A NUMERICAL STUDY OF THE USE OF PATH INDEPENDENT INTEGRALS IN ELASTO-DYNAMIC CRACK PROPAGATION

T. NISHIOKA† and S. N. ATLURI‡
Center for the Advancement of Computational Mechanics, School of Civil Engineering, Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.

Abstract—The use of the path independent J' integral for dynamic crack propagation, which has the physical meaning of energy release rate is numerically studied by the finite element method. Other path independent integrals are also investigated along with the J' integral. Numerical results show that the combined use of the J' integral and the finite element method is a useful tool to obtain the fracture parameters such as the stress intensity factors and the energy release rates. The use of the several other types of path-independent integrals, despite their lack of a direct interpretation as energy release rates, is also demonstrated. This is so, because the alternate path-independent integrals have been explicitly expressed in terms of the time-dependent K-factors, or the energy release rate, in the present work.

1. INTRODUCTION

Recently Atluri[1] has derived many types of path independent integrals for cracks in elastic and inelastic solids, on the basis of very general conservation laws. For an elastodynamically propagating crack, it was found[1] that the path-independent integral in Ref.[1] has the meaning of the rate of change of Lagrangean per unit crack extension in a solid in dynamic equilibrium.

In the previous paper[2], the present authors have, through a simple modification of the integral in[1], derived a new path-independent integral J' which has the meaning of energy release rate for dynamic crack propagation. In Ref.[2] other types of path independent integrals derived by Bui[3] and Kishimoto et al.[4, 5] were also carefully examined. Using the general solutions of the asymptotic fields near the tip of an elastodynamically propagating crack under combined Mode I, II and III conditions, the relations of the path independent integrals derived by Atluri[1] and Kishimoto et al.[4, 5] to the stress intensity factors were established in Ref.[2]. These relations naturally differ from those of the energy release rates to the stress intensity factors[2].

In the present paper, the practical utility of the above path independent integrals, especially the J' integral, for the determination of the crack-tip parameters such as the stress intensity factors and the energy release rates is demonstrated. Numerical evaluations of the path independent integrals, using the finite element method, are performed for (i) self-similar, constant velocity, crack propagation in a square plate, and (ii) prediction type fracture simulation (non-constant velocity crack propagation) in a DCB specimen. From the structure of the path independent integral's relation to the kinetic energy, the existence of many path independent integrals similar to the J' integral is also demonstrated.

2. PATH INDEPENDENT INTEGRALS FOR DYNAMIC CRACK PROPAGATION

We now consider a contour integral path including the bounded volume as shown in Fig. 1. The radius ε is considered to be very small and shrunk to zero in the limiting process. Thus, Γ may be referred to as the limiting path or near-field path while Γ may be called the far-field path. It is noted that the far-field path presently considered is spatially fixed although the crack propagates with the velocity C(t).

For a crack propagating at an angle θc measured from the X1 axis (see Fig. 1), the energy release rate G can be written as[1]:

\[ G = (C_1/C)G_h = G_1 \cos \theta_c + G_2 \sin \theta_c \]

where \( C_h \) denotes the component of the crack velocity in the X_h direction. Using the limiting path,
Atluri [1] defines the $G_k$ components as

$$G_k = \lim_{e \to 0} \int_{\Gamma_c} [(W + T)n_k - t_i u_{i,k}] \, dS \tag{2}$$

where $W$ and $T$ are the strain and kinetic energy densities respectively, $n_k$ the outward normal direction-cosines, $t_i$ the traction, $u_i$ the displacement, and $( )_k$ denotes $\frac{\partial( )}{\partial X_k}$. The strain energy density is related to the stress $\sigma_{ij}$ and strain $\epsilon_{ij}$ by $W = \frac{1}{2} \sigma_{ij} \epsilon_{ij}$. Denoting by a superposed dot the material time derivative, the kinetic energy density is expressed by $T = \frac{1}{2} \rho u_i \dot{u}_i$ in which $\rho$ is the mass density ($\dot{u}_i$ being the total or material velocity).

