MULTIPLE SURFACE CRACKS IN PRESSURE VESSELS

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Abstract—An alternating method, in conjunction with the finite element method and a solution for multiple coplanar elliptical cracks in an infinite solid, is used to determine stress intensity factors for semi-elliptical surface flaws in cylindrical pressure vessels.

The solution technique for multiple cracks in an infinite body has recently been developed by the present authors which implements a well-known analytical solution for a single crack in an infinite body. The present finite element alternating method leads to a very inexpensive procedure for routine evaluation of accurate stress intensity factors for flawed pressure vessels.

Numerical examples are presented for the situation of two equal surface cracks in a pressure vessel. Comparison is made between these results and the procedure for multiple cracks in the ASME Boiler and Pressure Vessel Code.

1. INTRODUCTION

In practice, the actual flaws in structural components such as pressure vessels or aircraft attachment lugs are often approximated by elliptical or part elliptical cracks[1]. Accurate estimates of stress intensity factors along the flaw border are needed for a reliable prediction for crack growth. For this reason, the problems of subsurface and surface elliptical cracks have been the focus of considerable attention by many researchers.

However, when interacting multiple elliptical cracks exist in a structural component, the analysis becomes extremely difficult. Thus, in literature, there seems to be a lack of information for the interaction behaviour of three-dimensional multiple cracks. In fact, very few solutions for multiple semi-elliptical surface cracks in a semi-infinite solid have been obtained[2, 3]. A few results are also available for multiple semi-circular cracks in a hollow cylinder[4, 5].

Since analytical solutions to such problems are not available due to the inherent complexity, many numerical techniques have been implemented to obtain stress intensity factors. Recently Nishioka and Atluri[6, 7] have developed a new finite element alternating method which has several significant advantages compared with the classical alternating method presented by Shah and Kobayashi[8, 9]. In the new alternating method, the complete analytical solution[7, 10] for an embedded elliptical crack in an infinite solid, subjected to arbitrary tractions on the crack surface, was implemented in conjunction with the finite element alternating method. The new alternating method was successfully applied to obtain accurate stress intensity factors for a semi-elliptical surface crack in finite thickness plates[7], in pressure vessels[11], and for a quarter elliptical corner crack emanating from a pin hole in plates and in aircraft attachment lugs[12].

It was demonstrated that the present alternating method is approximately one order of magnitude less expensive in computing costs as compared to those other numerical methods reported in the literature. These include three dimensional finite element techniques which were used by Atluri and Kathiresan[13, 14], McGowan and Raymund[15], Newman and Raju[16], and Miyazaki et al.[5], while the boundary integral equation method was implemented by Heliot et al.[17].

In the present paper, using the finite element alternating method, an extension is made to deal with the problem of interacting multiple cracks in a finite solid and in particular a pressure vessel. Earlier the present authors developed a solution technique for the case of multiple coplanar elliptical cracks in an infinite body[18]. It is a combination of this procedure and the finite element

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method that is used to evaluate the appropriate stress intensity factors due to the interacting cracks. The results presented here include the stress intensity factors of interacting internal elliptical surface flaws for both thick and thin cylinders. Comparison is also made with the recommendations for multiple surface cracks as laid down in the ASME Boiler and Pressure Vessel Code[1].

2. ANALYTICAL SOLUTION FOR AN ELLIPTICAL CRACK IN AN INFINITE SOLID WITH ARBITRARY CRACK FACE TRACTION

In this section only the Mode I problem is considered. The complete general solution including Modes II and III is given in Refs. [7, 10]. Suppose that \( x_1 \) and \( x_2 \) are Cartesian coordinates in the plane of the elliptical crack and \( x_3 \) is normal to the crack plane such that

\[
\left( \frac{x_1}{a_1} \right)^2 + \left( \frac{x_2}{a_2} \right)^2 = 1 \quad a_1 > a_2
\]

