the reference midsurface is

$$d N_N^a = \left[ \frac{E^a}{\sqrt{\mathbf{g} \mathbf{E}^T}} \frac{1}{\sqrt{\mathbf{g} \mathbf{g}^T}} \right] d \xi^3 \cdot \tau^a \mathbf{g} \mathbf{e}_u \mathbf{e}_u$$  \hspace{1cm} (26)

and accordingly the following forms of $\mathbf{E}$ are available:

$$\mathbf{E} = (B \cdot U_0 - \xi^3 \mathbf{b} \cdot B \cdot U_0) \cdot A_a G_a^* + (B \cdot N N)$$  \hspace{1cm} (16)

$$= (A_a + u_a + (B \cdot N)_a \xi^3) G^* + (B \cdot N) N.$$  \hspace{1cm} (17)

while the corresponding stress resultant tensor is
Similarly, the Cauchy stress-couple, or moment, resultant per unit \( \zeta \) in the deformed configuration is

\[
\mathbf{M}^* = \int_0^1 \mathbf{n} \times \frac{1}{\sqrt{g}} \frac{\partial \mathbf{E}^*}{\partial \zeta} \, d\zeta^3
\]

and the corresponding tensor components are

\[
\mathbf{m}^* = \mathbf{M}^*/\sqrt{a}, \quad \mathbf{r}^* = \mathbf{R}^*/\sqrt{a}.
\] (30)

As before, the Cauchy stress-couple tensor is:

\[
\mathbf{r} = a_k (\mathbf{m}^*).
\] (31)

Similar stress resultants, in appropriate base vectors, can be derived for other stress measures [2]. Jaumann stress resultants and couples are logically defined as follows:

\[
\mathbf{R} = (\mathbf{A} \cdot \mathbf{B} + \mathbf{B}^T \cdot \mathbf{A}^T)/2
\] (32)

\[
\mathbf{L} = (\mathbf{A} \cdot \mathbf{B} + \mathbf{B}^T \cdot \mathbf{A}^T)/2.
\] (33)

Different stress measures and resultants have been used by other researchers, but this particular set forms the basis of subsequent theory. Furthermore, it transpires that \( \mathbf{n} \) is most useful in components of the undeformed basis \( \mathbf{A} \), thus:

\[
\mathbf{n} = \mathbf{n}^* + \mathbf{n}^\alpha \mathbf{A}^\alpha + \mathbf{n}^\beta \mathbf{A}^\beta.
\] (34)

Linear and angular momentum balance conditions for first Piola–Kirchhoff stress and moment resultants are [2]

\[
\frac{\partial}{\partial \zeta^3} \{\sqrt{\gamma}(A) \mathbf{n}^*\} + \{\sqrt{\gamma}(A) \mathbf{n}^\alpha\} + \{\sqrt{\gamma}(A) \mathbf{n}^\beta\} = 0.
\] (35)

wherein the linear form of eqn (35) makes \( \mathbf{R} \) ideally suited to a mixed-hybrid finite element algorithm.

In infinitesimal deformations, the strain energy density for a linear elastic isotropic solid in plane stress is given by

\[
W_0 = \frac{E_v}{2(1-\nu)} \{\mathbf{e} \cdot \mathbf{e}^*\} + \mu \mathbf{e} \cdot \mathbf{e}^*,
\] (37)

and by choosing the large deformation shell strain measure in this case as

\[
\epsilon = (\mathbf{U}_0 - L) - \xi^P(\mathbf{U}_0 - \mathbf{B})
\] (38)

it is possible to define the following "semilinear" elastic isotropic strain energy density:

\[
W_0 = W_0(\mathbf{E}^T, \mathbf{E})
\]

\[
= \frac{E_v}{2(1-\nu)} \{(\mathbf{U}_0 - L) - \xi^P(\mathbf{U}_0 - \mathbf{B})\}^2
\]

\[
+ \mu \{(\mathbf{U}_0 - L) - \xi^P(\mathbf{U}_0 - \mathbf{B})\}\{(\mathbf{U}_0 - L)
\]

\[
- \xi^P(\mathbf{U}_0 - \mathbf{B})\}. \] (39)

