ANALYSES OF LARGE QUASISTATIC DEFORMATIONS OF INELASTIC BODIES BY A NEW HYBRID-STRESS FINITE ELEMENT ALGORITHM

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A new hybrid-stress finite element algorithm, suitable for analyses of large, quasistatic, inelastic deformations, is presented. The algorithm is based upon a generalization of de Veubeke's complementary energy principle. The principal variables in the formulation are the nominal stress rate and spin, and the resulting finite element equations are discrete versions of the equations of compatibility and angular momentum balance.

The algorithm produces true rates, time derivatives, as opposed to 'increments'. There results a complete separation of the boundary value problem (for stress rate and velocity) and the initial value problem (for total stress and deformation); hence, their numerical treatments are essentially independent. After a fairly comprehensive discussion of the numerical treatment of the boundary value problem, we launch into a detailed examination of the numerical treatment of the initial value problem, covering the topics of efficiency, stability and objectivity. The paper is closed with a set of examples, finite homogeneous deformation problems, which serve to bring out important aspects of the algorithm.

1. Introduction

In his 'Treatise on the Mathematical Theory of Elasticity', Love [1, p. 2] observed that "When the general equations had been obtained, all questions of the small strain of elastic bodies were reduced to a matter of mathematical calculation." To this day, that 'matter of mathematical calculation' figures prominently in applied mechanics.

The early mechanicians realized that the general equations of elasticity were too difficult to solve except in a few special cases, so a large part of their effort was focused on methods for finding approximate solutions to problems of technological interest. Some of the techniques they used in deriving approximate theories for rods, plates, and shells are, in fundamental ways, very similar to the finite element technique.

Today it is well understood that the classical theories of rods, plates and shells may all be systematically derived from elasticity theory by introduction of approximations for the displacement to the principle of virtual work. Kirchhoff is the first person mentioned by Love [1] as having used this methodology, and in using it he managed to give a clear interpretation of the boundary conditions in the plate theory with which his name is now associated. In two respects Kirchhoff's methodology is the same as the finite element methodology. First of all, he made kinematic approximations, and secondly, he used an energy principle to maintain 'consistency' between his generalized stresses and strains, and to arrive at the correct boundary conditions. The principal difference between Kirchhoff's methodology and the finite
The finite element methodology lies in the degree to which the kinematic field is approximated. Because of the similarities in the construction of the classical rod, plate and shell theories to the construction of finite element equations, the successes and failures of the classical structural theories reflect, at least qualitatively, upon the performance of the finite element method.

No special theory in the realm of solid mechanics has enjoyed greater success than that of elastic beams, for there the general equations of elasticity are effectively replaced by a single ordinary differential equation. The theory is not only reasonably accurate, but extremely easy to understand because of its displacement based derivation. The classical plate and shell theories provide equations less easy to understand and less easy to solve than the beam equations, but still regarded as simpler than the general equations of elasticity.

A major failing of the classical theories of beams, plates and shells is their inability to account for the effects of ‘transverse shear stress’; that is, the shear stress acting on plane sections through the thickness of the structure. As a direct consequence, those theories always give a higher estimate of the stiffness of a structure than does the general theory. Secondly, the twisting moment and shear force are coupled on the edge of such a plate or shell. In spite of these shortcomings, it was not until after Reissner’s [2] investigation into the effect of shear stress on the bending of plates that satisfactory alternatives to the classical theory were widely accepted. But Reissner’s paper has had as great an impact on the methods used in applied mechanics as did his plate theory of itself. In its derivation, his theory is distinguished from the classical theories by the fact that both assumed stresses and displacements are used. Since that time the use of assumed stresses in the derivation of plate and shell theories has become common.

It is not surprising that the finite element method has evolved along similar lines. The finite element method in which one introduces kinematic approximations to the virtual work principle is the direct counterpart of Kirchhoff’s rod and plate theories. The same types of advantages and defects are inherent.

The principal advantage of the displacement based finite element methods is their conceptual simplicity. For application to beams, the simplicity rivals the simplicity of the beam theory itself. In the cases of plates and shells though, it proves difficult to construct ‘compatible’ shape functions for the displacement. Finite elements for thin plates based on kinematic approximations sometimes overestimate the stiffness of the plate so badly that they are described as ‘locking’. As a means of avoiding locking, and just for simpler construction of shape functions, some researchers have presented ‘incompatible’ plate bending elements, elements which do not satisfy slope continuity at element interfaces. A second drawback of displacement based finite element methods (in general) is their inability only to satisfy traction boundary conditions accurately (analogous to the coupling of moment and shear force at a Kirchhoff plate’s edge).

It was Pian’s [3] investigation into the derivation of element stiffness matrices that brought widespread attention to the potential advantages of introducing the stress as an independent variable. By his formulation, which was based on the complementary energy principle of linear elasticity, a viable alternative to incompatible elements was made available. Also, as was the case in Reissner’s plate theory, the stress formulation made possible considerably more accurate satisfaction of traction boundary conditions. Finally, Pian observed a marked acceleration in the convergence of the components of the stiffness matrix when the stress method was used. Since that time, the study of finite element methods related to Pian’s (which have
come to be known as ‘hybrid stress methods’) has produced a number of special methods which may be applied where conventional displacement based finite elements fail.

One particular class of problems in which the conventional displacement based finite element method fails is composed of problems involving incompressible or nearly incompressible bodies. The constitutive equation for such bodies is nearly or precisely singular for the mode of dilatation. The shape functions for the displacement used in the conventional finite element method are incapable of producing any motion other than pure shearing which does not contain (loosely speaking) ‘excessive’ dilatation. As a consequence, the conventional finite element method drastically overestimates the resistance to deformation of nearly and precisely incompressible bodies. In a key paper by Herrmann [4], it was shown that the difficulty could be avoided if only the mean stress were introduced as an independent variable.

Problems involving finite deformations of strain-softening bodies resemble problems involving nearly incompressible bodies in the sense that the body’s shear compliance is much greater than its bulk compliance. For the most part, finite element analyses of such deformations have been accomplished only at considerable expense, even when the pressure is introduced as an independent variable. No hybrid stress finite element algorithm for finite deformations was known.

The door to stress based finite element analysis of finite deformation problems was opened in 1972 by Fraeijs de Veubeke [5] with his presentation of a complementary energy principle for finite deformation elasticity. The stationary conditions of this principle are both the equations of compatibility and angular momentum balance. To date, variants of the principle have been used by de Veubeke and Millard [6], Sander and Carnoy [7], Koiter [8], Wunderlich and Obrecht [9], Murakawa [10], Murakawa and Atluri [11, 12], Murakawa et al. [13] and Atluri and Murakawa [14], in problems ranging from elastic membrane theory to beam, plate and shell theories.

A considerable generalization of de Veubeke’s principle was given by Atluri [15]. His reformulation of de Veubeke’s principle for stress rate and spin opened the way for the current work, that of developing a stress-rate based finite element algorithm for analysis of large deformations of inelastic bodies. It appears that the sole other analysis of large deformations of inelastic bodies by any similar algorithm is that presented by Atluri and Murakawa [14], in which necking of an elastic-plastic bar and postbifurcation analysis of a thin elastic-plastic plate was performed. The finite element algorithm used by those researchers was based on stress increments, rather than stress rates, and the motion of the elastic-plastic body was found by summation of increments. It was assumed that the accumulated error in this procedure could be kept small by the method of ‘residual loads’. This procedure has a firm foundation for problems involving elastic bodies (whose deformations were the subject of Murakawa’s earlier research), but is of questionable validity when the body is not elastic. In their assessment of incremental solution methods for inelastic rate problems, Argyris et al. [19] conclude that:

“Inelastic rate processes are in general path-dependent; therefore, the drift (i.e. the accumulation of numerical integration errors) cannot be eliminated by residual load iteration, e.g. at the end of each time step.”

1 An invalid principle was presented by Levinson [16], and again by Zubov [17]. The failure of that principle is discussed by Dill [18].
Moreover, when the body exhibits relaxation effects, this solution technique’s numerical stability becomes extremely sensitive to the time step size. Hughes and Taylor [20] observe that the time steps required for stability in the explicit time stepping technique are much smaller than required for accuracy when only quasistatic deformations are to be analyzed.

A final objection to ‘incremental’ finite element formulations may be raised on the grounds that there always results an artificial coupling between the boundary value problem and the initial value problem. When dealing with rate-type constitutive equations it is possible to treat the boundary value problem (for the rates) and the initial value problem (for the total stress and deformation) separately. Typically the boundary value problem for the rates is either precisely linear, or equivalent to a linear problem (without approximation). All of the nonlinearity falls into the initial value problem. Nonlinear initial value problems are, perhaps, of all nonlinear problems, the single type which we are best equipped to treat numerically. In any case, we are better equipped to treat them than we are nonlinear boundary value problems. The incremental approach has the effect of carrying the nonlinearity of the initial value problem into the boundary value problem, where it is dealt with, less efficiently, by some iterative technique.

The objective of the present work is to develop a stress-rate based finite element algorithm for analysis of large quasistatic deformations of inelastic bodies. In doing so, we discard the notion of ‘increments’ entirely. As a direct result, the boundary value problem and the initial value problem may be treated separately. The algorithm which results is applied to analyze large deformations of inelastic bodies.

As is true of ‘stress formulations’ in general, the development of the boundary value and initial value problems is more complicated than it is in a ‘velocity formulation’. The first part of this paper is devoted to presenting, with reasonable completeness, the development of the boundary value problem and associated variational principles. It follows that the finite element algorithm is more complicated, in that it involves more computation, and more attention to detail, than velocity based algorithms. However, from the results it is clear that the improvement in accuracy over velocity based methods is substantial; so much so that, in view of the difficulties encountered in the application of velocity based methods to finite deformation problems, the present stress based algorithm must be regarded as a ‘viable alternative’.

2. Kinematics, dynamics, rate-type constitutive equations

2.1. Kinematics

We represent natural space as a three-dimensional Euclidean space $\mathcal{E}$. Consider a motion of a body through space. The image of the body in $\mathcal{E}$ at time $t$ is the configuration $C(t)$. As time passes, the configuration changes, and we say that the body deforms. Let $X$ be the position in $\mathcal{E}$ that was occupied by a certain particle of the body at time $\tau$, and let $x$ be the position occupied by that same particle at time $t$; then our notion of deformation relative to the configuration $C(\tau)$ is expressed by

$$x = \chi(x)(X, t).$$

The mapping \(\chi\) embodies every aspect of the body’s motion.
As is customary, we denote by $F_r$ the deformation gradient and by $J_r$ its determinant:

$$F_r(X, t) = [\nabla x_r(X, t)]', \quad J_r(X, t) = \det F_r(X, t).$$

(2.1.2)

A particle's velocity is given by the time derivative of $x_r$:

$$v_r(X, t) = \frac{d}{dt} x_r(X, t),$$

(2.1.3)

and the spatial velocity distribution is found by putting $x_r^{-1}(x, t)$ for $X$ in $v_r(X, t)$:

$$v(x, t) = v_r(x_r^{-1}(x, t), t).$$

(2.1.4)

We caution the reader by pointing out that $v_r$ and $v$ are entirely different functions. Throughout this paper it is crucially important for the reader to keep such distinctions in mind.

We denote by $L$ the velocity gradient and by $J$ its trace:

$$L(x, t) = [\nabla v(x, t)]', \quad J(x, t) = \text{tr} L(x, t) = \nabla \cdot v(x, t).$$

(2.1.5)

$J_r, J_r$ and $J$ are related by Euler's expansion formula (see [21, p. 32])

$$J = J_r J_r.$$

(2.1.6)

The symmetric and skew-symmetric parts of $L$,

$$\varepsilon = \frac{1}{2}(L + L') \quad \text{and} \quad \omega = \frac{1}{2}(L - L'),$$

(2.1.7)

have the physical significance of stretching and spin, and are thus named.

