A HYBRID FINITE ELEMENT METHOD FOR STOKES FLOW: 
PART I—FORMULATION AND NUMERICAL STUDIES

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A hybrid finite element scheme, based on assumed deviatoric fluid stresses in the element and continuous velocity fields at the element-boundaries, is presented. The deviatoric stress and hydrostatic pressure field are subject a priori to the constraints of balance of momenta. The advantages of the present scheme are discussed, and its versatility is demonstrated through a few numerical examples. Studies of convergence and stability of the method are included in Part II of the paper.

1. Introduction

Computational fluid mechanics based on the finite element method has been developed (a) by using a stream function or a stream function vorticity formulation, and (b) by using a primitive variables formulation. The first approach developed as a natural extension of computational fluid dynamics based on the finite difference method wherein the Navier–Stokes equations are transformed into stream function and vorticity transport equations. The approach (b) is a result of experiences in solid mechanics wherein, most commonly, the finite element methods are based on displacements being the primal variables.

Conceptually, formulation (a) appears attractive since it allows the Navier–Stokes equations to be transformed into a set of simpler, decoupled equations which are solved sequentially. However, the computational process has certain intrinsic limitations, especially in the implementation of boundary conditions, and in the extension of the method to 3-dimensional problems.

Approach (b) above makes use of velocity and pressure as basic variables. Early applications of this approach yielded severely distorted pressure solutions while velocity solutions were found to be ‘acceptable’. Numerical experiments led Hood and Taylor [1] to develop a mixed interpolation procedure wherein the pressure variable is interpolated by a lower order polynomial than the velocity field and the residuals in the Galerkin process weighted differently. Several explanations have been given for the phenomena of spurious pressure solutions when equal-order interpolations are used for both velocity and pressure. Hood and Taylor [1] provided arguments in terms of ‘balancing of residuals’ from momentum and continuity equations. However, this explanation was considered inadequate by Zienkie-
wicz [2] and by Olson and Tuann [3]. Recently a constraint analysis was independently proposed by Bratianu and Atluri [4] and by Gresho et al. [5].

Even when mixed-order interpolations are employed, numerical difficulties are still encountered [6]. For example, piecewise linear approximation for velocity and piecewise constant approximation for pressure have been found to work poorly in some cases. The solutions sometimes display acceptable velocities, but totally spurious pressures which are afflicted with the ‘checkerboard syndrome’, wherein pressure oscillations occur which are frequently one sign on all ‘black’ elements and of the opposite sign on all ‘red’ elements. A great deal of research has been done to device computational ‘filters’ for extracting ‘good’ pressures from ‘polluted’ numerical results in certain cases [7]. Schneider [8] and Schneider et al. [9] have shown that equal order interpolation procedures for both velocity and pressure can be used if the continuity equation is replaced by the pressure Poisson equation.

In the past few years, a great deal has been published concerning the use of selective reduced integration and penalty methods (SRIP-methods) in the analysis of incompressible fluids [10]. These methods have been primarily developed by way of numerical experimentation. The basic idea is to incorporate the incompressibility condition into the constitutive equations via a penalty parameter. Thus, the finite element equation system is obtained solely from the momentum equations. Recent theoretical investigations of these methods by Oden, Kikuchi and Song [11–13] indicate that some of the RIP-methods are, in fact, unstable. Attempts have been made at ‘averaging’ or ‘filtering’ the pressures to stabilize these methods, but the methods are, in general, still very sensitive to singularities, and distortions of the mesh. These ‘drawbacks’ notwithstanding, there are some RIP methods which do work well for certain problems, and have the advantage that pressure is eliminated as a direct solution variable with the result that the finite element system of equations is ‘smaller’ in size.

In the following, we present an alternate solution scheme with the deviatoric fluid stresses along with the element-boundary velocity field as the assumed field variables. The hydrostatic pressure and deviatoric stress fields within each element are subject to the constraint of linear momentum balance. Thus the hydrostatic pressure field within each element is determined from the assumed deviatoric stress field to within an arbitrary constant. The assumed element-boundary velocity field serves as a Lagrange-multiplier to enforce the constraint of interelement traction reciprocity. Hence is the name ‘hybrid’ associated with the present finite element method [13]. It will be shown that the final finite element system of equations would have, as unknowns: (i) the finite-element nodal velocities; and (ii) the ‘constant term’ in the arbitrarily varying pressure field over each element. Thus, if the spatial domain consists of $N$ finite elements, the present method would have $N$ additional equations for the entire domain, as compared to the popular RIP methods. It will be demonstrated here that the present method, wherein ‘exact’ integration may be used, leads to the desired accuracies in both velocities and pressure. A mathematical study of the convergence and stability of the present method are being reported on in Part II of this paper.