(1)

describes the border of the elliptical crack of aspect ratio \( a_1/a_2 \), as in Fig. 1. The foregoing geometry is more conveniently described in an ellipsoidal coordinate system. The necessary ellipsoidal coordinates \( \xi_\alpha (\alpha = 1, 2, 3) \) are the roots of the cubic equation

\[
w(s) = 1 - \left( \frac{x_1^2}{a_1^2 + s} \right) - \left( \frac{x_2^2}{a_2^2 + s} \right) - \left( \frac{x_3^2}{s} \right) = 0.
\]

(2)

Let the normal traction along the crack surface be expressed in the form

\[
\sigma_{33}^{(0)} = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{m=0}^{M} \sum_{n=0}^{m} A_{j,m}^{(i,j)} x_1^{2m-2i} x_2^{2n+j}
\]

(3)

where the \( A \)'s are undetermined coefficients and the parameters \( i \) and \( j \) specify the symmetries of the load with respect to the axes of the ellipse \( x_1 \) and \( x_2 \).

The solution corresponding to the load expressed by eqn (3) can be assumed in terms of the potential function:

\[
f_1 = \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{k} \sum_{l=0}^{l} C_{j,k-l}^{(i,j)} F_{2k-2l+i,2l+j}
\]

(4)

where

\[
F_{2k-2l+i,2l+j} = \frac{\partial^{2k+i+j}}{\partial x_1^{2k-2l+i} \partial x_2^{2l+j}} \left[ \int_\xi \frac{[w(s)]^{2k+i+j+1}}{\sqrt{Q(s)}} ds \right]
\]

(5)

![Fig. 1. Elliptical crack in an infinite solid.](image-url)
Multiple surface cracks in pressure vessels

and

\[ Q(s) = s(s + a_2^2)(s + a_3^2) \]  

and the \( C \)'s are also undetermined coefficients.

The components of displacement \( u_i \) and the stress \( \sigma_j \) in terms of \( f_3 \) are given by

\[
\begin{align*}
\sigma_{11} &= 2\mu(f_{3,31} + 2yf_{3,32} + x_3f_{3,31}) \\
\sigma_{22} &= 2\mu(f_{3,32} + 2yf_{3,31} + x_3f_{3,32}) \\
\sigma_{12} &= 2\mu(f_{3,31} + 2yf_{3,32} + x_3f_{3,31}) \\
\sigma_{33} &= 2\mu(f_{3,33} + x_3f_{3,33}) \\
\sigma_{31} &= 2\mu x_3 f_{3,331} \\
\sigma_{32} &= 2\mu x_3 f_{3,332}
\end{align*}
\]

where \( \mu \) and \( \nu \) are the shear modulus and Poisson's ratio, respectively. For convenience, the stresses given by eqn (8) through eqns (4)-(6) are expressed in a matrix form:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\vdots \\
\sigma_N
\end{bmatrix} = 
\begin{bmatrix}
P_1 & \cdots & P_N
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_N
\end{bmatrix}
\]

where \([P]\) is a function of the coordinates \((x_1, x_2, x_3)\) and \(N\) is the total number of coefficients \(A\) or \(C\).

Satisfying the boundary condition on the crack surface, the relation between the coefficients \(A\) and \(C\) can be summarized in a matrix form:

\[
\begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_N
\end{bmatrix} = 
\begin{bmatrix}
B_1 & \cdots & B_N
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_N
\end{bmatrix}
\]

The detailed complete expression of components of \([B]\) is given in Ref. [7].

For a complete polynomial loading expressed by eqn (3), the maximum degree of polynomial \(M_e\) and the number of coefficients \(N\) can be expressed respectively by \(M_e = 2M + 1\) and \(N = (M + 1)(2M + 3) \times 3\). For an incomplete polynomial loading in which the symmetries of the problem are accounted, the maximum degree of polynomial and the number of coefficients depend not only on \(M\) but also on parameters \(i\) and \(j\).