Of course not any tensor field $L$ is the gradient of a velocity field; the condition of integrability (henceforth called compatibility equation) is

$$\nabla \times (L') = 0.$$

(2.1.8)

Likewise, if $\varepsilon$ is a symmetric tensor field and $\omega$ is a skew-symmetric tensor field, and $(\varepsilon - \omega)$ satisfies the compatibility equation

$$\nabla \times (\varepsilon - \omega) = 0,$$

(2.1.9)

then there is a twice differentiable vector field $v$ for which

$$\varepsilon = \frac{1}{2}(\nabla v' + \nabla v) \quad \text{and} \quad \omega = \frac{1}{2}(\nabla v' - \nabla v).$$

(2.1.10)

It is worthy of special mention that (2.1.9) is precise as well as linear.

\*In Appendix A the notations of this paper are explained.
2.2. Dynamics

The two fundamental principles of dynamics are called *balance of linear moment* (LMB) and *balance of angular momentum* (AMB). Under ordinary circumstances, the principles are equivalent to

\[ T_r(n_r) + T_r(-n_r) = 0 , \]
\[ T_r(n_r) = n_r \cdot t_r , \]
\[ \nabla_r \cdot t_r + \rho_r b_r = \rho_r \dot{v}_r , \]
\[ (F_r \cdot t_r) - (F_r \cdot t_r)' = 0 . \]

In (2.2.1) through (2.2.4) \( T_r \) is the nominal traction and \( t_r \) is the nominal stress; \( \rho_r, n_r, \) and \( b_r \) are defined by

\[ \rho_r(X) = \rho(X, \tau) , \]
\[ n_r(X) = n(X, \tau) , \]
\[ b_r(X, t) = b(\chi_r(X, t), t) \]

where \( \rho(\cdot, t) \) is the mass density over \( C(t) \), \( n(\cdot, t) \) the outward-directed normal to a material surface in \( C(t) \), and \( b(\cdot, t) \) the intensity of body force over \( C(t) \). Equations (2.2.1) through (2.2.4) are referred to as ‘traction reciprocity’, ‘the stress principle’, ‘linear momentum balance’ and ‘angular momentum balance’, respectively.

Using the stress principle (2.2.2) and Nanson’s relation (see [21, p. 181)

\[ n_r \cdot dS_r = J_r(\xi) n_r \cdot F_r^{-1}(\xi) dS_r , \]

we obtain the equation relating the nominal stresses \( t_r(t) \) and \( t_\zeta(t) \):

\[ t_r(t) = J_r(\xi) F_r^{-1}(\xi) \cdot t_\zeta(t) . \]

In the special case that \( \xi = \eta \), we recover

\[ t_r(t) = J_r(t) F_r^{-1}(t) \cdot \tau(t) , \]

the equation relating \( t_r \) to the true stress \( \tau \).

\[ ^3 \text{A careful account of the derivation of these equations, including qualifications, is given by Truesdell [22].} \]

\[ ^4 \text{There are two forms which the stress principle may take: } T_r(n_r) = n_r \cdot t_r \text{ and } T_r(n_r) = t_r' \cdot n_r. \text{ The “nominal” [23] stress } t \text{ follows from the former, the “first Piola-Kirchhoff” [22] stress } t^* \text{ from the latter. The nominal stress is also called the “Lagrange” stress [24, 16, 25], or the “Piola” stress [18, 8, 5, 26, 17].} \]
We now set (2.2.1) through (2.2.4) for traction rates and stress rates. By ordinary differentiation, we obtain

\[ \dot{T}_r(n_r) + \dot{T}_r(-n_r) = 0, \]  
(2.2.11)
\[ \dot{T}_r(n_r) = n_r \cdot \dot{t}_r, \]  
(2.2.12)
\[ \nabla \cdot \dot{t}_r + \rho \dot{b}_r = \rho \dot{\omega}_r, \]  
(2.2.13)
\[ (\dot{F}_r \cdot t_r + \dot{F}_r \cdot \dot{t}_r) - (\dot{F}_r \cdot t_r + \dot{F}_r \cdot \dot{t}_r) = 0. \]  
(2.2.14)

From (2.2.9) and (2.2.10) we obtain upon differentiation (and then choosing \( \xi = t \) in (2.2.9) and \( \tau = t \) in (2.2.10)),

\[ \dot{t}_r = J_r F_r^{-1} \cdot \dot{t} \]  
(2.2.15)

and

\[ \dot{t} = - (\epsilon + \omega) \cdot \tau + J \tau + \dot{\tau}. \]  
(2.2.16)

Equation (2.2.16), which effectively defines the nominal stress rate \[23\], will be used when we make a 'change of variables' in the constitutive equation; (2.2.15) will prove essential in the eventual integration of the stress rate.

If the motion is quasistatic, (2.2.11) through (2.2.14), for \( \xi = t \), become

\[ \dot{T}_r(n_r) + \dot{T}_r(-n_r) = 0, \]  
(2.2.17)
\[ \dot{T}_r(n_r) = n_r \cdot \dot{t}, \]  
(2.2.18)
\[ \nabla \cdot \dot{t} + \rho \dot{b} = 0, \]  
(2.2.19)
\[ (\epsilon + \omega) \cdot \tau + \dot{t} - \dot{t} - \tau \cdot (\epsilon - \omega) = 0. \]  
(2.2.20)

It is this form of the dynamic equations that we use henceforth. The reader should note that (2.2.17) through (2.2.20) are precise as well as linear.

It is instructive to examine the equation relating the nominal and true traction rates:

\[ \dot{T}_r = \dot{T} + (\dot{J} - n \cdot \epsilon \cdot n) T. \]  
(2.2.21)*

In rate and incremental type finite element algorithms the (interelement) traction reciprocity equation

\[ \dot{T}_r(n_r) + \dot{T}_r(-n_r) = 0 \]  
(2.2.22)

*We emphasize that 'rate' means 'time derivative'. Emphasis is necessary because some authors use the word 'rate' interchangeably with the word 'increment'.

**Hill \[23, \text{p. 53}\] gives the kinematical formulas needed to construct (2.2.21) from \( \dot{T} = \dot{n} \cdot \tau + n \cdot \dot{\tau} \). It should be clear that \( \dot{J} - n \cdot \epsilon \cdot n \) measures the surface expansion or contraction.
is usually solved approximately. According to (2.2.21), this is equivalent to solving

\[ \frac{d}{dt} \left[ T(n) + T(-n) \right] + (\dot{J} - n \cdot \varepsilon \cdot n)[T(n) + T(-n)] = 0 \]  
(2.2.23)\(^7\)

in an approximate manner. The traction imbalance grows or attenuates as \((\dot{J} - n \cdot \varepsilon \cdot n)\) is negative or positive [27]; that is, as the interelement surface contracts or expands. In practice the traction imbalance may be kept small by addition of a residual, \([- [T(n) + T(-n)]/h]\), to the right-hand side of (2.2.23), where \(h\) is the time step size.

Finally, we note that the general solution of the linear momentum balance (2.2.19) is of the form

\[ \dot{i} = \dot{i}^0 + \dot{i}^b \]  
(2.2.24)

where \(\dot{i}^0\) is the solution of the homogeneous equation \(\nabla \cdot \dot{i} = 0\),

\[ \dot{i}^0(x, t) = \nabla \times \Phi(x, t), \]  
(2.2.25)

and \(\dot{i}^b\) is any particular solution of \(\nabla \cdot \dot{i} = -\rho \dot{b}\). A particular solution \(\dot{i}^b\) may be constructed in cartesian coordinates using indefinite integrals [15]

\[ \dot{i}^b_{ij} = -\delta_{ij} \int \left\{ \rho \left[ \frac{\partial}{\partial t} b_i + \nu \cdot \nabla b_i \right] \right\} dx, \]  
(no sum).  
(2.2.26)

Notice that \(\dot{i}^b\) will depend upon \(\nu\) unless \(\nu \cdot \nabla b = 0\). If \(b\) is spatially constant, as gravitational force is usually presumed to be, then \(\nu \cdot \nabla b = 0\). However, if D'Alembert's principle is used, or if \(b\) is due to motion through an electromagnetic field, then \(\nu \cdot \nabla b\) generally does not vanish, and \(\dot{i}^b\) depends upon \(\nu\).

The angular momentum balance (2.2.20) involves the stretching, spin and true stress, as well as \(i\). For this reason there are no second-order stress functions analogous to those for the true stress; the function \(\Phi\) in equation (2.2.25) is called a "first order stress function" [15].

### 2.3. Rate-type constitutive equations

Our aim in this section is only to discuss a certain 'change of variable' possible in rate-type constitutive equations. We consider only materials whose mechanical behavior may be adequately described\(^8\) by a relation of the form:

\[ \ddot{\sigma}^* = \nabla : \varepsilon + \Sigma. \]  
(2.3.1)

which we call 'rate-type'. In (2.3.1) \(\ddot{\sigma}^*\) is the corotational rate of the Kirchhoff stress\(^9\), defined by

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\(^7\)For isoparametric elements, \((\dot{J} - n \cdot \varepsilon \cdot n)\) is continuous across interelement boundaries.

\(^8\)The notion of 'adequacy' of a rate-type equation relative to specific data has been discussed by the authors [28].

\(^9\)This stress rate seems to have been introduced by Hill [29] as a means of making a certain b.v.p. self-adjoint.
\[
\dot{\sigma}^* = (\dot{\tau} - \omega \cdot \tau + \tau \cdot \omega) + J \tau ,
\]

(2.3.2)

and \( V \) and \( \Sigma \) are independent of \( \varepsilon \).\(^{10}\) If \( V_{ijkl} = V_{klji} \), then we describe \( V \) as symmetric. The parenthecized quantity in (2.3.2) is the well-known Zaremba–Jaumann stress rate.\(^{11}\)

The following variable change leaves the constitutive equation in the form needed for our discussion of variational principles. According to (2.2.16) and (2.3.2), the symmetric and skew-symmetric parts of \( i \) are

\[
\frac{1}{2}(i + i') = \dot{\sigma}^* - \frac{1}{2}(\varepsilon \cdot \tau + \tau \cdot \varepsilon) - \frac{1}{2}(\tau \cdot \omega - \omega \cdot \tau) ,
\]

(2.3.3)

\[
\frac{1}{2}(i - i') = -\frac{1}{2}(\varepsilon \cdot \omega) \cdot \tau - \tau \cdot (\varepsilon - \omega) .
\]

(2.3.4)

Equation (2.3.4) is merely a rearrangement of the angular momentum condition (2.2.20).\(^{12}\) Using (2.3.1) to eliminate \( \dot{\sigma}^* \) from (2.3.3) yields

\[
\dot{\mathbf{r}} = W : \varepsilon + \Sigma ,
\]

\[
W_{ijkl} \equiv V_{ijkl} - \frac{1}{2}(\tau_{ik} \delta_{kj} + \delta_{ik} \tau_{kj}) ,
\]

(2.3.5)

\[
\dot{\mathbf{r}} = \frac{1}{2}(\mathbf{t} + \tau \cdot \omega - \omega \cdot \tau + i') .
\]

According to Hill\(^{13}\), the stress rate \( \dot{\mathbf{r}} \) was introduced by Biot (see [32, p. 62; 33])\(^{14}\). We note that if \( V \) is symmetric, then \( W \) is symmetric also.

