INSTABILITY ANALYSIS OF SPACE TRUSSES USING EXACT TANGENT-STIFFNESS MATRICES

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Abstract. A simple (exact) expression for the tangent-stiffness matrix of a space truss undergoing arbitrarily large deformation, as well as member buckling, is given. An arc-length method is used to solve the tangent-stiffness equations in the post-buckling range of the structural deformation. Several examples to illustrate the viability of the present approaches in analyzing large space structures, simply, efficiently, and accurately, are given.

Introduction

Currently, there is an enormous interest in deploying large structures in outerspace for a variety of reasons. These structures are, in general, of very low mass and very high flexibility. A pressing technical problem in the design of these structures is the need for active or passive control of transient dynamic (traveling wave type) response. Since these structures are highly flexible, there is the inherent need to account for large deformations. The transient dynamic response equations for the space structure may be written as

\[ M\ddot{d} + C\dot{d} + S(d) = f_c + Q_E, \]

where \( M \) is the mass matrix, \( C \) is the matrix of viscous damping which arises due to a deliberate design of the structural joints, among other reasons, \( S \) is the vector of nodal restraining forces which depend nonlinearly on the vector of nodal displacements \( d \), \( f_c \) is the vector of control forces to \( d \) determined from a properly formulated feedback control strategy, \( Q_E \) is the vector of externally applied dynamic nodal loads, and \( \dot{d} \) and \( \ddot{d} \) are, respectively, the vectors of velocity and acceleration. In transient dynamic response calculations, it is customary to linearize equation (1) as

\[ M^{(N+1)}\ddot{d} + C^{(N+1)}\dot{d} + K^{(N)}\Delta d = f_c^{(N+1)} + Q_E^{(N+1)} - R^{(N)}, \]

where, now, \( K^{(N)} \) is the so-called tangent-stiffness matrix at state \( N \) (or at time \( t_N \) in a time-integration scheme), \( \Delta d \) is the incremental displacement vector, and \( R^{(N)} \) are internal restraining forces at \( t_N \).

In the usual finite element analysis, much of the effort is usually expended in evaluating the tangent-stiffness matrix \( K^{(N)} \), which accounts for the effects of initial displacements and initial stresses. Usually, one assumes basis functions and integrates over the element the appropriate strain energy terms (that depend nonlinearly on the displacements) to obtain \( K^{(N)} \). The

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objective of the present research is to obtain an explicit expression for \( (N)K \) for large space structures, without the use of assumed element basis functions and element integrations. The aim is to obtain analytical solutions to the appropriate nonlinear ordinary differential equations and to use them to derive exact (closed form) expressions for stiffness matrices. It should be noted that, while the concepts of tangent stiffness finite element method and arc-length solution method were not used, several interesting analytical studies of stability of space-trusses were earlier presented by Britvec [1].

Depending on the design of the joints, each member of a tree-dimensional (space) structure may be considered as a 'truss' member or a 'frame' member. The truss member carries only an axial force, and the kinematics of deformation is characterized by the three displacements of each node. The frame member carries bending moments, a twisting moment, and lateral forces, in addition to an axial force; and the deformation is characterized by the three displacements and three rotations at each node.

The present paper is limited to a large deformation, post-buckling analysis of large space trusses under quasi-static loading using explicit expressions for \( (N)K \). Thus, in (2), \( \dot{d}, \phi, f_c \) are set to zero. The nonlinear tangent-stiffness equations are then solved by using an arc-length method. Several examples are given to illustrate the simplicity of the present approach which renders the large deformation analysis of reasonably large-sized space trusses suitable for currently available personal computers.

In the next section, a derivation of the tangent-stiffness matrix is given, the third (short) section deals with a brief description of the arc length method, and the final section deals with several examples.

Derivation of an explicit tangent-stiffness matrix for finite-deformation, post-buckling analysis of space trusses

The space truss structures discussed herein are assumed to remain elastic. Also, only a conservative system of concentrated loads at the nodes of the space truss structures is considered.

Relation between stretch and axial force in a truss member

Consider a typical slender truss member spanning between nodes 1 and 2 as shown in Fig. 1. This member is considered to have a uniform cross section, and its length before deformation is \( l \). The coordinates \( x_1, x_2, \) and \( x_3 \) are the member's local coordinates, while \( u_1, u_2, \) and \( u_3 \) denote the displacements at the centroidal axis of a member along the coordinate directions \( x_1, x_2, \) and \( x_3, \) respectively.