2. A sketch of the method of analysis

We consider the familiar ‘Stoke’s flow’ of an incompressible viscous fluid in a domain with spatial cartesian coordinates $x_i$. We use the notation: $\rho$, the fluid density; $\bar{F}_i$ are body forces
(excluding inertia) per unit mass; \( \sigma_{ij} \) the fluid stress; \( \sigma'_{ij} \) the deviatoric stress; \( p \) the hydrostatic pressure; \( v_i \) the velocities; \( V_{ij} \) the velocity strains; \( \bar{t}_i \) the prescribed tractions on a boundary segment \( S_1 \); \( \bar{v}_i \) the prescribed velocities on a boundary segment \( S_v \); \( (\ )_i \) denotes partial differentiation w.r.t. \( x_i \). The field equations are:

\[
v_{i,i} = 0 \quad \text{in} \quad V \quad (\text{incompressibility}), \tag{1}
\]

\[
\sigma_{ij} + \rho \bar{F}_i = 0; \quad \sigma_{ii} = \sigma_{ii} \quad \text{in} \quad V \quad (\text{momentum balance}), \tag{2}
\]

\[
V_{ij} = v_{i(i,j)} = \frac{1}{2}(v_{ij} + v_{ij}) = V_{ji} \quad (\text{compatibility}), \tag{3}
\]

\[
\sigma_{ij} = \partial A/\partial V_{ij} \quad (\text{constitutive law}), \tag{4}
\]

where

\[
A = A(p, V_{lm}) = -p V_{kk} + \mu V_{lm} V_{lm}, \quad \mu = \text{coefficient of viscosity}, \tag{5}
\]

and thus

\[
\sigma_{ij} = -p \delta_{ij} + 2\mu V_{ij}, \tag{6}
\]

\[
\sigma_{ij} n_j = \bar{T}_i = \bar{\bar{T}}_i \quad \text{at} \quad S_1 \quad (\text{traction b.c.}), \tag{7}
\]

\[
v_i = \bar{v}_i \quad \text{at} \quad S_v \quad (\text{velocity b.c.}). \tag{8}
\]

In (7), \( n_j \) are components of a unit outward normal to \( S_1 \). We note further that

\[
\sigma_{ij} = -p \delta_{ij} + \sigma'_{ij} \tag{9}
\]

thus,

\[
\sigma'_{ij} = 2\mu V_{ij}. \tag{10}
\]

It is well known that for all compatible velocity fields \( v_i \), which satisfy (8) a priori, but need not satisfy (1) a priori, the conditions of stationarity of the functional

\[
P(p, v_i) = \int_V [A(p, V_{ij}) - \rho \bar{F}_i v_i] dv - \int_{S_1} \bar{T}_i v_i ds \tag{11}
\]

for all arbitrary variations \( \delta p \), and admissible variations \( \delta v_i \) (i.e., \( \delta V_{ij} - \delta v_{i(i,j)} \)), lead to the Euler–Lagrange equations which are the above given (1), (2) and (7) respectively. Suppose now that the constraints (3) and (8) are relaxed through Lagrange multipliers \( \sigma_{ij} \) and \( T_i \) respectively. Thus, a general functional whose stationarity conditions lead to (1), (2), (3), (7) and (8) is given by

\[
G(p, V_{ij}, v_i, \sigma'_{ij}) = \int_V \{ -p V_{kk} + \mu V_{lm} V_{lm} - \sigma_{ij} (V_{ij} - v_{i(i,j)}) - \rho \bar{F}_i v_i \} dv
- \int_{S_1} \bar{T}_i v_i ds + \int_{S_v} \sigma_{ij} n_j (\bar{v}_i - v_i) ds. \tag{12}
\]

In (12), it is understood that \( \sigma_{ij} = -p \delta_{ij} + \sigma'_{ij} \).