When \( W \) is symmetric a rate potential \( W \) exists for \( \dot{\mathbf{r}} \):

\[
\dot{\mathbf{r}} = \partial \varepsilon W'(\varepsilon, \tau) , \quad W'(\varepsilon, \tau) = \frac{1}{2} \varepsilon : W : \varepsilon + \varepsilon : \Sigma .
\]

(2.3.6)

If \( W \) is also non-singular\(^{15}\), then (2.3.5) may be inverted

\[
\varepsilon = W^{-1} : (\dot{\mathbf{r}} - \Sigma)
\]

(2.3.7)

and there exists a rate potential \( \mathcal{R} \) for \( \varepsilon \):

\[
\varepsilon = \partial \varepsilon \mathcal{R}(\tau, \dot{\mathbf{r}}) , \quad \mathcal{R}(\dot{\mathbf{r}}, \tau) = \frac{1}{2}(\dot{\mathbf{r}} - \Sigma) : W^{-1} : (\dot{\mathbf{r}} - \Sigma) .
\]

(2.3.8)\(^{16}\)

\(^{10}\)A particularly simple kind of exception is made to this rule so that (2.3.1) may include classical plasticity.

\(^{11}\)The work of Key et al. [30] suggests that distinction of the two rates may not be necessary for nearly incompressible materials. However Bazant [31] has shown that estimates of buckling loads can be very sensitive to a change of stress rate.

\(^{12}\)Biot’s [32, p. 59] “alternative” stress, in the present notation, is \((\dot{\mathbf{t}} - \omega \cdot \tau)\); he gave the same interpretation of the two-dimensional counterpart to (2.3.4).

\(^{13}\)Hill [23, p. 20], actually credits Biot for introducing a stress, whose rate \((\dot{\mathbf{r}})\) Biot gave.

\(^{14}\)Hill and Hutchinson [34] refer to it as Biot’s “peculiar ‘symmetrized stress’”.

\(^{15}\)When \( W \) is non-singular iff \((W : \varepsilon = 0) \Rightarrow \varepsilon = 0\).

\(^{16}\)Atluri [15] considers more complicated inverse relations than (2.3.7); \( \mathcal{R} \) may still be defined by a Legendre, or contact transformation.
3. Boundary value problems, initial value problems

3.1. Scope of section

Thus far we have treated kinematics, dynamics and material behavior as separate subjects. Presently, we regard the equations of compatibility (2.1.9), linear momentum balance (2.2.19), angular momentum balance (2.2.20) and the constitutive equations (2.3.1) and (2.3.7), as a system of coupled partial differential equations. For ease of reference we collect these equations below:

\[ \nabla \times (\varepsilon - \omega) = 0 \quad \text{in} \; V; \quad s \cdot (-\varepsilon + \omega + \nabla v) = 0 \quad \text{on} \; S; \quad (3.1.1) \]

\[ \nabla \cdot \dot{\mathbf{v}} + \rho \dot{\mathbf{b}} = 0 \quad \text{in} \; V; \quad n \cdot \dot{\mathbf{v}} = \dot{\mathbf{T}} \quad \text{on} \; S; \quad (3.1.2) \]

\[ \frac{1}{2}[(\varepsilon + \omega) \cdot \mathbf{T} + \dot{\mathbf{v}} - \dot{\mathbf{v}} \cdot (\varepsilon - \omega)] = 0 \quad \text{in} \; V; \quad (3.1.3) \]

\[ \dot{\mathbf{v}} - \nabla : \varepsilon - \varepsilon \cdot \mathbf{T} + \varepsilon \cdot \omega + \Sigma; \quad (3.1.4) \]

\[ \mathbf{e} = \mathbf{W}^{-1}; \left[ \frac{1}{2}[(\varepsilon + \omega) \cdot \mathbf{T} + \dot{\mathbf{v}} - \dot{\mathbf{v}} \cdot (\varepsilon - \omega)] - \Sigma \right]; \quad (3.1.5) \]

\[ \mathbf{v} = \mathbf{v} \quad \text{on} \; S_v; \quad (3.1.6) \]

\[ \dot{\mathbf{T}} = \dot{\mathbf{T}} \quad \text{on} \; S_v. \quad (3.1.7) \]

We call this set of equations the general boundary value problem.

In this section we first present and compare the specializations of the general boundary value problem obtained by systematic use of (2.1.10) (the general solution of (3.1.1)) and (2.2.24) (the general solution of (3.1.2)). Then variational statements of the same boundary value problems are derived and discussed. Finally, under the assumption that a solution of the general boundary value problem is known, we show that an initial value problem must be solved to find the total deformation and stress.

3.2. Specializations of the general boundary value problem

If the general solution of the compatibility equation is used to eliminate \( \varepsilon, \omega \) and \( \dot{\mathbf{v}} \) from the general boundary value problem (by virtue of the constitutive equation 3.1.4), then we obtain a single, second-order, partial differential equation for the velocity field \( \mathbf{v} \):

\[ \nabla \cdot \dot{\mathbf{v}}(\nabla \mathbf{v}) + \rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = 0 \quad (3.2.1) \]

\[ ^{17} \text{Throughout this section it is implied, rather than expressly stated, that} \; \varepsilon \; \text{is symmetric and} \; \omega \; \text{is skew-symmetric.} \]

\[ ^{18} \text{The vector} \; s \; \text{is an arbitrary tangent on} \; S. \; \text{The boundary condition governs in-surface components of} \; \varepsilon \; \text{and} \; \omega \; \text{only; it is the formal counterpart to the stress principle.} \]

\[ ^{19} \text{This equation follows directly from (2.2.16), (2.3.1) and (2.3.2).} \]

\[ ^{19} \text{A solution is made up of the fields} \; \varepsilon, \; \omega \; \text{and} \; \dot{\mathbf{v}}. \]
and the boundary conditions:

\[ \mathbf{v} = \bar{v} \text{ on } S_v ; \quad n \cdot (\nabla \mathbf{v}) = \hat{T}_t \text{ on } S_\sigma . \] (3.2.2)

In analogy to its counterpart in linear elasticity, we call (3.2.1) "Navier's" equation [24, p. 155]. It is important to note that the angular momentum balance (3.1.3) is satisfied implicitly so long as \( \mathbf{v} \) satisfies

\[ (\mathbf{V} : \varepsilon) - (\mathbf{V} : \varepsilon)' = 0 \] (3.2.3)

for all symmetric \( \varepsilon \).

By a procedure parallel to that above, we use the general solution of the linear momentum balance to eliminate \( t \) and \( \varepsilon \) from the general boundary value problem, thereby obtaining

\[ \nabla \times [\varepsilon(t^0, \omega) - \omega] = 0 , \] (3.2.4)\(^{21}\)

\[ [\varepsilon(t^0, \omega) + \omega] \cdot \tau + t^0 + t^b = \text{symmetric} , \] (3.2.5)

and the boundary conditions

\[ s \cdot [-\varepsilon(t^0, \omega) + \omega] + (s \cdot \nabla) \bar{v} = 0 \quad \text{for all } s \text{ on } S_v , \]

\[ n \cdot \bar{t} = \hat{T}_t \text{ on } S_\sigma . \] (3.2.6)

In analogy to its counterpart in linear elasticity, we call (3.2.4) the "Beltrami–Michell" equation [24, p. 160]. In contrast to the situation in linear elasticity, where the general simultaneous solution of linear and angular momentum balance equations are known, here the angular momentum balance equation must be retained.\(^{22}\)

Both Navier's and Beltrami–Michell equations are linear and second-order; the stronger appeal of Navier's equation stems from the clear physical significance of the principal unknown, the velocity field. The boundary conditions are simple, and possess a natural interpretation. In contrast, the significance of a stress function is difficult to grasp; in terms of it, boundary conditions become complicated and defy an easy interpretation. Now, it is a fact that 'semi-inverse' techniques are the most widely understood of all methods of solution, and their use generally requires some intuition. So it is no puzzle as to why Navier's equation is encountered so much more frequently than the Beltrami–Michell equation.

3.3. Variational principles

The first step of the finite element method consists of generalizing the equations of the boundary value problem. It is the generalized problem upon which the approximate scheme is

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\(^{20}\)Contrary to the usual claims, \( \mathbf{v} \) need not satisfy \( V_{ijkl} = V_{ikjl} \). Consider, for example, the tensor \( A_{ijkl} = \delta_{ik} \delta_{jl} \); \( A \) certainly satisfies (3.2.3), but still \( A_{1212} \neq A_{2112} \).

\(^{21}\)We write \( t^b \) for \( \mathbf{v} \times \mathbf{\Phi} ; \mathbf{v} \cdot t^b = -\rho \mathbf{b} \).

\(^{22}\)This has been a stumbling block for a number of workers; cf. [15, 18, 22].
In engineering this generalization is accomplished by finding a variational principle 'equivalent' to the original problem. In solid mechanics, the construction of variational principles was first systematized by Washizu [26]. The formalism he introduced led not only to unification of the classical energy principles, but also to the abstraction of those principles for inelastic materials. Thus, the variational principles discussed below are called 'virtual work', 'potential energy', etc., so as to remind the reader of the corresponding principle of linear elastostatics.

We begin by deriving the generalization of the linear momentum balance equation (3.1.2). Let us momentarily regard $\delta v$ as a Lagrange multiplier. Then a stress rate $\mathbf{t}$ and a traction rate $\mathbf{T}$ satisfy (3.1.2) if

$$\int_V [\mathbf{t} \cdot \nabla \delta v - \rho \dot{\mathbf{b}} \cdot \delta v] \, dV - \int_S (\mathbf{T} \cdot \mathbf{n}) \cdot \delta v \, dS = 0, \quad (3.3.1)$$

for arbitrary $\delta v$. In this equation $\mathbf{t}$ apparently must be differentiable, but $\delta v$ need not even be continuous. Now, by formally integrating by parts, (3.3.1) is transformed to

$$\int_V [\mathbf{t} \cdot \nabla \delta v - \rho \dot{\mathbf{b}} \cdot \delta v] \, dV - \int_S \mathbf{T} \cdot \delta v \, dS = 0, \quad (3.3.2)$$

to be satisfied for arbitrary differentiable $\delta v$. In (3.3.2) the stress rate $\mathbf{t}$ need not even be continuous. Any stress rate $\mathbf{t}$ admissible to the differential form of LMB (3.1.2) is also admitted by (3.3.2), but the converse is not true. Therefore we call (3.3.2) the 'generalized' linear momentum balance.

A modification of the general boundary value problem is now possible: we simply use (3.3.2) in place of (3.1.2). If we proceed to eliminate $\mathbf{e}$, $\dot{\mathbf{e}}$ and $\mathbf{t}$ from this modified boundary value problem, just as we did in deriving Navier's equation, we obtain

$$\int_V [\mathbf{t} \cdot \nabla \delta v - \rho \dot{\mathbf{b}} \cdot \delta v] \, dV - \int_S \mathbf{T} \cdot \delta v \, dS = 0, \quad (3.3.3)$$

for $v = \delta v$ on $S_v$, $\mathbf{T} = \hat{T}$ on $S_v$. If we admit only $v = \delta v$ on $S_v$, $\mathbf{T} = \hat{T}$ on $S_v$, and $\delta v = 0$ on $S_v$, then (3.3.3) is reduced to a functional of the velocity field alone; that any solution of the general boundary value problem also causes this functional to vanish comprises a statement of the principle of virtual work. Most finite element algorithms used in engineering today are founded upon the principle of virtual work.

By a procedure parallel to that above we may derive the generalization of the equation of

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23This is in sharp contrast to the finite difference method, in which the differential operator plays the fundamental role.

24A unified treatment of variational principles for rate-type materials has been given by Atluri [15].

25The stress rate $\mathbf{t}$ and traction rate $\mathbf{T}$ are independent.

26Throughout this section integration by parts is formal.