Recently the far-field path independent integral which has the physical meaning of energy release rate has been derived as [2]:

$$J_k = \lim_{e \to 0} \int_{\Gamma_c} [(W + T)n_k - t_i u_{i,k}] \, dS = \lim_{e \to 0} \int_{\Gamma_c} [(W + T)n_k - t_i u_{i,k}] \, dS + \int_{\Gamma_c} \int_{V - V_c} [\rho \ddot{u}_i u_{i,k} - \rho \dot{u}_i \dot{u}_{i,k}] \, dV \tag{3}$$

Comparing the limiting integrals in eqns (2) and (3), it is seen that $J_k = G_k$. The identity $T_{x,k} = \rho \dot{u}_i \dot{u}_{i,k}$ should be noted in eqn (3) although $T_{x,k}$ was used in Ref. [2].

Other path independent integrals derived by Atluri [1] and Kishimoto et al. [4, 5] are respectively given by

$$J_k = \lim_{e \to 0} \int_{\Gamma_c} [(W - T)n_k - t_i u_{i,k}] \, dS = \lim_{e \to 0} \int_{\Gamma_c} [(W - T)n_k - t_i u_{i,k}] \, dS + \int_{\Gamma_c} \int_{V - V_c} [\rho \ddot{u}_i u_{i,k} + \rho \dot{u}_i \dot{u}_{i,k}] \, dV \tag{4}$$
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and

\[ J_k = \lim_{\varepsilon \to 0} \int_{\Gamma_{k\varepsilon}} [W_{nk} - t_i u_{ik}] \, dS \]
\[ = \lim_{\varepsilon \to 0} \int_{\Gamma_{k\varepsilon}} [W_{nk} - t_i u_{ik}] \, dS + \int_{V-V_{k\varepsilon}} \rho u_i u_{ik} \, dV. \quad (5) \]

Thus, the path independent integrals are related to each other by

\[ J_k - J_l = -2 \int_{\Gamma_{k\varepsilon}} T_{nk} \, dS = -2 \int_{\Gamma_{l\varepsilon}} T_{nk} \, dS + 2 \int_{V-V_{k\varepsilon}} \rho u_i u_{ik} \, dV \quad (6) \]
\[ J_k - J_l = \int_{\Gamma_{k\varepsilon}} T_{nk} \, dS = -\int_{\Gamma_{l\varepsilon}} T_{nk} \, dS + \int_{V-V_{k\varepsilon}} \rho u_i u_{ik} \, dV. \quad (7) \]

The identities of the near-field and far-field integrals in eqns (6) and (7) can be easily shown as follows:

\[ \int_{V-V_{k\varepsilon}} \rho u_i u_{ik} \, dV = \int_{V-V_{k\varepsilon}} (1/2 \rho u_i u_{ik}) \, dV = \int_{\Gamma_{k\varepsilon}} T_{nk} \, dS. \quad (8) \]

Thus, let

\[ J_{\kappa \ell} = \lim_{\varepsilon \to 0} - \int_{\Gamma_{\kappa \varepsilon}} T_{nk} \, dS = \lim_{\varepsilon \to 0} - \int_{\Gamma_{\ell \varepsilon}} T_{nk} \, dS + \int_{V-V_{\kappa \varepsilon}} \rho u_i u_{ik} \, dV. \quad (9) \]

Using the asymptotic solution\,[2] for a dynamically propagating crack, eqn (9) can be expressed directly by the instantaneous stress intensity factor or energy release rate.

\[ J_{\kappa 2} = \frac{G_{\kappa}}{D(C)} \left[ 2\beta_1 (1 - \beta_2) \frac{(1 - \beta_1)}{2\beta_1} (1 + \beta_2)^2 - 2(\beta_1 - \beta_2) \sqrt{(1 + \beta_1)(1 + \beta_2)} \right] \]
\[ + \frac{G_{2\ell}}{D(C)} \left[ 2\beta_2 (1 - \beta_2) - \frac{(1 - \beta_2)}{2\beta_2} (1 + \beta_1)^2 - 2(\beta_2 - \beta_1) \sqrt{(1 + \beta_2)(1 + \beta_1)} \right] + G_{\mu \mu \mu} \frac{1 - \beta_2}{2\beta_2} \quad (10) \]