We now consider the familiar Legendre contact transformation to express the stress working density of the fluid in terms of stresses. Thus, let
$-B(\sigma_{km}) = A(p, V_{km}) - \sigma_{km}V_{km}$  

such that

$V_{ij} = \partial B/\partial \sigma_{ij}$.

Using (5) and (10) in (13) we obtain

$B(\sigma_{km}) = (1/4\mu)\sigma'_{km}\sigma'_{km}$.

We rewrite (2) in the form

$(\sigma'_{ij})_j - p_j + \rho F_i = 0; \quad \sigma'_{ij} = \sigma^*_{ij}$.

We designate the fields $\sigma'_{ij}$ and $p$ that satisfy (16) and (7) as being ‘admissible’ in the present formulation. From the general functional in (12), one can eliminate as independent variables (i) $V_i$ by using the contact transformation as in (13) and (15), and (ii) $v_i$ by satisfying (7) and (16) a priori. We thus obtain the reduced functional

$C = -\int_V (1/4\mu)\sigma_{ij}\sigma'_{ij}dv + \int_{S_i} \sigma_{ij}\delta_i\delta_j ds$.

We now show that, among all admissible fluid stress fields, the actual solution, i.e., one which satisfies the compatibility condition (3), the incompressibility condition (1), and the velocity b.c. (8), is characterized by the stationary value of the functional $C$ of (17). To this end, note that the variations $\delta\sigma'_{ij}$ and $\delta p$ satisfy the constraints

$\delta\sigma'_{ij} - \delta p_j = 0; \quad \delta\sigma'_{ij} = \delta\sigma^*_{ij}$ in $V$.

$\delta\sigma_{ij}n_j = 0$ at $S_i$.

From (17) we see that

$0 = \delta C = -\int_V (1/2\mu)\sigma_{ij}\delta\sigma'_{ij}dv + \int_{S_i} n_i\delta\sigma_{ij}\delta_i ds$

$= -\int_V \{[(1/2\mu)\sigma_{ij} - f_{(i,j)}]\delta\sigma'_{ij} - [f_{,i}][\delta p] \} dv + \int_{S_i} n_i(\delta_i - f_i)(\delta\sigma'_{ij} - \delta p\delta_{ij}) ds$

wherein (18), (19) have been used. It is seen from (20) that when $C$ attains a stationary value, the following conditions hold:

(i) the velocity strains, $(1/2\mu)\sigma'_{ij}$, derived from the admissible stress field correspond to a compatible strain field,

(ii) the velocity field is subject to the incompressibility constraint, and

(iii) the velocity b.c. is obeyed.

In using the variational statement of (20) in conjunction with a finite element method, the
assumed stress field, in order to be admissible, must be such that it satisfies a priori not only
the constraints of (16) and (7), but also the constraint that at the inter-element boundaries, the
tractions must be reciprocated (Newton’s 3rd law!) between adjoining elements. This inter-
element traction reciprocity for two spatial finite elements may be satisfied either ‘exactly’ or
in an integral average sense. Exact satisfaction may be achieved by selecting the stress field in
each of the two neighbouring elements such that along their common boundary-segments, the
traction field is a unique-order polynomial expressed in terms of tractions at a finite number of
nodes at the common boundary. Complete traction reciprocity is then enforced by equating
the nodal tractions of the two neighbouring elements at their common boundary, by using
discrete (nodal) Lagrange multipliers. These discrete Lagrange multipliers will have the
physical meaning of integral averages of the element-boundary velocity field. Alternatively the
traction reciprocity condition can be satisfied in an average sense, by choosing an arbitrary-
order admissible stress field in each of the two neighbouring elements, and by choosing a
continuous field of Lagrange multipliers at the interelement boundary. This continuous
Lagrange-multiplier field will have the physical interpretation of the continuous velocity field
at the interelement boundary. We shall use the latter approach for purposes of illustration
here. The element-boundary velocity field may be chosen to satisfy (8) a priori. However, if
the boundaries are curved, satisfaction of (7) may, in general, be inconvenient. In as much as a
velocity field is being introduced at the element boundary as a Lagrange multiplier, we may, in
fact, treat (7) as an a posteriori constraint. With this motivation, we consider the problem of
finding the stationary value of the functional