27In particular, (3.3.2) admits discontinuous stress rate fields, where as (3.1.2) does not.
compatibility (3.1.1). Let us momentarily regard the stress function $\delta \Phi$ and the surface tangents $\delta s'$ as Lagrange multipliers. We write $(\delta s' e)$ as $(n \times \delta \Phi)$. Then the stretching $\epsilon$, spin $\omega$ and velocity $v$ satisfy (3.1.1) if

$$\int_V [\nabla \times (-\epsilon + \omega)] : \delta \Phi \, dV + \int_S (n \times \delta \Phi) : (-\epsilon + \omega + \nabla \nu) \, dS = 0$$

(3.3.4)

for arbitrary $\delta \Phi$. Just as for the linear momentum balance, we formally integrate by parts to relax the smoothness requirements on $\epsilon$ and $\omega$, obtaining

$$\int_V [-\epsilon + \omega] : (\nabla \times \delta \Phi) \, dV + \int_S (n \times \delta \Phi) : (\nabla \nu) \, dS = 0,$$

(3.3.5)

to be satisfied for arbitrary differentiable $\delta \Phi$. For the same reasons that we called (3.3.2) the generalization of linear momentum balance, we call (3.3.5) the generalization of compatibility. When expressed in this form it is easy to see that the velocity field enters the compatibility equation only so far as to determine the stretching and spin of the bounding surface of the body. This fact is not brought out in the literature, and is obscured by the conventional form for generalized compatibility, which we now give. Using the formula (integration by parts)

$$\int_S (n \times \delta \Phi) : (\nabla \nu) \, dS = \int_S n \cdot (\nabla \times \delta \Phi) \cdot \nu \, dS$$

and identifying $(\nabla \times \delta \Phi)$ as $\delta \hat{t}$ (in accordance with (2.2.24)), equation (3.3.5) becomes

$$\int_V [-\epsilon + \omega] : \delta \hat{t} \, dV + \int_S n \cdot \delta \hat{t} \cdot \nu \, dS = 0.$$

(3.3.6)

It is clear from this derivation that the stress rate variations $\delta \hat{t}$ are subject to no constraint except $\delta \hat{t} = \nabla \times \delta \Phi$.²⁹

We now consider the general boundary value problem as modified by replacing (3.1.1) with (3.3.6). If we proceed by eliminating $\epsilon$ from the modified problem just as we did in deriving the Beltrami-Michell equation, then we obtain

$$\int_V [\epsilon'(\hat{t}, \omega) \cdot \omega] : \delta \hat{t} \, dV + \int_S n \cdot \delta \hat{t} \cdot \nu \, dS = 0 \quad \text{for all } \delta \hat{t} = \nabla \times \delta \Phi ;$$

$$[\epsilon'(\hat{t}, \omega) \cdot \tau + \hat{t}^0 + \hat{t}^b] = \text{symmetric} ;$$

$$v = \hat{v} \quad \text{on } S_v ; \quad n \cdot (\hat{t}^0 + \hat{t}^b) = \hat{T}_t .$$

(3.3.7)

If we admit to (3.3.7) only $v = \hat{v}$ on $S_v$, $n \cdot (\hat{t}^0 + \hat{t}^b) = \hat{T}_t$ on $S_v$, $n \cdot \delta \hat{t} = \theta$ on $S_v$, and

²⁸In fact, we find no counterpart of (3.3.5) in the literature, though our search has not been exhaustive.

²⁹In the generalized compatibility equation of infinitesimal strain theories, stress variations must also satisfy AMB.
combinations of \((\hat{t}^0 + \hat{t}^b)\) which satisfy AMB, then we can reduce (3.3.7) to a single functional. That any solution of the general boundary value problem also satisfies this functional to vanish comprises a statement of the principle of complementary virtual work. Except for pathological cases, e.g. \(S = S_0, \tau = 0\), construction of such \((\hat{t}^0 + \hat{t}^b)\) and \(\omega\) is impracticable.

The problems associated with use of the complementary virtual work principle in its 'pure' form may be avoided by treating the angular momentum balance and conditions on \(S_0\) as constraints. We introduce the Lagrange multipliers \(\delta_\omega\) (for the angular momentum balance) and \(\delta v\) (for the traction boundary condition). Then (3.3.7) may be restated in the form

\[
\int_V \left[ -\epsilon(\hat{t}^0, \omega) + \omega \right] \cdot \delta t \, dV + \int_S n \cdot \delta \hat{t} \cdot v \, dS = 0 \quad \text{for all } \delta t = \nabla \times \delta \Phi,
\]

\[
\int_V \left[ (\epsilon(\hat{t}^0, \omega) + \omega) \cdot \tau + \hat{t}^0 + \hat{t}^b \right] \cdot \delta \omega \, dV = 0 \quad \text{for all } \delta \omega: \quad \delta \omega + \delta \omega' = 0,
\]

\[
\int_{S_0} \left[ n \cdot (\hat{t}^0 + \hat{t}^b) - \hat{T}_l \right] \cdot \delta v \, dS = 0 \quad \text{for all } \delta v \text{ on } S_0,
\]

\[
v = \tilde{v} \text{ on } S_0.
\]

That any solution of the general boundary value problem also satisfies (3.3.8) is the statement of the complementary virtual work principle upon which our finite element algorithm is based.

The most important property of the generalized linear momentum balance and compatibility equations is that they admit functions \(v, \epsilon, \omega\) and \(t\) less smooth than did their partial differential equation counterparts. A second property to be noted is their independence from any constitutive equation. In deriving the generalized Navier's equation (3.3.3) and the generalized Beltrami-Michell equation (3.3.8) we tacitly restricted our attention to materials whose constitutive equations are expressible in the forms \(t = \epsilon(\nabla \sigma)\) and \(\epsilon = \epsilon(\tau, \omega)\), respectively. Now, by inspection of (3.3.3), we see that an 'energy' principle exists if potentials \(U, \Psi\) and \(\psi\) exist such that

\[
i = \partial_L U, \quad -\dot{b} = \partial_\epsilon \Psi, \quad -\dot{T}_l = \partial_\omega \psi.
\]

Henceforth we assume that \(\partial b / \partial v = 0\) and \(\partial \dot{T}_l / \partial v = 0\) so that \(\Psi = -b \cdot v\) and \(\psi = -\dot{T}_l \cdot v\). From (2.1.7) and (2.3.3) through (2.3.5) we see that

\[
i : \delta L' = \dot{\epsilon} : \delta \epsilon - (\delta \epsilon : \tau \cdot \omega + \epsilon : \tau \cdot \delta \omega) - \frac{1}{2} \tau : (\omega \cdot \delta \omega + \delta \omega \cdot \omega),
\]

so the potential \(U\) exists whenever the potential \(W\) (2.3.6) exists; i.e. when \(V\) is symmetric. Henceforth we shall assume that \(V\) is symmetric. Then \(U\) may be expressed as

\[
U(\epsilon, \omega, \tau) = W(\epsilon, \tau) - \epsilon : \tau \cdot \omega - \frac{1}{2} \tau : (\omega \cdot \omega), \quad (3.3.10)
\]

We now introduce the potential \(U\) to (3.3.3) to obtain

\[
\epsilon = \frac{1}{2}(\nabla v' + \nabla v), \quad \omega = \frac{1}{2}(\nabla v' - \nabla v), \quad v = \tilde{v} \text{ on } S_0, \quad (3.3.11)
\]
\[ \delta \pi(v, \varepsilon, \omega) = 0 , \]
\[ \pi(v, \varepsilon, \omega) = \int_V [\mathcal{W}(\varepsilon, \omega) - \rho b \cdot v] \, dV - \int_{S_v} \hat{T}_i \cdot v \, dS . \]  

(3.3.12)

There are two ways to deal with the subsidiary conditions (3.3.11). It is an easy exercise to reduce (3.3.11) and (3.3.12) to a functional of the velocity field alone. Alternatively we may 'enforce' (3.3.11) by use of Lagrange multipliers.

The first course of action transforms (3.3.11) and (3.3.12) to

\[ \delta \pi_p(v) = 0 , \quad \pi_p(v) = \pi(v, \frac{1}{3} \nabla v^i + \nabla v^i), \frac{1}{3}(\nabla v^i - \nabla v)) . \]

(3.3.13)

Any solution of the general boundary value problem is a solution of (3.3.13); this comprises the principle of stationary potential energy.

The alternative course of action leads us to a Hu-Washizu energy principle. Let us momentarily regard \( \hat{T}_i \) and \( \iota \) as Lagrange multipliers. Then equations (3.3.11) and (3.3.12) may be replaced by

\[ \delta \pi_{hw}(v, \varepsilon, \omega, \hat{T}_i, \iota) = 0 , \]
\[ \pi_{hw}(v, \varepsilon, \omega, \hat{T}_i, \iota) = \pi(v, \varepsilon, \omega) + \int_V \iota : (\nabla v - (\varepsilon - \omega)) \, dV - \int_{S_v} \hat{T}_i \cdot (v - \bar{v}) \, dS . \]

(3.3.14)

Any solution of the general boundary value problem necessarily satisfies (3.3.14). We write out the stationary conditions for future reference:

**LMB:**

\[ \int_V [\iota : \nabla \delta v - \rho b \cdot \delta v] \, dV - \int_{S_v} \hat{T}_i \cdot \delta v \, dS - \int_{S_v} \hat{T}_i \cdot \delta v \, dS = 0 ; \]

(3.3.15)

**constitutive equation:**

\[ \int_V [(\partial \varepsilon \mathcal{W}) - \frac{1}{3}(\iota + \tau \cdot \omega - \omega \cdot \tau + \iota')] : \delta \varepsilon \, dV = 0 ; \]

(3.3.16)

**AMB:**

\[ \int_V \left[ \frac{1}{3}(\varepsilon + \omega) \cdot \tau + \iota - \iota' + \tau \cdot (\varepsilon - \omega) \right] : \delta \omega \, dV = 0 ; \]

(3.3.17)

**velocity boundary condition:**

\[ \int_{S_v} \delta \hat{T}_i \cdot (\bar{v} - v) \, dS = 0 ; \]

(3.3.18)
compatibility:

\[ \int_V [\nabla \mathbf{v} - (\mathbf{e} - \mathbf{\omega})] : \delta \mathbf{i} \, dV = 0 . \quad (3.3.19) \]

Notice that the stationary condition for \( \delta \mathbf{v} \) (3.3.15) is the generalized linear momentum balance (3.3.2).

Now, solely by rearrangement of terms, \( \pi_{\text{HW}} \) may be written in the form

\[
\begin{align*}
\pi_{\text{HW}}(\mathbf{v}, \mathbf{\varepsilon}, \mathbf{\omega}, \mathbf{T}, \mathbf{i}) &= \int_V \left\{ W(\mathbf{e}) - \frac{1}{2}(\mathbf{i} + \mathbf{\tau} \cdot \mathbf{\omega} - \mathbf{\omega} \cdot \mathbf{\tau} + \mathbf{i}') : \mathbf{e} \right\} - \frac{1}{2} \mathbf{\tau} : \mathbf{\omega} + \mathbf{i} : \mathbf{\omega} \right\} dV \\
&\quad + \int_V [\mathbf{i} : \nabla \mathbf{v} - \rho \mathbf{\dot{b}} \cdot \mathbf{v}] dV - \int_{S_v} \mathbf{\dot{T}} \cdot \mathbf{v} \, dS - \int_{S_v} \mathbf{T} \cdot \mathbf{v} \, dS + \int_{S_v} \dot{T} \cdot \mathbf{v} \, dS.
\end{align*}
\]  \quad (3.3.20)

If the constitutive equation (2.3.7) is used to eliminate \( \mathbf{\varepsilon} \) as a variable from \( \pi_{\text{HW}} \), then, defining

\[
-\mathcal{R}(\mathbf{i}, \mathbf{\omega}) = W(\mathbf{e}(\mathbf{i}, \mathbf{\omega})) - \frac{1}{2}(\mathbf{i} + \mathbf{\tau} \cdot \mathbf{\omega} - \mathbf{\omega} \cdot \mathbf{\tau} + \mathbf{i}') : \mathbf{e}(\mathbf{i}, \mathbf{\omega}),
\]

we obtain a *Hellinger–Reissner energy principle*:

\[
\delta \pi_{\text{HR}}(\mathbf{v}, \mathbf{\omega}, \mathbf{T}, \mathbf{i}) = 0 , \quad (3.3.21)
\]

\[
\begin{align*}
\pi_{\text{HR}}(\mathbf{v}, \mathbf{\omega}, \mathbf{T}, \mathbf{i}) &= \int_V \{ -\mathcal{R}(\mathbf{i}, \mathbf{\omega}) - \frac{1}{2} \mathbf{\tau} : (\mathbf{\omega} \cdot \mathbf{\omega}) + \mathbf{i} : \mathbf{\omega} \} dV + \int_{S_v} \mathbf{T} \cdot \mathbf{v} \, dS \\
&\quad + \int_V [\mathbf{i} : \nabla \mathbf{v} - \rho \mathbf{\dot{b}} \cdot \mathbf{v}] dV - \int_{S_v} \mathbf{\dot{T}} \cdot \mathbf{v} \, dS - \int_{S_v} \mathbf{T} \cdot \mathbf{v} \, dS.
\end{align*}
\]  \quad (3.3.22)

Any solution of the general boundary value problem is also a solution of (3.3.21). The stationary conditions are LMB (3.3.15), AMB (3.3.17), velocity boundary condition (3.3.18) and compatibility (3.3.19). In the stationary conditions \( \mathbf{e} \) appears only as a function of the stress rate and spin.