By suitable approximation and integration through the thickness, it can be shown that the resulting strain energy per unit undeformed mid-surface area decouples into stretching and bending components thus:

\[
\mathcal{W}_0 = \frac{E_h}{(1-\nu)} \{\sqrt{2}(\mathbf{U}_0 - L)\}_{L}^2
\]

\[
+ \frac{(1-\nu)}{2} \{(\mathbf{U}_0 - L)\}_{L}\}
\]

\[
+ \frac{E_h}{12(1-\nu)} \{\sqrt{2}(\mathbf{U}_0 - \mathbf{B})\}_{L}^2
\]

\[
+ \frac{(1-\nu)}{2} \{(\mathbf{U}_0 - \mathbf{B})\}_{L}\}
\]

\[
= \mathcal{W}_{00}(\mathbf{U}_0) + \mathcal{W}_{0b}(\mathbf{U}_0). \] (40)

Two significant Legendre contact transformations, necessary in the elimination of \( \mathbf{U}_0 \) and \( \mathbf{U}_0^* \) strain measures, can be established from eqn (40). These are

\[
\mathcal{W}_{00}(\mathbf{a}) = \mathbf{a} : \mathbf{U}_0 - \mathcal{W}_{0b}(\mathbf{U}_0) \] (41)

\[
\mathcal{W}_{0b}(\mathbf{r}) = -\mathbf{r} : \mathbf{U}_0^* - \mathcal{W}_{0b}(\mathbf{U}_0^*). \] (42)

Functional \( F_i \) of eqn (43) corresponds [2] to a version, in the current shell kinematic scheme, of a general Hu–Washizu mixed principle for elastic materials. The success of this principle, which was first proposed by Fraeijs de Veubeke [3] and gener-
alized by Atluri and Murakawa [12], lies in the independence of stress and rigid rotation variables.

$$F_i(U_0, \hat{u}^*, u, \hat{\theta}, \hat{\theta}, \hat{\theta})$$

$$= \sum_i \left\{ \int_{\Omega} \left( W_{\alpha}(U_0, \hat{u}^*) + \lambda \cdot T : \left( (A_u + u_a) A^{*} - \hat{\theta} \cdot u, \sqrt{(A)} d \xi^1 d \xi^2 \right) \right. \right.$$  

$$- \int_{\Omega} \left[ [N \cdot \hat{u} + \lambda \cdot \left( (B - L) \cdot N \right) \sqrt{(A)} d \xi^1 d \xi^2 \right.$$  

$$- \int_{\Omega} \left[ v_0 \left[ \right. \right.$$  

$$+ \lambda \cdot \left( (B - L) \cdot N \right) \sqrt{(A)} d \xi^1 d \xi^2 \right.$$  

This expression contains the traction (c_m), displacement (c_m), and interelement boundaries (p_m) of a conventional finite element discretization. Inter-element displacements \( \hat{u} \) and rotations \( \hat{\theta} \) act as Lagrange multipliers, in accordance with the fundamental philosophy of the mixed-hybrid approach. After appropriate rearrangement, a variation of \( F_i \) with respect to independent variables yields the constitutive law, compatibility, local momentum balance, traction, and displacement boundary conditions, as well as interelement continuity and traction reciprocity.

From a computational standpoint, functional \( F_i \) has too many variables and can be usefully modified. One of the merits of the Jaumann stress formulation is that it generates a uniquely invertible constitutive law thus:

$$\left( U_0 - L_0 \right) = \left( \frac{1 + v}{E_h} \right) \left[ \lambda \left( \frac{1 + v}{1 + v} \right) \left( \lambda : L_0 \right) \right]$$  

$$\left( \hat{u}^* - \hat{\theta} \right) = - \frac{12}{h^2} \left( \frac{1 + v}{E} \right) \left[ \lambda : B - \left( \frac{v}{1 + v} \right) \left( \lambda : \hat{L}_0 \right) \right]$$  

Furthermore, the decoupled stretching and bending complementary energy densities per unit undeformed midsurface area [eqns (41) and (42)] become

$$W_{\alpha}(\lambda) = \lambda : \lambda^{*} + \left( \frac{1 + v}{2Eh} \right) \left[ \lambda : \lambda^{*} - \left( \frac{v}{1 + v} \right) \left( \lambda : \lambda^{*} \right) \right]$$  