$$C^* = \sum_n \left\{ -\int_{V_n} (1/4\mu)\sigma' \cdot \sigma' dv + \int_{\partial V_n} n_i (\sigma'_{ij} - p\delta_{ij}) \tilde{v}_i ds - \int_{S_m} \bar{T}_i \tilde{v}_i ds \right\}$$

where $V_n$ is the $n$th finite element, $\partial V_n$ its boundary, $S_m$ is the part of $\partial V_n$ where tractions are
prescribed, and $\tilde{v}_i$ is the velocity field (Lagrange multiplier) at the boundary $\partial V_n$. The
condition that $C^*$ is stationary leads, in addition to (1) and (3), to the following a posteriori
constraints

$$[(\sigma'_{ij} - p\delta_{ij})n_i]^+ + [(\sigma'_{ij} - p\delta_{ij})n_i]^* = 0 \quad \text{at } \rho_n,$$

$$\sigma'_{ij} n_i - \bar{T}_i @ S_m.$$  (23)

In (22), $\rho_n$ is the inter-element boundary, and $+$ and $-$ refer, arbitrarily to the two sides of $\rho_n$.

We shall now employ the following notation. A boldface symbol denotes a vector; ~ under a
boldface symbol denotes a matrix, and ($'$ denotes a transpose. We denote the symmetric
stress tensor by the $(6 \times 1)$ vector $\sigma'$; and the matrix of fluid compliance by a diagonal matrix $C$
(with either $(1/2\mu)$ or $(1/\mu)$ as elements). Thus, we rewrite (21) as

$$C^* = \sum_n \left\{ -\int_{V_n} \frac{1}{2} \sigma' \cdot \sigma' dv + \int_{\partial V_n} T^i \cdot \tilde{v} ds - \int_{S_m} \bar{T}^i \tilde{v}_i ds \right\}$$

where $\sigma'$ is the $(6 \times 1)$ vector representation of the symmetric deviatoric stress tensor $\sigma'_{ij}$; and
$T$ is the $(3 \times 1)$ vector representation of the traction field $(\sigma'_{ij} - p\delta_{ij})n_i$; $\tilde{v}$ is the $(3 \times 1)$ vector of
the assumed boundary velocity field.
We first assume an arbitrary polynomial approximation in each $V_n$ for the deviatoric stress field $\sigma'$ as:

$$\sigma' = B(x)\beta \quad \text{in each } V_n,$$

(25)

where $\beta$ represents $N_p$ undetermined parameters, and $B$ are functions of spatial coordinates $x$. Thus, the stress-energy of the element $V_n$ is

$$\int_{V_n} \frac{1}{2} \sigma' \cdot \sigma' = \frac{1}{2} \beta H \beta.$$

(26)

From the structure of $c$, it is immediately evident that $H$ is positive definite for any assumed $\sigma'$.

We now assume the hydrostatic pressure $p$ in each element, to satisfy the constraint (16). Assuming for simplicity that $F_i = 0$, it is seen from (16) that $p$ in each $V_n$ can be determined from the chosen $\sigma'$ in $V_n$, to only within an arbitrary constant, $\alpha_n$. Thus,

$$p = \alpha_n + D\beta \quad \text{in } V_n.$$

(27)

Thus, $p$ in $V_n$ may, in general, involve $N_p$ undetermined coefficients. From the specific element-developments described later in this paper, it may be seen that, in general, $N_p$ need not be equal to $N + 1$.

Note that $D$ of (27) is related, in general, to $B$ of (25) through the constraint (16), and that $\alpha_n (n = 1, 2, \ldots)$ is different for each element $V_n (n = 1, 2, \ldots)$.

Finally, the boundary velocity field is assumed as

$$\hat{v} = L q \quad \text{at } \partial V_n,$$

(28)

where $L$ are functions of the boundary-coordinates at $\partial V_n$, and $q$ is the vector of $N_q$ number of nodal velocities.