If we admit to \( \pi_{\text{HR}} \) only stress rates \( \mathbf{i} \) and traction rates \( \mathbf{T} \), which satisfy the generalized linear momentum balance (3.3.2), then we obtain the *complementary energy principle* of Atluri [15]:

\[
\delta \pi_{\text{c}}(\mathbf{\omega}, \mathbf{i}) = 0 , \quad (3.3.22)
\]

\[
\begin{align*}
\pi_{\text{c}}(\mathbf{\omega}, \mathbf{i}) = & \int_V \{ -\mathcal{R}(\mathbf{i}, \mathbf{\omega}) - \frac{1}{2} \mathbf{\tau} : (\mathbf{\omega} \cdot \mathbf{\omega}) + \mathbf{i} : \mathbf{\omega} \} dV + \int_{S_v} \mathbf{n} \cdot \mathbf{i} \cdot \mathbf{\varepsilon} \, dS, \\
\mathbf{i} = & \nabla \times \mathbf{\Phi} + \mathbf{i}^v, \\
\mathbf{n} \cdot \mathbf{i} = & \mathbf{T}, \quad \text{on } S_v .
\end{align*}
\]  \quad (3.2.23)
Any solution of the general boundary value problem is a solution of \((3.3.22)\) and \((3.3.23)\). The stationary conditions are \(\text{AMB} (3.3.17)\), velocity boundary condition \((3.3.18)\) and compatibility \((3.3.19)\). In the stationary conditions \(\varepsilon\) appears only as a function of the stress rate and spin.

It is generally not practicable to construct stress rates satisfying the traction boundary condition \((3.3.23)\). As we have done several times before, we introduce the Lagrange multiplier \(\nu\) on \(S_{\nu}\) to ‘enforce’ \((3.3.23)\). The result is a modified complementary energy principle,

\[
\delta \pi_{c}^{*}(\nu, \omega, \dot{t}) = 0, \tag{3.3.24}
\]

\[
\pi_{c}^{*}(\nu, \omega, \dot{t}) = \pi_{c}(\omega, \dot{t}) + \int_{S_{\nu}} (n \cdot \dot{T}_{t}) \cdot \nu \ dS, \quad \dot{t} = \nabla \times \Phi + \dot{t}^{b}.
\]

Any solution of the general boundary value problem is a solution of \((3.3.24)\). The stationary conditions are the same as those of the ‘pure’ complementary energy \(\pi_{c}\), except that the traction boundary condition follows from \(\delta \pi_{c}^{*}/\delta \nu = 0\).

### 3.4. Initial value problems

At this point we assume that a solution of the general boundary value problem can be constructed in any assigned configuration \(C(t)\) once the stress \(\tau(x, t)\) and boundary conditions are specified. A ‘solution’ consists of the values of \(\varepsilon(x, t), \omega(x, t)\) and \(t(x, t)\) over \(C(t)\) at the instant \(t\). The velocity may be found by integration:

\[
v(x, t) = \ddot{v}(\bar{x}, t) + \int_{\bar{x}}^{x} \left[ \varepsilon(x, t) + \omega(x, t) \right] dx \tag{3.4.1}
\]

where \(\bar{x}\) is a point on \(S_{\nu}\). For the purposes of the discussion in this section, we formally indicate the dependence of \(v\) and \(t\) on \(x, \tau\), and the time-dependent prescribed loads by introduction of integral operators \(\mathcal{F}\) and \(\mathcal{G}\) such that

\[
v = \mathcal{F}[x, \tau, t], \tag{3.4.2}
\]

\[
t = \mathcal{G}[x, \tau, t]. \tag{3.4.3}
\]

In principle \(\mathcal{F}\) and \(\mathcal{G}\) could be found by an integral transform method \([35]^{30}\), since the boundary value problem is linear.

Now consider the following transformation:

\[
x = \chi_{r}(X, t),
\]

\[
\tau(\chi_{r}(X, t), t) = J_{r}^{-1}(X, t) \mathcal{F}_{r}(X, t) \cdot t_{r}(X, t), \tag{3.4.4}
\]

\[
v(\chi_{r}(X, t), t) = \dot{\chi}_{r}(X, t),
\]

\[
\dot{t}(\chi_{r}(X, t), t) = J_{r}^{-1}(X, t) \mathcal{F}_{r}(X, t) \cdot \dot{t}_{r}(X, t).\]

\(^{30}\text{Fung} [24] \text{ discusses integral transforms for boundary value problems of linear elasticity.}\)
Introduction of (3.4.4) for $x$, $\tau$, $v$ and $\dot{t}$ in equations (3.4.2) and (3.4.3) gives

$$\dot{x}_\tau = \mathcal{F}_\tau \{x, \tau, t, t\},$$

(3.4.5)

$$\dot{\dot{t}} = \mathcal{G}_\tau \{x, \tau, t\},$$

(3.4.6)

where $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ are defined by

$$\mathcal{F}_\tau \{\cdot, \cdot, t\} \equiv \mathcal{F} \{\cdot, J^{-1}_\tau \mathcal{F}_\tau \cdot, t\},$$

(3.4.7)

$$\mathcal{G}_\tau \{\cdot, \cdot, t\} \equiv J_\tau F^{-1}_\tau \cdot \mathcal{G} \{\cdot, J^{-1}_\tau \mathcal{F}_\tau \cdot, t\}. $$

(3.4.8)

Equations (3.4.5) and (3.4.6), along with initial values for $x_\tau$ and $t_\tau$, constitute an initial value problem.

The initial value problem described above is ill-posed for two reasons. First, construction of the operators $\mathcal{F}$ and $\mathcal{G}$ is impracticable.\textsuperscript{31} Second, $\mathcal{F}_\tau$ and $\mathcal{G}_\tau$ govern the evolution of functions defined continuously over $C(\tau)$; excepting cases in which $\mathcal{F}$ and $\mathcal{G}$ are linear,\textsuperscript{32} no analytical methods have been devised for treating such initial value problems. We shall see that neither of these problems cause any difficulty in the numerical treatment of the problem.

4. The finite element algorithm

4.1. Shape functions

Equation (3.3.8) is the basis of the finite element algorithm presented here.\textsuperscript{33} The finite element equations are obtained by introduction of polynomial representations for $v$, $\omega$ and $\dot{t}$ (satisfying $\omega + \omega^t = 0$, $\dot{t} = \dot{t}^t + \dot{t}^b$, $v = \dot{v}$ and $\delta v = 0$ on $\Sigma_\tau$ a priori) to that functional and performing the assigned integrations.\textsuperscript{34} On the $N$th element let $v$, $\omega$ and $\dot{t}$ be represented by

$$v = \sum_{i=1}^{NO} N_i \omega_i^t, \quad N_i: \text{isoparametric shape functions [38]}.$$

(4.1.1)

$$\omega = \sum_{i=1}^{NW} QW_i \alpha_i^t, \quad \text{where } QW_i + QW_i^t = 0,$$

(4.1.2)

$$\dot{t} = \sum_{i=1}^{NT} QT_i \beta_i^t + \dot{t}^b, \quad \text{where } QT_i = \nabla \times \Phi_i, \quad \nabla \cdot \dot{t}^b = -\rho \dot{b}$$

(4.1.3)

\textsuperscript{31}Indeed, their counterparts in linear elasticity are known only for bodies of infinite extent [24].

\textsuperscript{32}In infinitesimal strain linear viscoelasticity the boundary value/initial value problem may be solved by successive use of integral transforms.

\textsuperscript{33}A more comprehensive exposition may be found in the author's dissertation [36].

\textsuperscript{34}Gaussian quadrature rules are used on each element, so the finite element equations depend only on the quadrature-point values of deformation and stress; cf. [37].
where \( NQ \) is the number of velocity parameters (on the \( N \)th element). \( NW \) is the number of spin parameters (on the \( N \)th element) and \( NT \) is the number of stress parameters (on the \( N \)th element). The shape functions used in the examples accompanying this paper are given in Appendix B.

### 4.2. The finite element equations

As is customary for 'hybrid stress' finite elements, the representation of the stress rate \( \dot{\mathbf{t}} \) is independent on each element, so to (3.3.8) we append the following statement of 'interelement traction reciprocity':

\[
\sum_{N=1}^{\text{NELM}} \left\{ \int_{S_N \cap (S_N \cap S_t)} (\mathbf{n} \cdot \dot{\mathbf{t}} \cdot \delta \mathbf{v}) \, dS - \int_{(S_N \cap S_t)} \dot{T}_i \cdot \delta \mathbf{v} \, dS \right\} = 0 \tag{4.2.1} \]

where \( \text{NELM} \) is the total number of elements. The finite element equations resulting from use of the representations (4.1.1) through (4.1.3) in equations (3.3.8) and (4.2.1) are

\[
\{ \delta \alpha \}^i \left\{ - [H^{11}] \{ \alpha \}^i + \{ P^{\alpha \beta} \} + \{ P^\alpha \Sigma \} \right\} = 0 , \tag{4.2.2}
\]

\[
\{ \delta \beta \}^i \left\{ - [H^{21}] \{ \beta \}^i + \{ P^{\beta \alpha} \} + \{ P^\beta \Sigma \} + \{ G \} \{ q_N \} \right\} = 0 , \tag{4.2.3}
\]

\[
\sum_{N=1}^{\text{NELM}} \left\{ \{ \delta q_N \}^i \left[ 0 \quad G_N \{ \beta \}^i - \{ \delta q_N \}^i \{ F_N \} \right] = 0 \right. \tag{4.2.4}
\]

The individual matrices in (4.2.2) through (4.2.4) are defined below:

\[
H^{11}_{ij} = \int_{V_N} \{ \mathbf{t} \cdot \mathbf{QW}_i \} : \mathbf{D} : \{ \mathbf{t} \cdot \mathbf{QW}_j \} \, dV , \tag{4.2.5}
\]

\[
H^{12}_{ij} = \int_{V_N} \{ \mathbf{t} \cdot \mathbf{QW}_i \} : \mathbf{D} : \{ \mathbf{QW}_j \cdot \mathbf{QW}_j \} \, dV , \tag{4.2.6}
\]

\[
H^{21}_{ij} = \int_{V_N} \{ \mathbf{QW}_i \} : \mathbf{D} : \{ \mathbf{QW}_j \} - \{ \mathbf{QW}_i \cdot \mathbf{QW}_j \} \, dV , \tag{4.2.7}
\]

\[
H^{22}_{ij} = \int_{V_N} \{ \mathbf{QW}_i \} : \mathbf{D} : \{ \mathbf{QW}_j \} \, dV , \tag{4.2.8}
\]

\[
G_{ij} = \int_{S_N} \mathbf{n} \cdot \{ \mathbf{QW}_i \} \cdot \{ \mathbf{N}_j \} \, dS , \tag{4.2.9}
\]

\[
F_i = \int_{(S_N \cap S_t)} \dot{T}_i \cdot \{ \mathbf{N}_i \} \, dS , \tag{4.2.10}
\]

\(^{35}\)This equation includes the traction boundary condition.