\[ J_{\kappa 3} = \frac{-4\beta_1 (1 + \beta_2)(1 + \beta_2)^2 + 2(1 + \beta_2)^2 \sqrt{(1 + \beta_2)(1 + \beta_1)}}{(4\beta_1 \beta_2 + (1 + \beta_2)^2)(2 + \beta_1 + \beta_2) - 4(1 + \beta_2)^2 \sqrt{(1 + \beta_2)(1 + \beta_1)}} G_{\ell \ell \ell} \quad (11) \]

where I, II and III denote the fracture modes, \( D(C) = 4\beta_1 \beta_2 = (1 + \beta_2)^2, \beta_1^2 = 1 - C^2/C_4^2, \beta_2^2 = 1 - C^2/C_4^2, \)
\( C_4 \) and \( C_4 \) are the dilatational and shear wave speeds respectively. For Mode I crack propagation both the plane stress and plane strain conditions the variations of \( J_{\kappa 1} \) with crack speed are shown in Fig. 2 along with the effect of Poisson's ratio \( \nu \).

One of the key features in elasto-dynamic crack propagation is that the limiting integral \( J_{\kappa \ell} \) does not vanish for any non-zero crack speed. As pointed out in Ref.\,[1, 2], for elasto-dynamic crack propagation the \( J_k \) integral is the rate change of Lagrangean, while \( \dot{J}_k \) does not have the meaning of either energy release rate or rate change of Lagrangean.

Since the \( J_{\kappa \ell} \) integral given by eqn (9) is path independent by itself, one may establish many types of path integrals by adding \( \pm mJ_{\kappa \ell} \) (\( m: \) integer number) to eqn (3).

The relations between the path independent integrals and time-dependent stress intensity factors \( K_I(t), K_{II}(t) \) and \( K_{III}(t) \) are expressed as\,[2]:

\[ J_1 = \frac{1}{2\mu} \{ K_I^2 A_1(C) + K_{II}^2 A_{II}(C) + K_{III}^2 A_{III}(C) \} \quad (12a) \]
\[ J_2 = -\frac{K_{II} K_{III}}{\mu} A_{IV}(C) \quad (12b) \]
\[ J_3 = \frac{1}{2\mu} \{ K_I^2 F_1(C) + K_{II}^2 F_{II}(C) + K_{III}^2 F_{III}(C) \} \quad (13a) \]
where $\mu$ is the shear modulus. The detailed expressions for the crack speed functions are given in Ref.[2]. We rewrite the functions here only for the Mode I case.

$$A_t(C) = \frac{\beta_1(1 - \beta_2)}{D(C)}$$  \hspace{1cm} (15)

$$F_t(C) = \frac{\beta_1(1 - \beta_2)}{(D(C))^2} \left\{ 4\beta_2 - \frac{1}{\beta_1} (1 + \beta_2)^2 - 4(\beta_2 - \beta_2) \frac{1 + \beta_2}{\sqrt{1 + \beta_2}(1 + \beta_2)} \right\}$$  \hspace{1cm} (16)

$$F_t(C) = \frac{\beta_1(1 - \beta_2)}{(D(C))^2} \left\{ 2\beta_2(1 + \beta_2) - \frac{(1 + \beta_1)}{2\beta_1} (1 + \beta_2)^2 - 2(1 - \beta_2) \frac{1 + \beta_2}{\sqrt{1 + \beta_2}(1 + \beta_2)} \right\}.$$  \hspace{1cm} (17)

Instead of the spatially fixed path as presently used, one may consider a far-field path that moves with the crack tip, although this moving path is difficult to use in the ordinary finite element method. In this case one can no longer expect the same physical interpretation of the far-field integrals as derived for the spatially fixed paths.

### 3. FINITE ELEMENT FORMULATION

The present authors[6,7] have previously developed a moving singular finite element procedure which leads to highly accurate solutions for dynamic crack propagation. The finite element method presently used is basically the same as that presented in Ref.[6,7]. However, the following standard principle of virtual work is used in the present paper whereas the special principle of virtual work for a dynamically propagating crack was used in Ref.[6,7].

$$\int_V \left[ \sigma_{ij} \delta e_{ij} + \rho \dot{u} \delta u_i \right] \, dV - \int_{S_v} \vec{t}_i \delta u_i \, dS = 0$$ \hspace{1cm} (18)

where $S_v$ is the boundary of the body subjected to the prescribed tractions $\vec{t}_i$. The standard principle
results in a simpler formulation while both the variational principles lead to almost identical numerical results for stress intensity factors. Further discussion and comparison of the standard principle with the special principle are given in Ref. [8].