Using (25), (27) and (28) in (24), we obtain,

$$C^* = \sum \{-\frac{1}{2} \beta H \beta + q^i G \beta - q^i S \alpha_n - Q_i q\}$$

(29)

with apparent definitions for $H$ (involving integration over $V_n$), $G$ and $S$ (involving integrations over $\partial V_n$), and $Q_i$ (involving integrations over $S_n$). It is noted that these integrations may be performed exactly by using the necessary order quadrature rules, as in the present numerical applications discussed later.

We observe that the vector $\beta$ as well as the constant $\alpha_n$ are arbitrary and independent for each element $V_n$ (i.e., not subject to nodal connectivity). The stationarity condition of $C^*$ with respect to $\beta$ leads to the solution

$$\beta = H^{-1} G' q$$

(30)
which is always possible since $H$ is positive (and symmetric). Using (30), we write (29) as

$$C^* = \sum \left\{ \frac{1}{2} q'^T k q - q'^T S \alpha_n - Q' \cdot q \right\}$$

(31)

where

$$k = GH^{-1}G^T.$$  

(32)

Denoting by $\alpha$ the vector of coefficients $\alpha_n \ (n = 1, 2, \ldots \text{number of elements})$, it is evident that the final system of finite element equations for the flow domain have the form

$$K q^* - S^* \alpha = Q,$$

(33a)

$$S^* q^* = 0.$$  

(33b)

The matrices and vectors in (33) have the usual meanings in the global sense.

Thus, as seen from (33a), even though the hydrostatic pressure can be assumed in each element $V_n$ to be an arbitrary-order polynomial, it is only the constant term $\alpha_n$ of this polynomial in each $V_n$ that has to be solved for, as an independent variable, from the global finite element equations. It is also seen from (33) that the additional number of unknowns (and the additional 'computing') in the present method, as compared to the popular RIP methods, is equal to the number of elements $V_n$ (each $V_n$ with a different $\alpha_n$).

### 3. Several element formulations

In the following we discuss several 2-dimensional element formulations. The variety of elements developed and tested is summarized as follows in Table 1.

The above elements were developed both for rectangular shapes as well as 'isoparametric' geometric shapes. This made the studies of effects of mesh distortions on the obtained

<table>
<thead>
<tr>
<th>Element</th>
<th>Hydrostatic pressure</th>
<th>Deviatoric stress</th>
<th>Boundary velocity</th>
<th>Nodes per element</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPLSE 4</td>
<td>Constant</td>
<td>Bilinear</td>
<td>Linear</td>
<td>4</td>
</tr>
<tr>
<td>CPQSE 4</td>
<td>Constant</td>
<td>Quadratic</td>
<td>Linear</td>
<td>4</td>
</tr>
<tr>
<td>LPOSE 4</td>
<td>Bilinear</td>
<td>Quadratic</td>
<td>Linear</td>
<td>4</td>
</tr>
<tr>
<td>CPOSE 4</td>
<td>Constant</td>
<td>Quadratic</td>
<td>Quadratic</td>
<td>8</td>
</tr>
<tr>
<td>LPQSE 8</td>
<td>Bilinear</td>
<td>Quadratic</td>
<td>Quadratic</td>
<td>8</td>
</tr>
<tr>
<td>QPQSE 8</td>
<td>Quadratic</td>
<td>Quadratic</td>
<td>Quadratic</td>
<td>8</td>
</tr>
</tbody>
</table>

However, it should be noted that in the present method, one needs to invert $H$ as in (32). It is also noted that $H$ in general, in the present method, is not fully populated.
velocity as well as pressure solutions possible. The required element integrations were performed either exactly where possible, or by using the necessary-order quadrature rules.

The details of element formulations are given below, for the convenience of readers who may wish to implement them in their own programs.

**CPLSE 4 Element.** This is the simplest element developed. The assumptions in each element arc as follows:

\[
p = \alpha_1, \\
\sigma'_{11} = \beta_1 + \beta_2 x + \beta_3 y, \quad \sigma'_{12} = \beta_4 + \beta_5 x - \beta_2 y, \quad \sigma'_{22} = \beta_6 + \beta_7 x - \beta_5 y.
\]  

The boundary velocity field is linear in terms of the two nodal velocities at each boundary segment.