\(^{36}\)The element index 'N' has been suppressed on the stress rate and spin parameters.
\[ P_{12}^{12} = \int_{V_N} \left\{ (QW_i) : \left[ i^b + (D : i^b) : \tau + \frac{1}{h} \tau \right] \right\} dV. \]  
(4.2.11)\(^{37}\)

\[ P_{12}^{12} = \int_{V_N} \left\{ (QT_i) : (D : i^b) \right\} dV. \]  
(4.2.12)

\[ P_{12}^{12} = \int_{V_N} \left\{ (\tau : QW_i) : D : \Sigma \right\} dV. \]  
(4.2.13)

\[ P_{12}^{12} = \int_{V_N} \left\{ (QT_i) : D : \Sigma \right\} dV. \]  
(4.2.14)

The tensor \( D \) is obtained from \( W^{-1} \) by symmetrization

\[ D_{ijkl} = \frac{1}{4}(W_{ijkl}^{-1} + W_{ikjl}^{-1} + W_{ijlk}^{-1} + W_{jilk}^{-1}), \]

and serves to reduce by a factor of four the number of multiplications required to compute the \( H \) matrices.

After simultaneous solution of (4.2.2) and (4.2.3), i.e. finding \( [H^{-1}G]H^{-1}P' \) such that

\[
[H] \begin{bmatrix} H^{-1}G & H^{-1}P \end{bmatrix} = \begin{bmatrix} 0 & P \end{bmatrix}
\]

(4.2.15)

where

\[
[H] = \begin{bmatrix} H^{11} & H^{12} \\ H^{21} & H^{22} \end{bmatrix}, \quad [P] = \begin{bmatrix} P_{12}^{12} + P_{12}^{12} \\ P_{12}^{12} + P_{12}^{12} \end{bmatrix},
\]

we may eliminate the stress rate and spin parameters from (4.2.4) to get

\[
\sum_{N=1}^{\text{NELM}} \{q_N\} \{k_N\} + \{G_N\} \{H^{-1}P_N\} - \{F_N\} = 0.
\]

(4.2.16)

Here we have identified the element stiffness matrix \( [k_N] \)

\[
[k_N] = \begin{bmatrix} 0 & G_N \end{bmatrix} \begin{bmatrix} H^{-1}G_N \end{bmatrix}.
\]

(4.2.17)

Now we introduce the global nodal velocity vector

\[ \{2\}, \quad \dot{2}_t = \dot{2}_t \quad \text{and} \quad \delta \dot{2}_t = 0 \quad \text{on} \quad S_v, \]

and the 'assembly matrices' \( [A_N] \). The \{q_N\} and \{\delta q_N\} are now expressible as

\[ \{q_N\} = [A_N]\{2\}, \quad \{\delta q_N\} = [A_N]\{\delta 2\}. \]

\(^{37}\)The last term in the integrand provides a residual for any imbalance of angular momentum [36]; \( h \) is the time step size.
Elimination of \( \{q_N\} \) and \( \{\delta q_N\} \) from (4.2.16) gives

\[
\{\delta \bar{\mathcal{I}}\} \{\mathcal{K}\} \{\mathcal{Z}\} + \{\mathcal{P}\} - \{\mathcal{F}\} = 0, \tag{4.2.18}
\]

in which the global stiffness \([\mathcal{K}]\), and global load vectors \(\{\mathcal{P}\}\) and \(\{\mathcal{F}\}\) are defined by

\[
[\mathcal{K}] = \sum_{N=1}^{N_{ELM}} [A_N]'[k_N][A_N],
\]

\[
\{\mathcal{P}\} = \sum_{N=1}^{N_{ELM}} [A_N]'[0 \ G_N']H^{-1}P_N,
\]

\[
\{\mathcal{F}\} = \sum_{N=1}^{N_{ELM}} [A_N]'F_N.
\]

The equation (4.2.18) can be solved by standard methods [39]. We note that (4.2.18) involves only the nodal velocities as unknowns. By a process of backsubstitution we may finally recover \(v(x, t)\), \(\omega(x, t)\) and \(f(x, t)\) [36].

### 4.3. Numerical stability criteria

In our presentation of the finite element equations it was tacitly assumed that the following conditions held:

1. \( W \) nonsingular at each quadrature point,
2. \([H_N]\) singular on each element,
3. \([G_N]\{q_N\} \neq \{0\}\) except for rigid translations,
4. \(\{\delta \bar{\mathcal{I}}\}\{\mathcal{K}\}\{\mathcal{Z}\} \neq \{0\}\) for any \(\{\mathcal{Z}\}\) satisfying the velocity boundary condition.

The first of these is satisfied by models for solids found in the engineering literature. If the last is not satisfied for some \(\{\mathcal{Z}\}\), then typically the solution of the initial value problem is at a physically significant bifurcation point. Satisfaction of the second and third conditions depends principally upon the functions \(N_i, QW_i, \) and \(QT_i\) used in forming the matrices \([H_N]\) and \([G_N]\). In this section we discuss conditions whose fulfillment is necessary for satisfaction of (4.3.2) and (4.3.3); these conditions we call 'numerical stability criteria'.

An analogue to the condition (4.3.3) arises in the hybrid-stress finite element algorithm of linear elastostatics. The analysis of Tong and Pian [40], with a minor modification, applies in the present case. The rank of the matrix \([G]\) is usually

\[
\min(NT, NQ - T)
\]

where \(NT\) is the number of stress rate parameters, \(NQ\) the number of boundary velocity parameters, and \(T\) the number of translational degrees of freedom of an element. It is well known that if \(NT < NQ - T\), then 'kinematic modes' (deformations to which the element offers no resistance) will occur. The 'rank' condition which is necessary for the satisfaction of (4.3.3) is
In the examples accompanying this work, the number of stress rate parameters always equalled or exceeded the number of boundary velocity parameters, and no kinematic mode was encountered.

A second type of kinematic mode is possible in any complementary work or energy based finite element algorithm. If a velocity shape function \( N_i \) vanishes everywhere on an element's boundary,\(^{38}\) then it is easy to see that the \( j \)th column of \( G_{ij} \) (defined in (4.2.9)) vanishes identically. In practice one must therefore use boundary-noded elements only.\(^{39}\)

The condition (4.3.2) turns out to be the most troublesome. Even when \([H]\) is not singular, it may be so ill-conditioned that an accurate solution of the matrix equation (4.2.15) can only be found by scaling, that is, adjusting the magnitude of the stress and spin functions to improve the condition of \([H]\). In any case, the problem may be overcome by replacing \( QW_i \) and \( QT_i \) in a trial and error process until nonsingular \([H]\) is found.

After a number of trials, it became apparent that when \([H]\) was singular, the spurious eigenmode consisted of a pure (but inhomogeneous) spin. Moreover, if a combination of functions \( [QW] \) and \( [QT] \) was found to be acceptable in the stress-free state, it remained so as the deformation progressed. Setting the initial stress to zero, the criterion sufficient for no spin mode to occur follows as

\[
[H^{a}_0] \{\delta \alpha \} \neq 0, \quad [H^{a}_0]\{\alpha\} = \int_{\Omega^t} (QT_i : QW_i) dV. \tag{4.3.5}
\]

A similar criterion (for a finite element model of an elastic membrane) was given by de Veubeke and Millard \cite{6}, but their conclusions differ from our own. A necessary condition for the satisfaction of (4.3.5) is that

\[
NT^* \geq NW
\]

where \(NT^*\) is the number of skew-symmetric tensors \((QT_i - QT'_i)\) that are linearly independent, and \(NW\) is the number of spin functions \(QW_i\).\(^{40}\)

De Veubeke and Millard suggest that the polynomial degree of the spin field \((m)\) be related to the polynomial degree of the stress field \((n)\) as \(m = n - 1\) \((NT^* > NW)\), calling the case \(m = n\) \((NT^* = NW)\) the 'classical equilibrium model'. However, experience with the present finite element algorithm indicates that if \(m\) is less than \(n\), then the angular momentum balance is not satisfied with reasonable (pointwise) accuracy. The degree of the spin field and the degree of the stress rate field were always the same in the examples accompanying this work.

5. Numerical treatment of the initial value problem

5.1. Definition of the initial value problem

The finite element algorithm described in the previous section produces an approximation

\(^{38}\) Examples include shape functions for interior nodes, and 'bubble functions'.

\(^{39}\) A more general examination of this problem is given in the author's dissertation \cite{36}.

\(^{40}\) It has been tacitly assumed that all of the \(QW_i\) (and \(QT_i\) and \(N_i\)) are linearly independent.
for the stress rate $\dot{\mathbf{t}}$ and spin $\mathbf{\omega}$ on the interior of each element, and the velocity $\mathbf{v}$ on the boundary of each element. The algorithm does not define the velocity on the interior of an element.\(^{41}\) We have chosen to assign the velocity on the interior of each element simply by interpolation of that element’s boundary velocity, and in doing so we effectively discard $(\mathbf{\varepsilon}(\mathbf{t}, \mathbf{\omega}) - \mathbf{\omega})$.\(^{42}\)

Now let us write $\{\mathbf{x}\} = \{x^1, x^2, \ldots, x^{ND}\}$ for the vector of nodal positions, and $\{\mathbf{v}\} = \{v^1, v^2, \ldots, v^{ND}\}$ for the vector of nodal velocities, ND being the total number of nodes. Similarly, we write $\{\mathbf{\tau}\} = \{\tau^1, \tau^2, \ldots, \tau^G\}$ for the quadrature point stresses and $\{\mathbf{\dot{t}}\}$ for the quadrature point stress rates, $G$ being the total number of quadrature points in the body. We indicate the dependence of $\{\mathbf{v}\}$ and $\{\mathbf{\dot{t}}\}$ on $\{\mathbf{x}\}$, $\{\mathbf{\tau}\}$, and the time dependent boundary conditions by introduction of functions $\mathbf{f}$ and $\mathbf{g}$ such that

\begin{align*}
\{\mathbf{v}\} &= \mathbf{f}(\{\mathbf{x}\}, \{\mathbf{\tau}\}, t), \quad (5.1.1) \\
\{\mathbf{\dot{t}}\} &= \mathbf{g}(\{\mathbf{x}\}, \{\mathbf{\tau}\}, t). \quad (5.1.2)
\end{align*}

The reader should note that the integral operators $\mathcal{F}$ and $\mathcal{G}$ (see (3.4.2) and (3.4.3)) are replaceable by functions $\mathbf{f}$ and $\mathbf{g}$ because the finite element equations depend exclusively upon the quadrature point values of $\mathbf{x}$ and $\mathbf{\tau}$. Performing the transformation (3.4.4), we finally obtain

\begin{align*}
\{\mathbf{x}_\tau\} &= \mathbf{f}_\tau(\{\mathbf{x}_\tau\}, \{\mathbf{t}_\tau\}, t), \quad (5.1.3) \\
\{\mathbf{t}_\tau\} &= \mathbf{g}_\tau(\{\mathbf{x}_\tau\}, \{\mathbf{t}_\tau\}, t). \quad (5.1.4)
\end{align*}

The functions $\mathbf{f}_\tau$ and $\mathbf{g}_\tau$ govern the evolution of discrete values of $\mathbf{x}_\tau$ and $\mathbf{t}_\tau$, so (5.1.3), (5.1.4) and appropriate initial values of $\{\mathbf{x}_\tau\}$ and $\{\mathbf{t}_\tau\}$, represent an initial value problem of ordinary differential equations.