The stage is set, therefore, for the a priori elimination of \( U_0 \) by eqns (41) and (42), \( \hat{u}^* \) by invoking compatibility [2], and \( u \) and \( p \) by enforcing equilibrium [eqn (35)]. The resulting functional

$$F_i(\lambda, \hat{u}, \hat{\theta}, \hat{\theta}) = \sum_i \left\{ \int_{\Omega} \left( - W_{\alpha}(\lambda) \right. \right.$$  

$$+ \int_{\Omega} \left( [N \cdot \hat{u} + \lambda \cdot \left( (B - L) \cdot N \right) \sqrt{(A)} d \xi^1 d \xi^2 \right.$$  

$$- \int_{\Omega} \left[ v_0 \left[ \right.$$  

$$+ \int_{\Omega} \left[ v_0 \left[ \right.$$  

represents moment resultants in terms of the rigid rotation tensor \( \hat{\theta} \). Hence, as secondary quantities, these are not likely to be accurately computed in the numerical algorithm. A superior procedure involves introducing eqn (42), in addition to eqn (45), and using the identity

$$\int_{\Omega} \left( \lambda \cdot \lambda^{*} : (B - L) \cdot N \right) \sqrt{(A)} d \xi^1 d \xi^2$$  

$$= \int_{\Omega} \left( \lambda \cdot \lambda^{*} \right) \frac{1}{E} \left( \lambda : L_0 \right)$$  

$$- \int_{\Omega} \left( [\lambda \cdot \lambda^{*} : \left( (B - L) \cdot N \right) \sqrt{(A)} d \xi^1 d \xi^2 \right.$$  

The stage is set, therefore, for the a priori elimination of \( U_0 \) by eqns (41) and (42), \( \hat{u}^* \) by invoking compatibility [2], and \( u \) and \( p \) by enforcing equilibrium [eqn (35)]. The resulting functional

$$F_i(\lambda, \hat{u}, \hat{\theta}, \hat{\theta}) = \sum_i \left\{ \int_{\Omega} \left( - W_{\alpha}(\lambda) \right. \right.$$  

$$+ \int_{\Omega} \left( [N \cdot \hat{u} + \lambda \cdot \left( (B - L) \cdot N \right) \sqrt{(A)} d \xi^1 d \xi^2 \right.$$  

$$- \int_{\Omega} \left[ v_0 \left[ \right.$$  

$$+ \int_{\Omega} \left[ v_0 \left[ \right.$$  

to incorporate \( \lambda \) directly, in the modified functional \( F_i \):

$$F_i(\lambda, \hat{u}, \hat{\theta}, \hat{\theta}) = \sum_i \left\{ \int_{\Omega} \left( - W_{\alpha}(\lambda) \right. \right.$$  

$$+ \lambda \cdot \lambda^{*} \cdot A^{*} \cdot [A] + \lambda \cdot \lambda^{*} \cdot (B - L) \cdot N \sqrt{(A)} d \xi^1 d \xi^2$$  

$$= \int_{\Omega} \left( \lambda \cdot \lambda^{*} \right) \frac{1}{E} \left( \lambda : L_0 \right)$$  

$$- \int_{\Omega} \left( [\lambda \cdot \lambda^{*} : \left( (B - L) \cdot N \right) \sqrt{(A)} d \xi^1 d \xi^2 \right.$$  

$$+ \int_{\Omega} \left[ v_0 \left[ \right.$$  

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$$= \sum_i \left\{ \int_{\Omega} \left( - W_{\alpha}(\lambda) \right. \right.$$  

$$+ \lambda \cdot \lambda^{*} \cdot A^{*} \cdot [A] + \lambda \cdot \lambda^{*} \cdot (B - L) \cdot N \sqrt{(A)} d \xi^1 d \xi^2$$  

$$+ \int_{\Omega} \left[ v_0 \left[ \right.$$  

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Angular momentum balance, compatibility and all relevant boundary conditions are the \textit{a posteriori} conditions of $F_1$.