**CPQSE 4 and CPQSE 8 elements.** Both these elements have constant pressure and quadratic stress distributions. Thus, in each of these elements,

\[
p = \alpha_1, \\
\sigma'_{11} = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \beta_6 y^2, \\
\sigma'_{12} = \beta_7 + \beta_8 x - \beta_2 y - 2 \beta_5 xy + \beta_9 x^2 - \frac{1}{2} \beta_4 y^2, \\
\sigma'_{22} = \beta_{10} + \beta_{11} x - \beta_8 y - 2 \beta_9 xy + \beta_{12} x^2 + \beta_5 y^2.
\]

The CPQSE 4 is used in conjunction with quadrilateral elements with 4 nodes, while the CPQSE 8 is used in conjunction with quadrilateral elements with 8 nodes. The boundary velocity fields for CPQSE 4 and CPQSE 8, are thus linear and quadratic, respectively.

**LPQSE 4 and LQPSE 8 elements.** Both these elements have bilinear pressure and quadratic stress distributions. Thus, in each of these elements,

\[
p = \alpha_1 + (\beta_2 + \beta_9) x + (\beta_8 + \beta_{13}) y + (\beta_4 + 2 \beta_{10}) xy, \\
\sigma'_{11} = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \beta_6 y^2, \\
\sigma'_{12} = \beta_7 + \beta_8 x + \beta_4 y - 2 \beta_5 xy + \beta_{10} (x^2 + y^2), \\
\sigma'_{22} = \beta_{11} + \beta_{12} x + \beta_{13} y + \beta_{14} xy + \beta_{15} x^2 + \beta_5 y^2.
\]

The LPQSE 4 and LQPSE 8 are: (i) developed in conjunction with quadrilateral elements with 4 and 8 nodes respectively, and (ii) thus have linear and quadratic boundary velocity fields, respectively.

**QPQSE 8 element.** This is the most versatile element developed. It is based on quadratic pressure and quadratic stress distributions. These fields are

\[
\sigma'_{11} = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy + \beta_5 x^2 + \beta_6 y^2, \\
\sigma'_{12} = \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} xy + \beta_{11} (x^2 + y^2), \\
\sigma'_{22} = \beta_{12} + \beta_{13} x + \beta_{14} y + \beta_{15} xy + \beta_{16} x^2 + \beta_{16} y^2, \\
p = \alpha_1 + (\beta_2 + \beta_9) x + (\beta_4 + \beta_{10}) y + (\beta_4 + 2 \beta_{11}) xy + \frac{1}{2} (2 \beta_5 + \beta_{10}) x^2 + \frac{1}{2} (\beta_{10} + 2 \beta_{16}) y^2.
\]
This model is used in conjunction with a quadrilateral element with 8 nodes, and has a quadratic boundary-velocity field.

From the studies on existence of the solution, presented in Part II of the paper, the rank conditions of the 'stiffness matrix' $k$ of the present method dictate certain restrictions on the number of pressure, velocity, and deviatoric stress parameters ($N_p$, $N_u$ and $N_d$, respectively) that may be employed. From the discussions in Part II of the paper, it is seen that all the above element developments, satisfy these criteria. It is also to be noted that such criteria are, in fact, trivial to be met. (In short this criterion amounts to requiring that $N_d \geq N_u - 4$ for planar elements.)

4. Numerical studies

We present here some representative (this overworked phrase is used in all honesty in the present context) results using the above elements are discussed here. The main objectives of these studies were:

(i) to demonstrate the feasibility of the new elements in solving steady, laminar flow of viscous incompressible fluids with negligible inertia,

(ii) to demonstrate the versatility of these new elements to yield accurate solutions for pressure, stress, and velocities, even with mesh distortions, etc.

The first example is that of a parallel flow through a straight channel. This is a rather simple, one-dimensional flow with a linear pressure variation and a parabolic velocity profile. The analysis domain and boundary conditions, are presented in Fig. 1(a). A two dimensional simulation, as shown in Fig. 1(b), is used for convenience only. First, rectangular elements with 4 and 8 nodes are considered. Then, these meshes are distorted arbitrarily and the program was reexecuted. Results from these numerical experiments are summarized in Fig. 2. It is noted that all the results were insensitive to mesh distortion.

![Fig. 1(a). Parallel flow in a straight channel. Domain and boundary condition definitions.](image-url)
Fig. 1(b). Different finite element meshes for problem in Fig. 1(a).