A great number of ‘time-stepping’ schemes are presently available for the numerical treatment of such problems.\(^{43}\) In principle, any scheme in the literature may be used. In practice, three factors, aside from stability, affect the choice of scheme.

1. **Storage requirements** for implementation of integration schemes can vary appreciably.

2. **Execution time** depends principally upon the number of evaluations of $\mathbf{f}_\tau$ and $\mathbf{g}_\tau$; the finite element equations must be formed and solved for each evaluation, as in a tangent stiffness formulation.

3. The functions $\mathbf{f}_\tau$ and $\mathbf{g}_\tau$ will generally be discontinuous when the solution lies on a yield surface.

A comparison of five methods by Gear [42, pp. 233–235] indicates that when the number of function evaluations must be kept very small, the classical fourth-order Runge–Kutta method gives lower errors than (implicit and explicit) multistep methods. Single step methods generally

\(^{41}\)Equation (3.4.1) is of no use since the finite element approximation for $[\mathbf{\varepsilon}(\mathbf{t}, \mathbf{\omega}) - \mathbf{\omega}]$ is not compatible.

\(^{42}\)Outwardly, the very arbitrariness of this procedure would appear to invalidate it: to any velocity field found by interpolation of the element’s boundary velocity one could add another velocity field which vanished on the element’s boundary. However, one must ask whether or not the addition of such a field would improve the ‘order of accuracy’ of the velocity representation (cf. [41, p. 169]); when it would not, the procedure of boundary velocity interpolation is apparently sound.

\(^{43}\)An up-to-date exposition has been given by Gear [42].
have smaller storage requirements, and since smoothing over several time steps is not ‘built in’ (as it is with multi-step methods), they can be expected to perform more favorably when the solution crosses a yield surface. Finally, in contrast to multistep schemes, single step schemes are easily started and the time step may be easily changed. In the examples accompanying this work the Euler, and second- and fourth-order explicit Runge–Kutta schemes (denoted RK2 and RK4, respectively) were used, cf. [43, Section 6.5]. The accuracy of a result was gauged by step-doubling [42, p. 81], or by comparison to the result found by application of a higher order scheme.

It is crucially important that the time stepping scheme not introduce errors which tend to unbalance the total stress \( t \), (or \( \tau \)). This is because the numerical scheme is centered about the (generalized) compatibility equation, not the linear momentum balance equation, so there is no way to form a linear momentum residual to check for or correct an unbalanced stress. This maintenance of balanced stress, necessary in stress-based finite element algorithms, is the counterpart of maintenance of compatible deformation, necessary in velocity-based algorithms. It can be shown [36] that LMB is maintained when the stress \( t \) is integrated explicitly, but not when other stresses (such as \( \tau \)) are integrated explicitly. Thus we integrate \( t \), (and \( x \)), and find \( \tau \) (afterwards) by the formula

\[
\tau = 1/JF_{r} \cdot t_{r}.
\]

5.2. Stability of numerical solutions of the initial value problem

It can happen that a small change in the initial condition or time step size leads to a very large change in the numerical approximation to a mathematically stable solution of the initial value problem. Problems involving materials with a relaxation time (or spectrum of relaxation times) are the most susceptible to such numerical instabilities, so our attention in this section is focused on them.

Cormeau [37] has shown that Euler's method, when applied to finite element-initial value problems of infinitesimal strain quasistatic elasto/viscoplasticity [44], yields a stable numerical solution only if the time step is within a certain bound. Argyris et al. [19] have noted that Cormeau's bound is the same as the bound necessary for stable integration of the stress if in the rate-type constitutive equation one prescribes the deformation. It seems plausible that a correspondence of time step bounds would also exist between the present finite element-initial value problem and the constitutive equation (2.3.1). Since a direct analysis of the present finite element initial value problem does not appear feasible, we give instead an analysis of the initial value problem

\[
\dot{\sigma} = \nabla : \epsilon + \Sigma, \quad \tau(0) = \tau_{0},
\]

in which \( \epsilon \) is prescribed, and \( \nabla \) and \( \Sigma \) depend upon \( \tau \). We assume that any time step bound thus obtained applies equally well to the finite element-initial value problem. This assumption is borne out by experience.

After application of Euler's method to (5.2.1) to generate two sequences of stresses, \( \{ \tau_{N} \} \n\)

\(^{44}\)It is clear that a time step bound must be invariant with respect to any ‘change of variables’ of the type described in Section 2.3.
and \( \{\tau_N + \Delta \tau_N\}, \Delta \tau_0 \neq 0 \), we find that their difference \( \{\Delta \tau_N\} = \{\tau_N + \Delta \tau_N\} - \{\tau_N\} \) satisfies

\[
\|\Delta \tau_{N+1}\| \leq \rho\{(1 - h\dot{J}) I + h\partial_\tau(\mathbf{V} : \epsilon + \Sigma)_{\tau=\tau_N}\} : \|\Delta \tau_N\|. \tag{5.2.2}\]

Here \( \rho\{A\} \) signifies the maximum eigenvalue of a fourth-order tensor \( A \). The sequence (numerical solution) \( \{\tau_N\} \) is stable with respect to perturbations of the initial condition if

\[
\|\Delta \tau_{N+1}\| \leq \|\Delta \tau_N\| \tag{5.2.3}
\]

[42, pp. 16–18], hence the time step \( h \) must be chosen so that

\[
(1 - h\dot{J}) + h\rho(\dot{\alpha}(\mathbf{V} : \epsilon + \Sigma)_{\tau=\tau_N}) \leq 1. \tag{5.2.4}
\]

In pure relaxation (\( \epsilon = 0 \) and \( \dot{J} = 0 \)), or when \( \mathbf{V} \) is independent of \( \tau \) and \( \dot{J} = 0 \), (5.2.4) reduces to Cormeau's result. In most problems of technological interest, the deformation is an unknown, as opposed to prescribed; equation (5.2.4) is inapplicable, so no least bound is found.\(^{46}\) However, satisfaction of the 'pure relaxation' bound is still necessary for general stability. Regarding those bounds, Hughes and Taylor [20] have commented:

"For slowly varying loads, or when equilibrium response is of prime interest, stability requires that time steps be selected which are much smaller than those necessary for accuracy."

Argyris et al. [19] remarked that the time step bound

"physically infers... that the inelastic strain increment... remains smaller than the elastic strain".\(^{47}\)

Since in metals the elastic strain is always infinitesimal, this implies that an intolerably large number of steps would be required for a finite strain analysis.

As a means of avoiding the stringent time step bounds associated with the use of the Euler method, several generalizations of the trapezoid (or midpoint) rule have been devised [49, 19, 46, 50, 20]. A clear survey of these methods, including discussions of stability and convergence requirements, has been given by Argyris et al. [46].\(^{48}\) The simplest of these they have called "the forward gradient scheme", and "the linear finite approximation scheme".

\(^{45}\)Our analysis has been simplified by choosing a 'corotational frame'; see [45, art. 147–148], and by assuming \( \mathbf{V} \) and \( \Sigma \) to be differentiable w.r.t. \( \tau \).

\(^{46}\)Argyris, Doltinsis and Willam [46] claim that in the case of pure relaxation "we obtain the most critical statements for stability."

\(^{47}\)For a material with a spectrum of relaxation times \( T_i \), the time step bound becomes \( h \leq 2T_{\min} \), or \( \frac{1}{2} < N_{\text{DE}} \), where \( N_{\text{DE}} = T_{\min}/h \) amounts to a numerical counterpart to the "Deborah number" [47] (whose precise definition is debated in the rheology literature [48]). The response to a stretching increment is predominantly elastic when \( N_{\text{DE}}^{*} = 1 \), fluid when \( (T_{\min}/h) < 1 \).

\(^{48}\)In an earlier paper, Argyris et al. [19] derived a time step bound which assured \( \|\tau_{N+1}\| \leq \|\tau_N\| \) in pure relaxation (see equations (3.22), (3.26) and (3.27) therein). The relevance of this bound to numerical stability is not clear; in fact, if in their equation (3.22) we replace \( \Delta \tau \) by \( \xi \Delta \tau \), then the critical time step obtained in the limit as \( \xi \to 0 \) does not agree with Cormeau's.
In the forward gradient scheme we replace the relaxation term $\Sigma$ in the constitutive equation (2.3.1) by $\Sigma + \theta h (\partial \Sigma) : \dot{\sigma}^*$, to obtain

$$\dot{\sigma}^* = [I - \theta h (\partial \Sigma)]^{-1} : (V_0 : \varepsilon + \Sigma)$$

(5.2.5)

($\Sigma$ and $\partial \Sigma$ are evaluated at $\tau_N$), or simply

$$\dot{\sigma}^* = V_\theta : \varepsilon + \Sigma_\theta.$$  

(5.2.6)

in which the definitions of $V_\theta$ and $\Sigma_\theta$ are apparent. Since (5.2.6) is of the same form as the original constitutive equation (3.2.1), we find the stability criterion simply by replacing $V$ and $\Sigma$ with $V_\theta$ and $\Sigma_\theta$ in (5.2.4). In pure relaxation we obtain:

$$|1 + h \rho (\partial \Sigma_\theta)| \leq 1.$$  

(5.2.7)

In the special case that $\Sigma = -(1/T) \tau'$, $T$ being a constant time of relaxation, $\partial \Sigma = -(1/T)(I - \frac{3}{2}\Pi)$. From (5.2.7) we get the following time step restriction

$$\left| \frac{1 - (1 - \theta)(h/T)}{1 + \theta(h/T)} \right| \leq 1.$$  

(5.2.8)

so, in this special case, unconditional stability is obtained for $\theta \leq 1/2$. For $\theta = 0$ we recover Cormeau's bound. To use the forward gradient scheme we derive $W_\theta$ from $V_\theta$ just as we derived $W$ from $V$ (see 2.3.5):

$$W_\theta = V_\theta - T, \quad T_{ijkl} = \frac{1}{2} (\tau_{ik} \delta_{lj} + \delta_{ik} \tau_{lj}).$$

(5.2.9)

When a material which exhibits relaxation is to be analyzed, $W_\theta$ and $\Sigma_\theta$ are used in place of $W$ and $\Sigma$ in forming the finite element equations. One effect of the forward gradient scheme is to lengthen all of the characteristic times so that the time step exceeds the least of them by no more than a factor of $(1/\theta)$. In the special case above, $T$ is effectively replaced by $(T + \theta h)$, so for $\theta \geq 1/2$, $h/(T + \theta h) \leq 1/\theta \leq 2.$

Two linear finite approximation schemes have been proposed in the literature [20, 19]. In practice both are very similar to the forward gradient scheme, above, differing mainly because of their implicitness. To facilitate their discussion, we introduce the 'plastic stretching' $\varepsilon^p = -V^{-1} : \Sigma$, and the intermediate stress $\tau_\theta$, defined by

$$\tau_\theta = (1 - \theta) \tau_N + \theta \tau_{N+1}, \quad 0 \leq \theta \leq 1.$$  

(5.2.10)

We assume that a function $g(\tau)$ may be defined such that $\varepsilon^p = g : \tau$, and we write $G$ for $\partial_\tau \varepsilon^p$; finally, $g_\theta = g(\tau_\theta)$, and $G_\theta = G(\tau_\theta)$.

49 A truncated series for $\Sigma$ about $\Sigma(\tau_N)$, in a corotational frame, is $\Sigma(\tau_N) + \theta h (\partial \Sigma) : \dot{\tau}$; using $\dot{\sigma}$ for $\dot{\tau}$ is a further approximation.