The use of global and local variables gives the mixed-hybrid approach its unparalleled versatility among finite element methods. With respect to global variables, continuity of nodal displacement and rotations parameters is guaranteed by a suitable element shape function interpolation. Nonetheless, slope-displacement compatibility must also be ensured, and two options exist for this purpose. In the first, both out-of-plane rotations are defined as derivatives of transverse displacements $w$ through a $C^1$ continuous Hermitian polynomial shape function field: while, in the second, only the tangential slope ($\partial w/\partial x^t$) is derived from the $C^0$ continuous Serendipity interpolation for $w$, and the normal slope is admitted as an additional nodal unknown. Numerical studies have indicated that the first technique is too stiff due to overconstraint by the Hermitian interpolation functions [13], but the second is satisfactory provided that independence of the extra variables is strictly maintained. Accordingly, the adopted discretization scheme has eight-noded quadratic isoparametric Serendipity shape functions for $w$, giving a constant-curvature capability, and lesser four-noded bilinear variations for inplane displacements $u$, $v$, and out-of-plane normal $\theta_1$, $\theta_2$. Furthermore, displacement and rotation fields on the boundary ($\theta$, $\bar{g}$) and in the interior ($u$, $\bar{g}$) of each element are described by the same shape function interpolations, as a result of which the distinction between them disappears completely.

Stress function, rotation, and moment resultant parameters are the local variables. The linear momentum balance expression [eqn (35)] can be identically satisfied by the choice:

$$\eta' = \left[ \varepsilon^{al} \frac{\partial F}{\partial x^l} + p' \right]$$

$$= \varepsilon^{al} \left[ \frac{\partial F}{\partial x^l} + F^t \Gamma^t_{lu} - F^t B^t_l \right] + p'^{as}$$

$$= \varepsilon^{as} \left[ \frac{\partial F}{\partial x^s} + F^t B^t_{us} \right] + p'^a. \quad (51)$$

All quantities are related to the undeformed mid-surface, the three stress functions $F'$, and particular solutions $p'$ being given by

$$F = F^t A_p + F^t N \quad (52)$$

$$p' = -\frac{1}{2\kappa (A)} \left[ \sqrt{(A)} d q^t + p'^{as} A_p + p'^{as} N \right]. \quad (53)$$

It is found, though, that some functional equations associated with the unknown coefficients of $F'$ are linearly dependent; and a second, non-nodal, rotation field is necessary to restore independence.

Specifically, the constant-stress terms in $\rho_{1,2}$ and $\rho_{2,1}$, which are unequal in general, produce identical complementary energy equations and require an inplane rotation $\phi_2$ of one polynomial degree lower to act as a Lagrange multiplier [14]. This rotation, however small, must always be incorporated in the formulation, and its role in the orthogonal rigid rotation tensor is now described.

\textit{A priori} orthogonality of the rigid rotation tensor can be achieved by means of a finite rotation vector [15]:

$$\Omega = (\sin \omega) \theta = \phi'A, \quad (54)$$

Unit vector $\theta$ has components $v^i$ in the covariant basis and generates an orthogonal $\bar{g}$ thus:

$$R = I + (\Omega \times I) + [(\Omega \times I)(\Omega \times I)]/2 \cos^2(\omega/2) \quad (55)$$

$$= [A_{sp} + (\sin \omega) v^i e_{sp} \sqrt{(A)} + (1 - \cos \omega) \times v^i u^m A^m_n e_{np} \sqrt{(A)} A^k_l A^p_q] \phi'A. \quad (56)$$

A more conventional computational form results when the treatment is specialized to the moderately large rotations $\omega$ of Von Karman large deformation plate theory, wherein the approximations $\sin \omega = \omega$, $(1 - \cos \omega) = \omega^2/2$ and $\phi' = \omega v'$ become plausible. Hence

$$\bar{g} = [A_{sp} + \phi' \sqrt{(A)} e_{sp} + \phi' \phi'^m A^m_n e_{np} \sqrt{(A)} A^k_l A^p_q, \quad (57)$$

and, in the case of a plate (i.e. $A_i = 0_i$):

$$R = \left[ \begin{array}{ccc}
1 & 1/2 & 1/2 \\
1/2 & 1 & 1/2 \\
1/2 & 1/2 & 1
\end{array} \right] \left[ \begin{array}{c}
\phi_i^2 + \phi_i^3 \\
\phi_i^2 + \phi_i^3 \\
\phi_i^2 + \phi_i^3
\end{array} \right] + \left[ \begin{array}{c}
\phi_i^2 \phi_i - 0_i \\
\phi_i^2 \phi_i - 0_i \\
\phi_i^2 \phi_i - 0_i
\end{array} \right]. \quad (58)$$

Finally, neglecting components $r_{13}$ and $r_{23}$, the $2 \times 2$ moment resultant tensor $M$ is locally approximated by complete quadratic polynomials for each of $r_{11}$, $r_{12}$, $r_{21}$, and $r_{22}$.