From the polar decomposition theorem, the relation between the total axial stretch and displacements of the member is

\[
\delta = \left[ (\bar{u}_1)^2 + (\bar{u}_3)^2 + (l + \bar{u}_3) \right]^{1/2} - l, \tag{3}
\]

where \( \delta \) is the total axial stretch, \( \bar{u}_1 = ^2u_1 - ^1u_1, \bar{u}_2 = ^2u_2 - ^1u_2 \) and \( \bar{u}_3 = ^3u_3 - ^1u_3 \). Equation (3) holds for both the pre- and post-buckled states of the member.

The incremental relation between the incremental total stretch and the incremental axial force in the member is written as

\[
\Delta N = k \cdot \Delta \delta, \tag{4}
\]
Fig. 1. Nomenclature for kinematics of deformation of a space truss member.

where

\[ \Delta N : \text{incremental axial force in the member,} \]
\[ \Delta \delta : \text{incremental total axial stretch in the member,} \]
\[ k = \frac{EA}{I} \text{ in the pre-buckled state,} \]
\[ = \frac{\pi^2 \cdot EI}{2I^3} \text{ in the post-buckled state (for the range of deformations considered),} \]
\[ E : \text{Young's modulus,} \]
\[ A : \text{cross sectional area of the member,} \]
\[ I : \text{moment of inertia.} \]

Equation (5a) simply follows from the linear-elastic (isotropic) stress-strain law of the material of the member.\(^1\) On the other hand, equation (5b) for the post-buckled state of the member is derived in Appendix A by simplifying and modifying the governing equations of the problem of the elastica, which is treated as a simply supported beam.

Here, one should note that \( N \) is in the direction of the straight line connecting node 1 and node 2 of the member after its deformation (see Fig. 1), and \( \delta \) is calculated from equation (3). Hence, equation (4) holds even when the rigid motion of the member is very large. Also, note that the stiffness-coefficient \( k \) is a constant in each of the two states, such as pre-buckled and post-buckled, of each member, of a space truss.

The condition for the buckling of a member, treated as a simply supported beam, is given by the following well-known equation:

\[ N = \varepsilon N, \]
where

\[ \varepsilon N = -\frac{\pi^2 EI}{2I^2}, \]

the negative sign being used to denote the compressive axial force.

\(^1\) While the material is assumed to be linear-elastic in the present, the subsequent derivations of the tangent stiffness matrix remain valid, with straightforward modifications, even when the material stress-strain law is of a Ramberg-Osgood type: \( \sigma = E\varepsilon + B\varepsilon^n \).
The only force acting on a truss-member is considered to be the axial force. Hence, the strain energy of the member, $U$, in either the pre- or post-buckled states of the member, is given by

$$U = \frac{1}{2} \int \left( EA \cdot e^2 + EI \cdot \kappa^2 \right) \, dx = \int N \, d\delta,$$

where $e$ is the point-wise axial stretch, and $\kappa$ the curvature, $\kappa = 0$ for the pre-buckled state, and $\kappa \neq 0$ for the post-buckled state.

The incremental form of equation (8) is represented, using equation (4), as

$$\Delta U = N \cdot \Delta \delta = \frac{1}{2} k (\Delta \delta)^2.$$  

The incremental form of equation (3) is given by

$$\Delta \delta = a \cdot \Delta \bar{u}_1 + b \cdot \Delta \bar{u}_2 + c \cdot \Delta \bar{u}_3$$

$$+ \frac{1}{2 \ell^*} \left[ (b^2 + c^2) \cdot \Delta \bar{u}_1^2 + (c^2 + a^2) \cdot \Delta \bar{u}_2^2 + (a^2 + b^2) \cdot \Delta \bar{u}_3^2 \right]$$

$$- \frac{1}{\ell^*} \left[ (a \cdot b) \Delta \bar{u}_1 \Delta \bar{u}_2 + (b \cdot c) \Delta \bar{u}_2 \Delta \bar{u}_3 + (c \cdot a) \Delta \bar{u}_3 \Delta \bar{u}_1 \right]$$

$$+ \text{higher order terms},$$

where

$$\ell^* = \left[ (\bar{u}_1)^2 + (\bar{u}_2)^2 + (1 + \bar{u}_3)^2 \right]^{1/2},$$

$a = \bar{u}_1/\ell^*$, \quad $b = \bar{u}_2/\ell^*$, \quad $c = (1 + \bar{u}_3)/\ell^*$,

$\Delta \bar{u}_1$, $\Delta \bar{u}_2$, and $\Delta \bar{u}_3$ represent the increments of $\bar{u}_1$, $\bar{u}_2$, and $\bar{u}_3$ respectively.