The virtual work equation as applicable to the moving singular element procedure may be written as:

\[
\sum_n \left[ \int_{V_{re}} \left( \sigma_{ij} \delta e_{ij} + \rho \delta u_i \right) dV - \int_{\partial V} \delta u_i dS \right] + \int_{V} \left( \sigma_{ij} \delta e_{ij} + \rho \delta u_i \right) dV = 0
\]  \hspace{1cm} (19)

where \( V_{re} \) and \( V \) are the volumes of regular and moving singular elements respectively (see Figs. 3 and 8). The field variables used for the finite element formulation are given in Ref. [6]. The additional variables required in calculation of the path independent integrals, such as \( u_{ik} \) and \( \dot{u}_{ik} \) can be easily derived from the finite element variables.

In the numerical calculation of the far-field integrals the following expressions are convenient:

\[
J_k = \int_{\Gamma} \left[ (W + T) n_k - t_i u_{ik} \right] dS + \int_{V} \left[ \rho \dot{u}_{i,k} + \rho \dot{u}_{i,k} \right] dV \quad (20)
\]

\[
J_k = \int_{\Gamma} \left[ (W - T) n_k - t_i u_{ik} \right] dS + \int_{V} \left[ \rho \dot{u}_{i,k} + \rho \dot{u}_{i,k} \right] dV \quad (21)
\]

\[
\dot{J}_k = \int_{\Gamma} \left[ W n_k - t_i u_{ik} \right] dS + \int_{V} \left[ \rho \dot{u}_{i,k} \right] dV \quad (22)
\]

Although the order of the volume integrand is \( O(r^{-2}) \), the existence of finite volume integrals can be verified as shown in Ref. [2].

In the present study the integral paths are chosen to be along the interelement boundaries, since the

![Contour Integral Path](image)

Fig. 3. Finite element mesh pattern and contour integral paths for cracked square plate.
volume integrals can be easily incorporated into the subroutine for the calculation of stiffness and mass matrices.

As demonstrated in Ref. [6, 7], the moving singular element formulation leads to a direct evaluation of the dynamic stress intensity factors, in as much as they are unknown parameter in the assumed basis functions for the singular element.

4. RESULTS AND DISCUSSION

(I) Self-similar, constant velocity, crack propagation in a square plate

The problem considered here is identical to that in Ref. [7]. Figure 3 shows the finite element breakdown for the initial configuration \( t = 0 \). A time-independent tensile stress \( \sigma \) acts at the edge of the specimen parallel to the crack-axis. Under the plane strain condition, the crack propagates...
symmetrically from the initial crack length \(a_0 = 0.2 \, W\) with a constant velocity \(C\). This problem may be considered to be similar to that treated by Broberg\[9\] except that Broberg treated an infinite body with a zero initial crack length. The material properties are \(\mu = 29.4\) GPa, \(\nu = 0.286\), \(\rho = 2450\) kg/m\(^3\). The shear wave speed is \(C_s = 3464.1\) m/sec. Five contour paths are considered as shown in Fig. 3. These paths are spatially fixed during the crack propagation. Three types of the far-field integrals defined by eqns (21)-(23) were estimated.

First, the results for \(C = 0.6 \, C_s\) is shown in Fig. 4. As explained earlier, the present singular element method directly gives a highly accurate solution for stress intensity factors. These stress intensity factors from the singular element were converted to the \(J_1\), \(J_2\), and \(J_3\) values using the formulae given by eqns (12)-(14) and shown by solid lines in Fig. 4.

Figure 4 clearly indicates the path independency of each type of integral. The results from the far-field integrals rapidly converge to those calculated from the stress intensity factors directly evaluated from the singular element.