Fig. 2. Computed results for problem in Fig. 1. (a) Velocity solution. (b) Pressure solution.

Fig. 3 presents results for a couette flow simulation. The shape of the velocity profile is determined by the dimensionless pressure gradient $P$ [14]. For $P > 0$, i.e., for a pressure decreasing in the direction of motion, the velocity is positive over the entire width of the channel. For negative values of $P$, the velocity may become locally negative (i.e., back flow may develop starting from the wall at rest).

Fig. 4 represents the analysis domain for 2-dimensional flow in a 90° corner. The vertical wall is stationary, while the horizontal wall is moving toward the corner with a constant
Fig. 3. Results of a couette flow simulation.

Fig. 4. Analysis domain for 2-dimensional flow in a 90° corner.
velocity $U$. Fluid in the region between the planes is set in motion, as might happen in a cylinder with a moving piston. For comparison purposes, the test problem used in [15] is studied here. Note that the stress solution has a singularity at $X = Y = 0$. (Since the stress field in the present development is arbitrary, it is possible to develop special elements that account for singularities. For a comprehensive account of hybrid elements for treating non-removable singularities, see [16].) However, in the present analysis, the region next to the corner point is not included in the finite element model, as in [15]. Boundary conditions are imposed in terms of velocity components at walls and tractions for the other two boundaries. The present numerical results are presented in Figs. 5 and 6.

Fig. 5. Computed pressure solution for problem in Fig. 4.

Fig. 6. Computed velocity solution for problem in Fig. 4.
As a final example, we consider the case of a plane slider bearing, treated by Schlichting [14] who gave the analytical prediction for velocity and pressure to be

\[
p(x) = p_0 + 6\mu U \frac{L}{h_1^2 - h_2^2} \frac{(h_1 - h)(h - h_2)}{h^2},
\]

\[
v_1(xy) = U \left(1 - \frac{y}{h}\right) - \frac{h^2}{2\mu h} \frac{p'}{y} \left(1 - \frac{y}{h}\right),
\]

where \(h - h(x)\) as shown in Fig. 7, and \(p'\) is given by

\[
p' = \frac{6\mu U}{h^2} \left(1 - \frac{1}{h} \frac{2h_1h_2}{h_1 + h_2}\right).
\]

Here \(U\) is the prescribed velocity of the guide surface, and \(v_1\) is velocity in \(x\)-direction. Two meshes of 16 and 32 8-noded elements were considered, as in Fig. 7. Two types of 8-noded elements, LPOSE 8 and QPQSE 8, were used. In each of the 4 cases computed, the velocity solution differed from the analytical solution, (43), by less than 1%, and each of these solutions (indistinguishable in the figure), is shown in Fig. 8.

The pressure solutions for each of the 4 types of mesh are shown in Table 2.

From Table 2, an engineering feel can be obtained for the convergence of the pressure solution with mesh refinement and/or with higher order polynomial approximation for pressure and stress in each element.

The above results, and others reported in [17], appear to indicate that the present hybrid elements ‘perform’ well, (i) with mesh distortions, (ii) in regions of singularity, and (iii) yield reliable predictions for both velocities and pressures.

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**Fig. 7.** Analysis domain in finite element mesh: plane slider bearing.
Fig. 8. Computed velocity profiles for problem in Fig. 7. —: analytical solution, O: computer solution.

Table 2

<table>
<thead>
<tr>
<th>Location $x$</th>
<th>Analytical solution $\times 10^4$</th>
<th>Numerical solutions $\times 10^4$</th>
</tr>
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5. Closure

Certain new hybrid elements have been presented to treat the constraint of incompressibility in a flow problem. These elements yield reliable accuracies for velocities as well as pressure. In developing these elements, exact analytical integrations, or numerical integration with necessary order quadrature, can be used. However, as compared to the currently popular
RIP methods, which involve only nodal velocities as unknowns, the present finite element system of equations is larger by the same number as the number of elements used, and, further, the present method involves inverting a matrix \( \mathbf{H} \) for each element.

The extensions of the present method to treat Navier-Stokes equations will be presented in a forthcoming paper. Also, the extension of the method to treat problems of flow with a slight compressibility, will be presented elsewhere shortly.

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References