Substituting equation (10) into equation (9), one finds that

$$\Delta U = N \left( a \cdot \Delta \bar{u}_1 + b \cdot \Delta \bar{u}_2 + c \cdot \Delta \bar{u}_3 \right) + \frac{1}{2} \left[ (b^2 + c^2) \cdot \frac{N}{\ell^*} + k \cdot a^2 \right] \Delta \bar{u}_1^2$$

$$+ \frac{1}{2} \left[ (c^2 + a^2) \cdot \frac{N}{\ell^*} + k \cdot b^2 \right] \Delta \bar{u}_2^2 + \frac{1}{2} \left[ (a^2 + b^2) \cdot \frac{N}{\ell^*} + k \cdot c^2 \right] \Delta \bar{u}_3^2$$

$$+ \left( k - \frac{N}{\ell^*} \right) \left[ (a \cdot b) \Delta \bar{u}_1 \Delta \bar{u}_2 + (b \cdot c) \Delta \bar{u}_2 \Delta \bar{u}_3 + (c \cdot a) \Delta \bar{u}_3 \Delta \bar{u}_1 \right]$$

$$+ \text{higher order terms}.$$  

Furthermore, neglecting terms of higher than the second order, the variation in the incremental strain-energy may be derived from equation (11) as

$$\delta (\Delta U) = \delta \Delta \bar{u}_1 (N \cdot a) + \delta \Delta \bar{u}_2 (N \cdot b) + \delta \Delta \bar{u}_3 (N \cdot c)$$

$$+ \delta \Delta \bar{u}_1 \left[ (b^2 + c^2) \cdot \frac{N}{\ell^*} + k \cdot a^2 \right] \Delta \bar{u}_1$$

$$+ \left( k - \frac{N}{\ell^*} \right) \cdot a \cdot b \cdot \Delta \bar{u}_2 + \left( k - \frac{N}{\ell^*} \right) \cdot c \cdot a \cdot \Delta \bar{u}_3$$

$$+ \delta \Delta \bar{u}_2 \left[ (c^2 + a^2) \cdot \frac{N}{\ell^*} + k \cdot b^2 \right] \Delta \bar{u}_2$$

$$+ \left( k - \frac{N}{\ell^*} \right) \cdot b \cdot c \cdot \Delta \bar{u}_3 + \left( k - \frac{N}{\ell^*} \right) \cdot a \cdot b \cdot \Delta \bar{u}_1.$$
where

\[ d^m: \text{vector of generalized nodal displacements}, \]
\[ R^m: \text{vector of internal forces}, \]
\[ K^m: \text{stiffness matrix of the element}, \]
\[ \Delta d^m = \begin{bmatrix} \Delta u_1; \Delta^2 u_1; \Delta u_2; \Delta^2 u_2; \Delta u_3; \Delta^2 u_3 \end{bmatrix}, \]
\[ \Delta d^m = \begin{bmatrix} (N \cdot a) \cdot \{I\} \\
(N \cdot b) \cdot \{I\} \\
(N \cdot c) \cdot \{I\} \end{bmatrix}, \]
\[ K^m = \begin{bmatrix} C_1[E] & C_4[E] & C_6[E] \\
C_6[E] & C_5[E] & C_3[E] \end{bmatrix}, \]
\[ C_1 = (b^2 + c^2) \frac{N}{l_e} + k \cdot a^2, \]
\[ C_2 = (c^2 + a^2) \frac{N}{l_e} + k \cdot b^2, \]
\[ C_3 = (a^2 + b^2) \frac{N}{l_e} + k \cdot c^2, \]
\[ C_4 = \left(k - \frac{N}{l_e}\right) \cdot a \cdot b, \]
\[ C_5 = \left(k - \frac{N}{l_e}\right) \cdot b \cdot c, \]
\[ C_6 = \left(k - \frac{N}{l_e}\right) \cdot c \cdot a, \]
\[ \{I\} = \begin{bmatrix} -1 \\
1 \\
1 \end{bmatrix}, \quad [E] = \begin{bmatrix} 1 & -1 \\
-1 & 1 \end{bmatrix}. \] \number{(12)}

One should note that equations (12) and (13) are written in the local coordinate system, so that it is necessary to transform the displacement vector from the local coordinate system to the global coordinate system in the usual fashion.