Once the coefficients \(C\) are determined by solving eqn (10) for a given loading on the crack surface, the stress intensity factor corresponding to this load is evaluated from the following equation

\[
K_I = 8\mu \left( \frac{\pi}{a_1a_2} \right)^{1/2} A^{1/4} \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{k=0}^{M} (-2)^{2k+i+j}(2k+i+j+1)!
\]

\[
\frac{1}{a_1a_2} \left( \frac{\cos \theta}{a_1} \right)^{2i-2+1/2} \left( \frac{\sin \theta}{a_2} \right)^{2i+j} C_{i,j}^{\theta}_{-i-l}
\]

where \(\theta\) is the elliptic angle measured from the \(x_i\) axis and

\[
A = a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta.
\]
3. FINITE ELEMENT ALTERNATING METHOD FOR MULTIPLE CRACKS

The alternating method for the single elliptical crack problem was originally developed by Shah and Kobayashi[8, 9]. In their method, the solution for an elliptical crack, subject to cubic polynomial distribution, is an infinite solid was implemented. Subsequently, Smith and Kullgren[19] introduced the finite element technique into the alternating method, employing the same solution[20] used by Shah and Kobayashi[8, 9]. The limitation to a cubic polynomial pressure, due to exhorbitant work in deriving the appropriate expressions for the chosen potentials, was one of the major impediments to obtaining accurate solutions through the alternating technique.

The present alternating method uses two basic solutions as follows[7]:

Solution 1: The complete general analytical solution for an elliptical crack subject to arbitrary loadings on the crack surface in an infinite solid as explained in the previous section and in Ref. [7].

Solution 2: A general numerical solution technique such as the finite element method or the boundary element method. In the present paper, the finite element method is used because of its simplicity. Use of the finite element method enables the alternating technique to be applied to more complex structural components.

It is important to note that since a number of cracks exist here, each crack will have separate sets of coefficients \( \{ A \} \) and \( \{ C \} \) in relation to Solution 1 above, where \( n \) is the number of the crack and there are \( N_c \) cracks in all. In the following, the subscript \( n \) will be used to denote crack number. The steps required in the present alternating method for the case of multiple cracks in a finite body are as follows. This procedure is summarized in Table 1.

(1) Solve the uncracked body under the given external loads by using the finite element method. The uncracked body has the same geometry as the given problem but for the cracks. To save computational time in solving the finite element equations repeatedly, an efficient equation solver OPTBLOK[21] which has a resolution facility was implemented as explained in Ref. [7]. In

Table 1. Flow chart for finite element alternating technique

<table>
<thead>
<tr>
<th>STEP 1</th>
<th>Solve the Uncracked Body Under External Loads by Using the Finite Element Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>STEP 2</td>
<td>Using FEM Solutions Compute Stresses at the Crack Locations</td>
</tr>
<tr>
<td>STEP 3</td>
<td>Add the Stresses in Step 2 to Those in Step 5</td>
</tr>
<tr>
<td>STEP 4</td>
<td>Are the Stresses in Step 3 Negligible? 1 Yes 1</td>
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<tr>
<td>STEP 5</td>
<td>Determine Coefficients ( A_n ) in the Applied Stresses by Fitting Crack Face Stresses in Step 3</td>
</tr>
<tr>
<td>STEP 6</td>
<td>Determine Coefficients ( C_n ) in the Potential Functions</td>
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<tr>
<td>STEP 7</td>
<td>Calculate the K-Factors for Each Crack for the Current Iteration.</td>
</tr>
<tr>
<td>STEP 8</td>
<td>For Each Crack, Calculate Residual Stresses (i) on External Surfaces and (ii) at All of the Other Crack Locations. Reverse Stresses (i) and Calculate Equivalent Nodal Loads. Add the Contributions to Both (i) and (ii) from Each Crack.</td>
</tr>
<tr>
<td>STEP 9</td>
<td>Consider the Nodal Forces in Step 8 as External Applied Loads Acting on the Uncracked Body</td>
</tr>
</tbody>
</table>

Add the K-factor Solutions of All iterations |

END
OPTBLOK, the reduction of the stiffness matrix is done only once although the reduction of load vector and back substitution may be repeated for any number of iterations, with only a small additional computational time.