Next, the far-field \(J^r\) integral values were converted to stress intensity factors using the relation given by eqn (12a). Among the results with five paths, Path 3 gave the best results as compared with the stress intensity factors directly computed from the singular element although the differences of all the other sets of results were within 0.7%. Figure 5 shows the comparison of the \(K_I\) values as calculated from the
J'_1 integral, with those directly evaluated from the singular element. Excellent agreement between both the results can be seen in the figure. The cases of slower and higher crack speeds C = 0.4 C_S and 0.8 C_S were also studied. Figures 6 and 7 show, respectively, the results for the case of C = 0.4 C_S and 0.8 C_S. For both the cases the K_I-values calculated from J' agree excellently with those directly evaluated from the singular element. The arrows in the figures indicate the readjustment of the finite element mesh to shift the moving singular element continuously[6]. For the higher crack speed, C = 0.8 C_S, the far-field integral values were somewhat affected by the mesh readjustment procedure while K_I values from the singular element (solid line) were fairly smooth.

(ii) Non-constant-velocity crack propagation in a DCB specimen

The second problem studied here is the prediction type simulation of fast fracture in a Double-Cantilever-Beam specimen. The same problem has been studied by the present authors in Ref.[10]. Figure 8 shows the specimen geometries and the finite element meshes at t = 0 and t = 20 μsec. The plane stress condition is postulated in the analysis because of the relatively thin nature of the specimen.
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RDCB Epoxy Resin (Araldite B) Specimen
Kalthoff et al [11]

Figure 9. Dynamic fracture toughness vs crack velocity relation for Araldite B DCB specimen.

Figure 9 shows the fracture toughness $K_{ID}$ vs crack velocity relation determined in the experiment [11] for Araldite B epoxy resin. In the prediction type fracture simulation the instantaneous crack velocity will be determined by using the $K_{ID}$ vs $C$ relation. In order to predict the average velocity between the times $t_0$ and $t_0 + \Delta t$, the $K_I$ value at time $t_0 + \Delta t/2$ is predicted by

$$K_{IP} = K_I(t_0) + \frac{\Delta t}{2} \bar{K}_I(t_0) + \frac{1}{2} \left( \frac{\Delta t}{2} \right)^2 \bar{K}_{II}(t_0) + R$$

where $R$ is a corrector of the predicted value. $K_I$, $\bar{K}_I$ and $\bar{K}_{II}$ are directly evaluated in the moving singular element at the previous time step. Since the present analysis is based on the standard variational principle, $R = 0$ was used, whereas the non zero corrector (see Ref. [10]) was used for the analysis based on the special variational principle.

In the calculation of the surface integral in eqn (21) the necessary quantities were calculated at the element boundaries inside of the paths, (see Fig. 10), as done in the previous analyses. The numerical results for the surface integral were found to oscillate with the time step while the volume integral values were in smooth variation. It was found that the source of oscillation is the $t_i\mu_{ik}$ term. In the finite element displacement model the traction reciprocity condition $t_i(t) = 0$ at interelement boundaries, with (+) and (−) denoting the two sides of these boundaries, is satisfied only in an average sense. In this particular modeling of DCB specimen and traction reciprocity condition was locally not well satisfied under the dynamic condition. To reduce the oscillation in the $t_i\mu_{ik}$ term, the average (from the two sides of the path) of the quantity $t_i\mu_{ik}$ was calculated at Path 1. For Path 2, $t_i\mu_{ik} = 0$ was used at the upper free end of the specimen.

Figure 10 shows the variations of the stress intensity factors and the crack velocities during the time of fast fracture. As seen from the figure the present fracture simulation gives an excellent prediction of the histories of the stress intensity factors, crack velocity (and the crack length), using the relation of fracture toughness and crack velocity (Fig. 9) and the crack initiation stress intensity factor for a blunted notch $K_{ID}$. The $K_I$ values calculated by the $J'$ integral are seen to be in good agreement with those extracted from the singular element. The influence of the different variational principle and the different assumed corrector can be seen in the comparison of the present Fig. 10 and Fig. 10 in Ref. [10]. However, one may find that the influence is negligible.
5. CONCLUDING REMARKS

The combined use of the path independent $J'$ integral and the finite element method has been shown to be an effective tool for determining the crack-tip fracture parameter such as the stress intensity factors as well as energy release rates in elasto-dynamic crack propagation.

The existence of many path independent integrals which do not have the physical meaning of energy release rate was also rationalized. Even though they do not have the meaning of an energy release rate, since their relations to the $K$-factors have been explicitly stated in this paper, these integrals also serve a useful purpose in dynamic fracture analysis.

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