It should be emphasized again that equations (12) and (13) (and, thus, the tangent-stiffness matrix and the load vector) are applicable for both the pre- and post-buckled states of the member, and that \( k \) has a constant value in each of the two states as given in equation (5). Consequently, if a member buckles, it is only necessary for the value of \( k \) to be changed. In view of this, it is seen to be very simple to derive the tangent-stiffness of the member, and, thus, of the structure as a whole.

Solution strategy: Arc-length method

Although a number of solution procedures is available for nonlinear structural analysis, a reliable approach to trace the structural response near limit points, and in post-buckling range, is the ‘arc-length’ method [2,3,4]. This method is the incremental/iterative procedure which represents a generalization of the displacement control approach. The arc-length method, in which the Euclidian norm of the increment in the displacement and load space is adopted as the prescribed increment, allows one to trace the equilibrium path beyond limit points such as in snap-through and snap-back phenomena.

A full description of the presently adopted procedure is already given in [4] and will not be repeated here.

Example problems of space trusses

The first example considered in this category is the shallow geodesic dome shown in Fig. 2. This structure, which exhibits a snap-through phenomenon, is subjected to one concentrated
load at the central node. Two initial configurations of the structure, one geometrically perfect and the other with slight imperfections, as specified in Table 1, are considered. This example was also analyzed, using a perturbation method, by Hangai and Kawamata [6] to study global stability. In the present study, however, the influence of member buckling on global stability is also examined.

Figs. 3 and 4, for the case of perfect geometry, show a typical snap-through phenomenon wherein the first limit point is reached at a load of $3.15 \times 10^{-4} \, EA$ (Kg). The present results are seen to be in good agreement with those of Hangai and Kawamata [6].

Table 1
Coordinates of the nodes of example (1)

<table>
<thead>
<tr>
<th>Node</th>
<th>(1) Perfect geometry</th>
<th></th>
<th>(2) Imperfect geometry</th>
</tr>
</thead>
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<tr>
<td></td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
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<td>0.0</td>
<td>0.0</td>
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</tr>
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<tr>
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<td>13</td>
<td>43.30</td>
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</tbody>
</table>
The influence of member buckling on global instability is illustrated in Figs. 5, 6, and 9 which indicate that a global behavior strongly depends on the buckling of a single member. In a practical design of a three-dimensional truss structure, this understanding is very essential and useful. Also, the effects of slight geometric imperfection are illustrated in Figs. 7 and 8, wherein the comparison results of Hangai and Kawamata [6] are also included.

Example (2) is also that of a shallow geodesic dome, analyzed earlier by Noor and Peters [7] and shown in Fig. 10. Two types of loading systems are considered: the first loading system...
consists of lateral concentrated loads $P_1$ over the entire dome; the second one, $P_2$, consists of concentrated lateral loads only over a quarter of the dome. An important difference between the present analysis and that of Noor and Peters [7] should be mentioned. Noor and Peters [7] ignore member buckling and assume each member of the truss to remain straight and stable. On the other hand, in the present analysis, local buckling of each member is allowed; and only for

Fig. 5. Vertical displacements of central node with and without the influence of local buckling of truss members.

Fig. 6. Horizontal radial displacements of non-central nodes with and without the influence of local buckling of truss members.
comparison purposes, results are also obtained using the present procedure with local buckling being intentionally suppressed.

Fig. 11 provides a comparison of the vertical displacement of the central node in the present and Noor and Peters' solutions for various combinations of $P_1$ and $P_2$, when local (member) buckling is ignored. The present results agree well with those of [7] except beyond the limit

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Fig. 7. Vertical displacements of central node under imperfect geometry without the influence of local buckling of truss members.

Fig. 8. Horizontal radial displacements of non-central nodes under imperfect geometry without the influence of local buckling of truss members.
The stability boundary, i.e., the combinations of load parameters $P_1$ and $P_2$ which render the structure unstable when local member buckling is ignored, is shown in Fig. 12. From which an excellent correlation of the present results (with member-buckling being suppressed) with those of [7] may also be noted.

Figs. 13 to 16 show the present results when local (member) buckling is considered. Fig. 13 shows the variation of vertical displacement of the central node; Fig. 14 shows the stability boundary under the various combinations of $P_1$ and $P_2$, and Fig. 15 shows the equilibrium path under the load system $P_2 = 0$ and $P_1 \neq 0$.