2) Using the finite element solution, compute the stresses at the locations of each of the original cracks in the uncracked solid.

3) Add the stresses in Step (2) for each crack to the stresses on the cracks from Step (8). The resultant stresses are the residual stresses on the crack surfaces. This step is skipped in the first iteration.

4) Compare the residual stresses on each crack calculated in Step (3) with a permissible stress magnitude. A suitable choice for this stress magnitude is one per cent of the maximum external applied stress.

Alternatively the convergence of the analysis is also checked with a norm of the stress intensity factor:

$$\|K_\theta\|_n = \sum_{i=1}^{L} |K_\theta(\theta_i)|_n/L$$

in which $L$ points are chosen along the $n$th crack front. The change in norm of stress intensity factor for each cycle of iteration is also monitored. For most cases, the change in norm between the 3rd and 4th iterations becomes less than one per cent at which stage it may be concluded that the analysis is complete.

5) To satisfy the stress boundary condition on the surface of each crack, reverse the residual stresses computed in Step (3). Determine the coefficients $\{A\}_n$ of eqn (3) for each crack using the following least squares fitting

$$I_n = \int_{S_{cn}} (\sigma_{33} - \sigma_{33}^{(0)}) \, dS, \quad n = 1, 2, \ldots, N_c$$

where $\sigma_{33}^{(0)}$ is the reversed residual stress for the $n$th crack and $S_{cn}$ is the region of fitting and $I_n$ is the corresponding functional to be minimized.

Rewriting eqn (3) in matrix form

$$\sigma_{33}^{(0)} = [L]^T \{A\}_n$$

and substituting eqn (15) into eqn (14), we obtain the relation between the coefficients $\{A\}_n$ and the residual stresses

$$\{A\}_n = [H]_n^{-1} \{R\}_n$$

where

$$[H]_n = \int_{S_{cn}} \{L\} [L]^T \, dS$$

$$\{R\}_n = \int_{S_{cn}} \{L\} \sigma_{33}^{(0)} \, dS$$

6) Determine the coefficients $\{C\}_n$ in eqn (4) for the potential functions by solving eqn (10) $\{C\}_n = [B]_n^{-1} \{A\}_n$.

7) Calculate the stress intensity factors at each crack front for the current iteration by substituting coefficients $\{C\}_n$ in eqn (11).

8) Now considering each crack as a single crack in an infinite body, calculate (i) the residual stresses on the external surface of the body and (ii) the stresses at each of the other crack locations due to the applied stresses in Step (5).

(i) To satisfy the stress boundary condition on the external surfaces of the body, reverse the residual stresses and calculate equivalent nodal forces. These nodal forces $\{Q\}_m$ can be expressed
in terms of the coefficients \( \{ C \}_n \):

\[
\{ Q \}_{mn} = -[G]_{mn} \{ G \}_n \quad \text{(no sum on } n) \tag{19}
\]

and

\[
[G]_{mn} = \int_{S_m} [N]^T [\hat{n}] [P]_n \, dS \tag{20}
\]

where \( m \) denotes the number for surface elements (see Fig. 2), \([N]\) is the matrix of isoparametric shape functions, \([\hat{n}]\) is the matrix of normal direction cosines, and \([P]_n\) is the basis function matrix for stresses as defined in eqn (9). The different sets of nodal force vectors \( \{ Q \}_m \) due to the \( N \) different cracks are added to get the overall residual forces \( \{ Q \}_m^* \).

In order to save computational time, the matrices \([G]_{mn}\) are calculated prior to the start of the iteration processes. Although the matrix \([P]_n\) has the singularity of order \( 1/|r| \) at the crack front, the magnitude of the stress decays rapidly with distance from the crack front. Thus the matrices \([G]_{mn}\) for the \( n \)th crack are calculated only at the surface elements which satisfy the following condition:

\[
r_{\min} < 5a_n
\]

where \( r_{\min} \) is the distance of the closest nodal point of each surface element from the center of the \( n \)th elliptical crack as shown in Fig. 2.