Choosing a linearized incremental approach to solution of the nonlinear variational problem, state variables in the $C^{N+1}$ state can be written in terms of the $C_1$st state as $\eta^{N+1} = \eta^1 + \Delta \eta$, etc., where $C_n$ is the reference state of the total Lagrangian description. Functional $F_3$, therefore, can be expanded in a truncated Taylor series about $C_0$ thus:

$$F_3(\eta^{N+1}, \xi^{N+1}, \alpha^{N+1}, B^{N+1})$$

$$= F_3(\eta^1, \xi^1, \alpha^1, B^1) + \Delta F_3(\Delta \eta, \Delta \xi, \Delta \alpha, \Delta B) + \Delta^2 F_3 \quad (59)$$
The foregoing variational field equations are also valid for increments of $F_i$ and provide continuity conditions, albeit in a weighted integral sense, which are necessary for existence of the Taylor series. $F_i$ and $F_j(x^r, z^m, u^k, E^n)$ are constant with respect to the incremental variables. Component $\Delta F_j$ contains first-order incremental terms, and its variation vanishes if the $C_n$th state fully satisfies the governing equations and boundary conditions. A variation of $\Delta^2F_j$ leads to conventional linear simultaneous equations, while third-order $\Delta^3F_j$ contributes nonlinear terms.

The solution methodology, an economical modified Newton-Raphson technique \[16\], involves using small load increments, so that higher-order terms are negligible, and solving $\delta[\Delta^2F_j] = 0$ for a linear approximation to the incremental unknowns. Since omission of high-order terms can cause accumulated error, it becomes necessary to retain $\delta[\Delta F_j] = 0$ as the basis of "compatibility mismatch" iterations in each increment. Although the simplicity of this procedure is its primary attraction, it is entirely analogous to rigorous mathematical imbedding techniques for nonlinear equation solutions \[17\].

**NUMERICAL RESULTS**

Intensive wartime research efforts into the behavior of aircraft sheet-stringer panels led to a set of truncated series solutions by Levy \[18, 19\] to the Von Karman thin plate equations under lateral pressure and a variety of boundary conditions. These partial differential equation solutions provide convenient comparisons for assessing the effectiveness of the finite element algorithm.

Of the modified functionals \[eqns (48) and (49)\], $F_3$ is chosen as the basis for an eight-noded isoparametric Serendipity element of constant thickness. The 56-parameter local variable field comprises cubic polynomials for stress functions $F^1$ and $F^2$ ($2 \times 9$), a quadratic variation for $F^3$ (5) \[eqn (52)\], linear terms for the three rotation components ($3 \times 3$), and a quadratic order for the four stress-couples ($4 \times 6$). The rank of this element is satisfactory when fully integrated; and in the corrective iteration, a convergence tolerance of 1% of the applied load increment gives acceptable accuracy over the deformation range of interest.

One quadrant of a laterally pressurized square plate is analyzed using four square elements (Fig. 3), the elastic properties of the material being comparable to typical aluminum alloys. With a span to thickness ratio $(a/h) = 100$, the plate is thin and its characteristics can be determined in terms of the nondimensional quantities: center deflection $(w_c/h)$, pressure ratio $(p_c/Eh)$, and stress ratio $(\sigma_{c}/Eh^2)$. In order to compare directly with the analytical solutions, all edges are maintained straight and pressure loadings are assumed to be independent of displacements.

For simply supported edges, the deflection results of functional $F_3$ (Fig. 4) agree quite well with the series solution initially but deviate on the stiff side at higher loadings. The deviation is smaller when in-plane restraint is applied to the edges (Fig. 5) and is least of all in the case of a rigidly clamped boundary (Fig. 6). Nonetheless, the agreement between finite element and series approaches is noteworthy in all cases. Convergence is usually achieved in one or two iterations.

The symmetrical second Piola-Kirchhoff stress is the preferred measure in Von Karman plate theory; so, to compare with Levy's mathematical treatment, it is necessary to transform the computed first Piola–Kirchhoff stress and moment resultants. (Actually, the deformation gradients in these problems are so small that first and second Piola–Kirchhoff stress measures are virtually identical anyway.) In the sim-
ply supported plate, the corner (A) and center (D) membrane stresses (Fig. 7) show relatively good agreement with Levy. However, of the membrane stresses $\sigma_{11}$ and $\sigma_{22}$ at midside B, only the longitudinal compression $\sigma_{11}$ has acceptable accuracy (Fig. 8). Similarly, the large twisting moment $\tau_{12}$ at A is good while the center moment $\tau_{11}$ is up to 30% too small at large deformations (Fig. 9).