From this numerical example (especially Fig. 14), it is clear that the decrease in the magnitude of critical loads for the structure, due to buckling of an individual member or members, i.e., the influence of local buckling on the response of the structure as a whole, is quite remarkable.

The third example of space trusses is that of a beam-like space truss (PACROSS Truss) subjected to axial and bending loads. The structure is that of a twelve-bay truss whose member properties...
Fig. 10. Schematic of shallow geodesic dome.

Fig. 11. Vertical displacements of central node under various combinations of loads, $P_1$ and $P_2$, without the influence of local buckling of truss members.

are shown in Figs. 17 and 19. In order to trigger the coupling between the axial and transverse displacements, which is characteristic of the buckling mode, in the case of only an axial-load application, a 'load imperfection' equal to $P/1000$ is added in the transverse direction at one of the end nodes, as shown in Fig. 17.
Fig. 12. Stability boundary under various combinations of loads. $P_1$ and $P_2$, with and without the influence of local buckling of truss members.

Fig. 13. Vertical displacements of central node under various combinations of loads. $P_1$ and $P_2$, with and without local buckling.
Fig. 14. Stability boundaries under various combinations of loads, \( P_1 \) and \( P_2 \), with and without the influence of local buckling of truss members.

Fig. 15. Vertical displacements of central node under load \( P \), with and without the influence of local buckling of truss members.
For the above predominantly axial-load case, Fig. 18 shows the relation between the magnitudes of the axial load and that of the transverse displacement at the loaded end, for two scenarios: (i) when local (member) buckling is suppressed and each member is assumed to remain straight and stable, and (ii) when each member is allowed to undergo local buckling. Fig. 18 clearly demonstrates the advantageous effect of controlling the local buckling deformations of individual members and forcing them to remain straight and stable. This leads one to the concept of active/passive control of member deformations.

Fig. 19 shows the schematic of the PACOSS Truss subject to predominantly bending loads. Fig. 20 shows the relation between the magnitudes of transverse (bending) load and transverse displacement, respectively, once again for two scenarios: (i) when local member buckling is suppressed, and (ii) when member buckling is allowed. Fig. 20 again demonstrates the beneficial effects of control of deformations of each member. Fig. 21 shows a computer plot of the deformed shape of the PACOSS Truss under bending loads.

It should be noted that in Figs. 18 and 20, the letters A, B, C, etc. indicate the stages at which the respective members, whose numbers are identified in Figs. 18 and 20 respectively, undergo local buckling.
**Closure**

While simple methodologies for large deformation analyses of large space structures (LSS) modeled as *trusses* are treated in this paper, similar methodologies for analyses of LSS modeled as *frames* (with each member carrying three moments and three forces) are objects of a forthcoming paper.
Appendix A. Post-buckling behavior of a truss member

In this appendix, equation (5b) for the post-buckled state of the truss member is derived. Consider the truss member being subjected to the compressive force \(-N\), as shown in Fig. 1. When \(N\) satisfies equation (6), this member undergoes bifurcation buckling. From the detailed treatment of the elastica problem given in [8], the post-buckling behavior of the member, treated as a simply-supported beam, is governed by the following equations:

\[
I = \frac{1}{f} \cdot F(\beta), \tag{A.1a}
\]

\[
I + \delta = \frac{2}{f} \cdot E(\beta) - I, \tag{A.1b}
\]

\[
\delta = \frac{2}{f} \cdot \beta, \tag{A.1c}
\]

where

\[
F(\beta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \beta^2 \sin^2 \phi}}, \quad E(\beta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \beta^2 \sin^2 \phi} \cdot d\phi,
\]

\[
f^2 = -\frac{N}{EI}, \quad \beta = \sin(\frac{1}{2}\alpha), \quad \alpha = \theta_{u=0} = -\theta_{u=I},
\]
and $F(\beta)$, $E(\beta)$ are the elliptic integrals of the first and the second kind, respectively. Also, $\delta$ is the stretch after the buckling of the member, and $\hat{\delta}$ is the lateral deflection at the middle of the centroidal axis of the element. Note that the total stretch $\delta$ is given by the sum of $\hat{\delta}$ and the stretch, $\alpha N \cdot l / EA$, before the buckling of the member. Also, it should be noted that in the derivation of equations (A.1), the change in the length of the member due to the compressive force is neglected.
Fig. 20. Deflection at free end under bending loads with and without the influence of local buckling of truss members.