(ii) Each crack location will have a contribution to the residual stresses from each of the other cracks. These stresses may be expressed as follows

\[
\sigma_{33pq} = [P]_{pq} \{ C \}_n, \quad q \neq n
\]

where \( \sigma_{33pq} \) are the stresses on the \( q \)th crack due to the \( n \)th crack. The stresses on each of the \( N_c \) cracks are summed over \( n \), and it is these stresses that in Step (3) will be added to those in Step (2).

(9) Consider the nodal forces \( \{ Q \}_n^* \) in Step (8) as external applied loads acting on the uncracked body.

Repeat all steps in the iteration process until the residual stresses on each crack become negligible (Step (4)). To obtain the final solution, add the stress intensity factors of all iterations.
Since the analytical solution for an elliptical crack in an infinite solid is implemented as Solution 1 in the case of surface cracks, it is necessary to define the residual stresses over the entire crack plane including the fictitious portion of the crack which lies outside of the finite body. Moreover, it is well known that the accuracy of the least squares' fitting inside of the fitting region can be increased with the increasing number of polynomial terms; however, the fitting curve may change drastically in the region outside of the fitting. For these reasons, numerical experimentation was done for arriving at an optimum pressure distribution on the crack surface extended into the fictitious region. For a semi-elliptical crack which lies in the region of \(-a_1 < x_1 < a_1\) and \(0 < x_2 < a_2\), it is concluded that the fictitious pressure, which for the region of \(-a_2 < x_2 < 0\) remains constant in the \(x_2\) direction but varies in the \(x_1\) direction, gives the best results among several numerical examples performed in Ref. [7]. This procedure is shown in Fig. 3, where the subscript \(n\), denoting the particular crack number, has been dropped for clarity. The procedure of the fictitious pressure distribution for a semi-elliptical surface crack was successfully used in the analysis of surface cracks in finite thickness plates subject to remote tension as well as remote bending[7] and in pressure vessels with a single crack[11].

4. RESULTS AND DISCUSSION

All problems considered here concern the linear elastic Mode I problems for coplanar semi-elliptic surface cracks in pressure vessels.

In these studies, to express the effects of boundary conditions, crack aspect ratio, cylinder thickness, curvature of cylinder and so on, a magnification factor (normalized stress intensity factor) \(F_p\) defined by the following equation is used for a pressurized cylinder

\[
F_p(\theta) = \frac{K_c(\theta)}{\sigma_1 \left( \frac{\pi a_2}{a_1} \right) A^{1/4}}
\]

where

\[
\sigma_1 = \frac{R_0^2 + R_i^2}{R_0^2 - R_i^2}
\]

\(E(k)\) is the complete elliptical integral of the second kind and \(A\) is given by eqn (12). The denominator of the r.h.s. of eqn (23) corresponds to the stress intensity factor for an elliptical crack,
with the pressure $\sigma_t$ on the crack surface, in an infinite solid. $\sigma_t$ is in fact the Lamé solution for the hoop stress in an internally pressurized cylinder at the inner surface.

The type of problem considered here consists of multiple internal surface cracks in cylindrical pressure vessels subject to a uniform internal pressure $p$. The cylinder was assumed to have two coplanar surface cracks of the same size as shown in Fig. 4. This situation leads to Mode I type fracture problems. Stresses $\sigma_{11}$, corresponding to the plane strain condition ($\sigma_{11} = -\nu(\sigma_{RR} + \sigma_{\phi\phi})$) (where the cylindrical coordinates consist of $(x_1, R, \phi)$ as shown in Fig. 4), were imposed on the end of the cylinder ($x_1 = L$) to account for the end condition of the problem.

Since the cylinder and crack geometry are symmetrical about the $X_1 = 0$ plane, it is only necessary to consider the region $X_1 > 0$. The plane of the crack is also a plane of symmetry; therefore, it is only necessary to analyze one quarter of the cylinder shown in Fig. 4.