Turning to the case of inplane edge restraint, excellent agreement is apparent for corner and center membrane stresses (Fig. 10), but midside membrane values deviate above by up to 23% (Fig. 11). Likewise, the twisting moment at A and bending moment at D differ from series values by ca ± 15% (Fig. 12).

The fully clamped plate gives particularly good finite element results; and of the stress and moment resultants in Fig. 13, the bending moment resultant $\tau_{11}$ at the center is the only one with a significant discrepancy.

A simply supported square plate subject to uniform uniaxial compression at two edges can also be modeled by the mesh of Fig. 3. This is an example of buckling from an irrotational fundamental state and contrasts with earlier loadings. A small lateral pressure of $(p_a^4/Eh^4) = 0.22$ is applied to induce buckling, and compression is achieved through progressively increasing prescribed edge displacements. The average compressive stress $\bar{p}$ is obtained from computed edge loadings after convergence.
Fig. 7. Membrane stresses at $A$ and $D$ in a simply supported square plate without edge restraint.

Fig. 8. Membrane stresses at $B$ in a simply supported square plate without edge restraint.

In the graphs of center transverse displacement (Fig. 14) and magnified early postbuckling behavior (Fig. 15), the agreement between finite element results and Levy's series solution is outstanding. Membrane compressive stresses (Fig. 16) and bending stresses (Fig. 17) at the corner ($A$) and center ($D$) are also presented. The compressive stress $\sigma_{11}$ and twisting moment $\tau_{12}$ at $A$ are the largest stresses in the plate, but these are underestimated by the finite element algorithm at large deformations. However, at $D$ the membrane compression $\sigma_{11}$ and bending moment $\tau_{11}$ are almost perfectly modeled.

CONCLUSIONS

Of the current discrete methods of analysis, only the finite element method has a full repertoire of nonlinear capabilities. The mixed-hybrid version [2], in particular, is unified by a hierarchy of well-founded variational functionals; and the technique of

Fig. 9. Extreme-fiber bending stresses at $A$ and $D$ in a simply supported square plate without edge restraint.

Fig. 10. Membrane stresses at $A$ and $D$ in a simply supported square plate with zero edge displacement.

Fig. 11. Membrane stresses at $B$ in a simply supported square plate with zero edge displacement.

Fig. 12. Extreme-fiber bending stresses at $A$ and $D$ in a simply supported square plate with zero edge displacement.

Fig. 13. Membrane and extreme-fiber bending stresses at $B$ and $D$ in a rigidly clamped square plate.
Lagrange multipliers, by relaxing certain physical constraints, lends versatility and accuracy to the solution. The various ad hoc stress and strain definitions of other authors are not necessary in this formulation, which is rigorously and successfully based upon stress functions, rigid rotation variables, and the symmetrized Biot-Luré (Jaumann) stress measure. Furthermore, a unique complementary energy density, which decouples conveniently into stretching and bending components, exists in this case. General functional $F_1$ can be modified to produce two secondary functionals, one of which ($F_2$) performs well in numerical applications. The diverse composition of $F_2$, with a combination of global displacements and rotations and local stress function, rotation, and moment resultant polynomials, makes it a particularly good model for the moderately large rotations of Von Karman plate theory. In comparison with truncated series solutions to laterally loaded plates under various boundary conditions, it gives excellent displacements, good membrane stresses, and somewhat inconsistent moment values, although the doubtful accuracy of moment values from a series solution may make any reasonable comparison impossible. Overall, the linearized incremental variational formulation, with modified Newton-Raphson iterative corrections, is simple, economical, and effective in the total Lagrangian description.

Optimal polynomial orders remain to be established for the field variable representations, and both functionals $F_2$ and $F_1$ feature a priori satisfaction of linear momentum balance through stress functions. It would be equally valid to relax this constraint through a Reissner-type formulation so that alternate stress measures with more complicated LMB expressions could be incorporated. The implementation and testing of these new algorithms would complete the variational treatment of this field of inquiry, leading on to problems of material nonlinearity, shear deformation, and dynamics of shells.

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