Equations (A.1) give the exact relations between $N$, $\delta$, and $\delta$ in the post-buckled range, except for the assumption concerning the length of the element. We now simplify and modify these relations to a form more useful for the present purposes of evaluating a tangent-stiffness matrix. To this end, we start by expanding $F(\beta)$, $E(\beta)$ in terms of $\beta$ (see [9]):

\[
F(\beta) = \pi + \frac{1}{2} \beta^2 \cdot S_2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \beta^4 \cdot S_4 + \cdots, \tag{A.3a}
\]

\[
E(\beta) = \pi - \frac{1}{2} \beta^2 \cdot S_2 - \frac{1}{2 \cdot 4} \cdot \beta^4 \cdot S_4 + \cdots, \tag{A.3b}
\]

where

\[
S_n = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^n \phi \cdot d\phi. \tag{A.4}
\]

We shall retain the terms of equations (A.3) up to the second order for the approximations of $F(\beta)$, $E(\beta)$:

\[
F(\beta) = \pi + \frac{1}{2} \pi \cdot \beta^2, \tag{A.5a}
\]

\[
E(\beta) = \pi - \frac{1}{2} \pi \cdot \beta^2. \tag{A.5b}
\]

The range of validity of these approximations will be demonstrated momentarily.
Then, equations (A.1a) and (A.1b), respectively, become

\[ l = \frac{1}{f} (\pi + \frac{1}{4} \eta \beta^2), \]  
\[ l + \delta = \frac{2}{f} (\pi - \frac{1}{4} \eta \beta^2) - l. \]
From equations (A.6) one obtains
\[ 4\theta + \delta = \frac{4}{f} \cdot \pi. \]  
(A.7)

Noting that \( f^2 = (-N/EI) \), one sees from equation (A.7),
\[ N = \frac{N^{(cr)}}{\left(1 + \frac{\delta}{4l}\right)^2}, \]  
(A.8)

where \( N^{(cr)} \) is the critical axial force for bifurcation buckling as given in equation (7).

For small values of \( -(\delta/l) \), equation (A.8) may be approximated as
\[ N = N^{(cr)} \left[1 - \frac{1}{2}(\delta/l)\right]. \]  
(A.9)

The incremental form of equation (A.9) results in equations (5). The linear relation (A.9), and its incremental counterparts, are useful in tangent-stiffness evaluations.

We now derive the relation between \( \delta \) and \( \delta \). This relation is not necessary for the construction of the tangent stiffness, but it is useful for the determination of maximum and/or minimum stress in each of the members.

Noting that \( \beta \) is nonnegative except for \( \alpha > 2\pi \), one obtains from equation (A.6a)
\[ \beta = 2\sqrt{\frac{1}{\pi} \cdot f \cdot 2 - 1}. \]  
(A.10)

Substituting equation (A.10) into equation (A.1c), it is seen that
\[ \delta = \frac{4}{\pi} \cdot \sqrt{\frac{1}{\pi} \cdot f \cdot l - 1}. \]  
(A.11)

![Fig. A.1. Axial stretches and lateral deflections (at the center of the span of the member) in the post-buckled range under an axial force.](image)
Substituting for \( f \) in terms of \( N \) and using equation (A.8), the following relation between \( \delta \) and \( \bar{\delta} \) is obtained:

\[
\frac{\delta}{l} = \frac{4}{\pi} \cdot \sqrt{-\frac{\delta}{4l} \left(1 + \frac{\delta}{4l}\right)}.
\]  

(A.12)

Thus, when the axial contraction \( \delta \) is solved from the finite element stiffness equation, equation (A.12) may be used to calculate the transverse displacement \( \bar{\delta} \) at midspan of the member; and from it, one may calculate the maximum or minimum stress in the member.

Fig. A.1 shows the relations between \( N, \delta, \) and \( \bar{\delta} \) as given by equations (A.9) and (A.12) and their comparisons with the exact solutions for the elastica problem. The dashed lines indicate the present solutions and the solid ones indicate the exact ones. From this figure it is seen that equations (A.9) and (A.12) are good approximations in the range of values for \(-\frac{\delta}{l}\) and \(\frac{\delta}{l}\) being smaller than about 0.15 and 0.25, respectively. It is also seen that this range of values for \(-\frac{\delta}{l}\) and \(\frac{\delta}{l}\) is typical in the problem of local (member buckling in a practical truss structure.

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References