The finite element breakdown for the uncracked cylinder is shown in Fig. 5. The mesh in this diagram consists of 160 twenty noded isoparametric elements with 2853 degrees of freedom (before imposition of the boundary conditions). It is worth emphasizing that since the finite element method is employed in analyzing the uncracked cylinder only, there is no need for any special singularity crack tip elements. However, in the region of the crack, the mesh must be sufficiently fine to determine accurate expressions for the residual stresses in this region. It can be noted that in the plane of the crack, the element surfaces are rectangular and so do not match the crack region exactly. This is different to earlier investigations[11, 12] where the elements in the region of the crack were curved to fit the crack region exactly. Numerical experimentation has been carried out to determine the optimum element dimensions and fitting region $S_C$ for the case of a single crack. The results obtained were in extremely close agreement with those for curved elements. The fitting region that is used in these problems is shown in Fig. 6.

Although the general procedure of the finite element alternating method for multiple cracks was presented in Section 3, in the example problems considered in this paper, we may treat a single crack imposing the symmetry condition with respect to $X_1 = 0$ (i.e. $\sigma_{1R} = \sigma_{1\phi} = 0$ and $u_1 = 0$ at $X_1 = 0$).

The matrices $[G]_{im}$, as given in eqn (19) are calculated on the surfaces of the cylinder (Fig. 4). It is noted that the subscript $n$ should be omitted since we utilize the solution for a single crack.
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Fig. 6. Mesh at crack surface with fitting region $S_1$, hatched.

These surfaces correspond to $R = R_1$, $R = R_2$, and $x_1 = L$; and it is only necessary to calculate them on those regions satisfying $r_{\text{min}} < 5a_i$. This is done prior to the start of the iteration process. In addition, since $x_1 = 0$ is a plane of symmetry, it may be seen that the shear stress is zero on this surface and so $[G]_m$ matrices are also calculated for this surface to reflect the zero shear condition. Also to save computation time, the length $L$, as shown in Fig. 4, is sufficiently large so that the $r_{\text{min}} < 5a_i$; and hence there is no need to calculate the $[G]_m$ matrices on this surface.

A search of current literature did not reveal any other techniques for multiple elliptical cracks in cylindrical pressure vessels. One of the few examples of interacting multiple cracks available is that of semi-elliptical surface flaws in a semi-infinite medium. This analysis was carried out by Murakami and Nemat Nasser[3] using a body force method. The alternating method was also applied to this situation with the semi-infinite body modeled by a body with large dimensions relative to the cracks. The results obtained by the alternating method were in excellent agreement with those in [3]. This is illustrated in Fig. 7.

The cases of both thick and thin cylinders have been considered in the following, and the results will demonstrate the interaction effects of two cracks and also the applicability of the results to the ASME Boiler and Pressure Vessel Code.

4.1 Interaction effects

Here the magnification factors $F_p(\theta)$ as defined in eqn (23) are compared for two cracks as described above and a single crack of the same size in a cylinder of identical geometry and under the same loading. Figure 8 shows such a situation for the case of a thin cylinder ($t/R_i = 1/10$) with $\theta = 0^\circ$ corresponding to the point $B$ and $\theta = 180^\circ$ corresponding to the point $A$. It may be seen that the magnification factor increases for all points on the crack surface with the largest increase occurring at $A$. This is to be expected since it is the point closest to the other crack and so it should
be the most critical. As increase of 21% over the case of a single crack was found at this point.

Figure 9 shows the variation of residual stress over surface $S_r$ with each cycle of iteration in the alternating technique. It is seen that the residual stress decreases rapidly and monotonically with the number of iterations. After the fifth iteration, the residual stress became 0.17% of the hoop stress $\sigma_r$. The angular distribution of the stress intensity factors for each iteration is plotted in Fig. 10, and this also demonstrates the convergence to the final solution.

In the present finite element alternating method, the 21 terms of the fifth order polynomial in eqn (3) were used for fitting the residual stresses in Step (5). It was found that a fifth order polynomial was sufficient to obtain accurate values for the stress intensity factors[7]. The C.P.U. time for this analysis was about 2650 sec on a Cyber 74.

Figure 11 shows another example of a thin cylinder. Here the crack depth is greater with $a_2/t = 2/3$, and the corresponding magnification factors are higher. A parametric study for different crack depths at various points on the crack surface is shown in Fig. 12. From this it is evident that the magnification factors increase as the crack depth increases.

An example of a thick cylinder ($t/R_i = 2/3$) is presented in Fig. 13. Here again the magnification factors increase due to the interaction of the two cracks. The interaction effects for this case are not as great as those for a thin cylinder.

4.2 ASME Code

Section XI of the ASME Boiler and Pressure Vessel Code recommends that two interacting surface flaws in a pressure vessel should be modeled by a single elliptical crack that covers both flaws. This can be expected to lead to a conservative estimate of the critical stress intensity factor. In this analysis, it is assumed that two interacting flaws are modeled firstly by two elliptical cracks and secondly a single long crack as laid down in the ASME code. The aspect ratio for this long crack is $a_2/c$ where $c = d/2 + a_1$, as defined in Fig. 14. The results for a thin cylinder ($t/R_i = 1/10$) are also shown in Fig. 14. Using the set of axes in the diagram, values of the magnification factor are plotted against the $X_r/c$ coordinate. The normalized stress intensity factor used here is defined.
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Fig. 9. Variation of residual stress on crack surface.

\[ \frac{a_2}{t} = \frac{1}{2} \]
\[ \frac{1}{R_1} = \frac{1}{10} \]
\[ \frac{a_2}{a_1} = \frac{2}{3} \]
\[ 2a_2/d = \frac{3}{4} \]

--- Fifth Iteration

--- 1st, 2nd, 3rd, 4th Iteration

Fig. 10. Convergence of magnification factor with successive iterations.
Fig. 11. Comparison of magnification factors for two cracks and single crack in thin cylinder ($a_2/t = 2/3$).

Fig. 12. Variation of magnification factors with the crack depth.
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![Diagram showing multiple surface cracks in pressure vessels]

$$\alpha_2/t = 1/2$$

$$t/R_i = 2/3$$

$$\alpha_2/a_i = 2/3$$

$$2a_i/d = 3/4$$

Fig. 13. Comparison of magnification factors for two cracks and single crack in thick cylinder ($a_2/t = 1/2$).

![Diagram showing comparison of normalized stress intensity factors for thick and thin cylinders]

$$\alpha_2/t = 1/2$$

$$t/R_i = 1/10$$

$$\alpha_2/a_i = 2/3$$

$$2a_i/d = 3/4$$

Fig. 14. Comparison of normalized stress intensity factors for two cracks and single long crack in thin cylinder ($a_2/t = 1/2$).
simply as

\[ M_F = \frac{K(\theta)}{\sigma_n(\pi a_2)} \]  

(25)

where \( \sigma_n \) has been defined in eqn (24) and \( a_2 \) is the length of the minor axis. It may be seen that the stress intensity factors for the single large crack are generally greater than those due to the two interacting smaller cracks. In particular the largest stress intensity factor occurs for a single long crack which implies the critical value for the single crack is greater than for the two interacting cracks. This indicates the conservative recommendations of the ASME code.

Figure 15 presents another example of a thin cylinder. Here \( t/R_i \) is increased and a similar trend results. The situation is the same for a thick cylinder with \( t/R_i = 2/3 \) and the results are presented in Fig. 16.

5. CONCLUDING REMARKS

The foregoing analysis demonstrates the usefulness of the finite element alternating technique for routine evaluation of accurate stress intensity factors in three-dimensional complex structural components. In addition, it was seen that it is possible to use rectangular elements in the region of the cracks without sacrificing any accuracy.

The results also show the significant effects on the stress intensity factors due to the interacting cracks particularly at the crack front regions that are close together. The recommendations of the ASME Boiler and Pressure Vessel Code in Section XI have also been investigated, and it may be concluded that they will tend to underestimate the design life of a flawed structure.
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Fig. 16. Comparison of normalized stress intensity factors for two cracks and single long crack in thick cylinder ($a_0/t = 1/2$).

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