50 Recall Cormeau's bound, $h/T_{\min} \leq 2$. 
Hughes and Taylor [20] replace $\mathbf{V}$ by

$$\mathbf{V}' = (\mathbf{V}^{-1} + \theta h \mathbf{G}_0)^{-1} = (\mathbf{I} + \theta h \mathbf{V} : \mathbf{G}_0)^{-1} : \mathbf{V},$$

(5.2.11)

and $\mathbf{e}^\theta(\tau_0)$ by $\mathbf{e}^\theta(\tau_0)$ in the finite element equations. The forward gradient scheme is used as a 'predictor', and the Hughes' and Taylors' scheme for a 'corrector'.

Argyris et al. [19] replace $\mathbf{V}$ by

$$\mathbf{V}' = (\mathbf{I} + \theta h \mathbf{V} : \mathbf{g}_0)^{-1} : \mathbf{V}$$

(5.2.12)

and $\mathbf{e}^\theta(\tau_0)$ by $\mathbf{g}_0 : \tau_0$ in the finite element equations. They do not say how the resulting 'corrector' equation should be started, by a 'guess' (such as $\tau_{N+1} = \tau_N$), by the forward gradient scheme, or by the simple 'initial load' scheme ($\tau = 0$).

We conclude this section by pointing out that for the constitutive equation (2.3.1) with constant $\mathbf{V}$ and $\mathbf{\Sigma} = -(1/T) \mathbf{\tau}'$ ($T$ being a constant relaxation time), the matrices $\mathbf{G}$ and $\mathbf{g}$ are constant and equal to each other. As a consequence, the scheme of Argyris et al. reduces identically to the forward gradient scheme. The scheme of Hughes and Taylor differs from the forward gradient scheme only in that the 'initial load' depends on $\tau_0$ instead of $\tau_n$. This circumstance is exploited in the sample problems accompanying this paper; when the forward gradient scheme is used to stabilize time integration, the results characterize the performance of all three integration schemes.

5.3. Numerical integration and objectivity

In order to be called objective a numerical approximation for a physical entity must transform between frames according to the same rule as the entity itself. An algorithm which produces an objective approximation will itself be called objective.

Recently, Hughes and Winget [51] proposed an algorithm whose use, they argued, ensured 'objective' numerical integration of rate-type constitutive equations. Key et al. [52] implemented midstep constitutive evaluation in the algorithm (in two dimensions), at the price of using trigonometric formulas to find the square root of the incremental rotation. Finally, Rubinstein and Atluri [53] gave a representation for the incremental rotation as a function of time, thereby making the evaluation of any root of the rotation increment trivial.\textsuperscript{51} The algorithm of Hughes and Winget, and its modifications, all embrace as basic the notion that 'proper invariance', or 'objectivity', of stress increments is achieved by "taking rigid body rotations of a material point relative to the spatial coordinates ... into account" [52].

Our approach to this subject is slightly different from that of previous workers in that numerical objectivity is treated without reference to any constitutive equation. As noted by previous workers, the source of inobjectivity is not the constitutive equation, but the inexactness of the numerical integration scheme. The entanglement of integration schemes and constitutive equations, which characterizes the earlier work, is not only unnecessary, but could very well lead to confusion between general objectivity of physical entities and the

\textsuperscript{51}It was recently learned that this representation was also given by Schwerdtfeger [54, p. 238].
Principle of Objectivity of Material Properties. Here we focus on integration schemes which produce `objective approximations' for physical entities.

Consider the motion of an isolated particle relative to some frame in which we describe its position and velocity by $x'(t)$ and $\dot{x}'(t)$, respectively. For the sake of illustration, let us say that the velocity $\dot{x}'$ is known precisely at each moment of time, and we wish to find the position $x'$ as a function of time. For this purpose we apply Euler's rule, and in the time step from $t_N$ to $t_{N+1}$ we get:

$$x'_{N+1} = x'_N + h\dot{x}'_N.$$ (5.3.1)

Now a second observer, in a frame whose origin coincides with that of our own, describes the position and velocity of that same particle by $x''(t)$ and $\dot{x}''(t)$. He applies Euler's rule to find the position of the particle, and in the time step from $t_N$ to $t_{N+1}$, he gets:

$$x''_{N+1} = x''_N + h\dot{x}''_N.$$ (5.3.2)

Between the two frames $x'(t)$ and $x''(t)$ are related by a rotation $Q(t)$ as

$$x''(t) = Q(t) \cdot x'(t)$$ (5.3.3)

and the velocities as

$$\dot{x}''(t) = \dot{Q}(t) \cdot \dot{x}'(t) + Q(t) \cdot \ddot{x}'(t).$$ (5.3.4)

If we assume that $x''_N = Q(t_N) \cdot x'_N$, then, at time $t_{N+1}$, $x'_{N+1}$ and $x''_{N+1}$ are related as

$$x''_{N+1} = Q(t_{N+1}) \cdot x'_{N+1} - Q(t_N) \cdot x'_N + h[\dot{Q}(t_N) \cdot x'_N + Q(t_N) \cdot \dot{x}'_N] - Q(t_{N+1}) \cdot [x'_N + h\ddot{x}'_N]
\quad = [Q(t_N) + h\dot{Q}(t_N) - Q(t_{N+1})] \cdot x'_N + h[Q(t_N) - Q(t_{N+1})] \cdot \ddot{x}'_N.$$ (5.3.5)

Since the right-hand side of (5.3.5) does not vanish identically, $x'_K$ and $x''_K$ do not obey (5.3.3) at times $t_K$ later than $t_N$, and therefore a result found by application of Euler’s rule is frame dependent.

At this point we pause to consider the result above, to clarify our objection to it. It is clearly futile to call upon the Principle of Objectivity of Material Properties, for no material property is involved. Our singular objection to the result above is that it is inobjective. Hughes and Winget seem to have been first to point this out. We now ask whether or not Euler's rule can be modified, or restricted in its application, in

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52 Hughes and Liu [55, Introduction], state that the incremental objectivity of their algorithm “is in keeping with the ‘objectivity’ of well-set finite-deformation constitutive equations”.

53 It is worthy of note that the result is objective with respect to all frame changes of the type $Q'(t) \cdot \dot{Q}(t) = 0$. Had we not chosen a common origin for the two frames, the result would have been objective only for frame changes of the type $x'' = (x''_0 + \dot{x}''_0) + Q \cdot x'$, $x''$, $\dot{x}''$, $\ddot{x}''$ and $Q$ being constant.
such a way that frame dependent results will be avoided. This may be done by specifying that Euler’s method may only be applied in ‘special’ frames. We might, for example, choose to apply Euler’s rule only in those frames whose angular velocity relative to an inertial frame vanishes. Between all such frames \( Q'(t) \cdot Q(t) = 0 \), and it follows immediately that results will be frame independent. In a frame of zero absolute angular velocity, \( w = 0 \), Euler’s rule gives us

\[
x_{N+1} = x_N + h\dot{x}_N.
\] (5.3.6)

Now we transform this equation, term by term, to a general frame (distinguishing entities in the general frame by primes)\(^{56}\):

\[
x_{N+1} = Q(t_{N+1}) \cdot x'_{N+1}, \quad x_N = x'_N,
\]

\[
\dot{x}_N = Q(t_N) \cdot \dot{x}_N + \dot{x}'_N,
\]

hence

\[
x'_{N+1} = Q'(t_{N+1}) \cdot [x'_N + h\dot{x}'_N], \quad \dot{x}' = \dot{x}' + w' \times x'.
\] (5.3.7)

The vector \( w' \) is the absolute angular velocity of the general frame. The orthogonal tensor \( Q(t_{N+1}) \) is the ‘incremental rotation’ that occurred between the defining frame \((w = 0)\) and the general frame \((w' \neq 0)\) during the time interval \((t_N, t_{N+1})\). The incremental rotation is found by solving the initial value problem

\[
\dot{Q}(t) = -Q(t) \cdot W(t), \quad Q(t_N) = I,
\] (5.3.8)

\( W(t) \) being the skew-symmetric tensor whose dual vector is the angular velocity.\(^{56}\) A similar formula (to 5.3.7) is found for the generalized trapezoid rule:

\[
x_{N+1} = x_N + h[(1 - \theta) \dot{x}_N + \theta \dot{x}_{N+1}]
\]

becomes

\[
x_{N+1} = Q'(t_{N+1}) \cdot [x_N + (1 - \theta) h\dot{x}_N] + \theta h\dot{x}_{N+1}
\] (5.3.9)

(for a general frame, \( w \neq 0 \)). As \( \theta \to 0 \) we recover (5.3.7). Letting \( \theta \to 1 \) in (5.3.9) we get an implicit ‘backward difference’ scheme.

We note that (5.3.7) and (5.3.9) are no more than discrete approximations to the ‘Jaumann integral’ of Goddard and Miller \([57]^{57}\)

\[
x(t) = Q'(t) \cdot x(t) + \int_t^{t'} Q'(t) \cdot Q(\zeta) \cdot \dot{x}(\zeta) d\zeta
\] (5.3.10)

\(^{54}\)It must be emphasized that such a choice is arbitrary. We could just as well choose as special those frames in which the particle’s velocity is always radial, \((x \times \dot{x} = 0)\). No matter what convention is used, we shall henceforth refer to the special frames as defining frames, after Zhong-Heng \([56]^{56}\).

\(^{55}\)We take \( Q(t) = I \) with no loss of generality.

\(^{56}\)\( W_{ij} = -\frac{1}{2} \epsilon_{ijk} w_k \) in cartesian tensor index notation.

\(^{57}\)These authors consider only the case in which \( W \) is put equal to the material spin.
where \( \dot{x} = \dot{x} + \omega \times x = \dot{x} - W \cdot x \). Putting \( \tau = t_N \) and \( t = t_{N+1} \), it is easily seen that (5.3.7) results if we approximate the integral

\[
\int_{t_N}^{t_{N+1}} Q(\zeta) \cdot \dot{x}(\zeta) d\zeta \quad \text{by} \quad h \dot{x}(t_N) \quad (\text{recall} \quad Q(t_N) = I)
\]

(5.3.11)

and (5.3.9) results if we approximate the same integral by

\[
h[(1 - \theta) \dot{x}(t_N) + \theta Q(t_{N+1}) \cdot \dot{x}(t_{N+1})].
\]

(5.3.12)

The relationship between the Jaumann integral and the objective numerical integration schemes is important because while the 'special' frames do play the role of defining frames for those numerical schemes, the algebra associated with their use can become tedious. The Jaumann integral is the vehicle by which general multistep schemes may be easily cast in an objective form. For example, we may write out, by inspection, the general formula for the objective Adams-Bashforth method:

\[
x_{N+1} = Q'(t_{N+1}) \cdot [Q(t_{N-p}) \cdot x_{N-p} + h \sum_{k=0}^{m} \gamma_p^k \Delta^k (Q(t_{N-k}) \cdot \dot{x}_{N-k})]
\]

(5.3.13)

where

\[
\gamma_p^k = \int_{-p}^{1} (-1)^k \binom{-s}{k} ds
\]

\( (\cdot^k) \) being the binomial function and \( \Delta^k \) is the \( k \)th order forward difference operator.\(^{58}\) A similar general formula can be given for implicit schemes.

Using the formula \( Q' \cdot Q = W \cdot W - W \) we can show that the generalized trapezoid rule, when applied to integrate the velocity from the acceleration, takes the objective form

\[
\dot{x}_{N+1} = Q'(t_{N+1}) \cdot [\dot{x}_N + h (1 - \theta) \dot{x}_N] + \theta h \dot{x}_{N+1}.
\]

(5.3.14)

By induction it may be shown that the objective generalized trapezoid formulas for higher accelerations are of the same form as (5.3.14).

One more problem must be considered before we can discuss algorithms for integration of the stress. Absolute objectivity is achieved by the schemes above only if we can construct the rotation increment \( Q(t_{N+1}) \) precisely.\(^{59}\) This is generally not possible. In the next few paragraphs we discuss schemes by which the rotation increment may be approximated.

The initial value problem for \( Q \) (5.3.8) may easily be reduced to a scalar problem by introduction of the 'angles of rotation' \( \theta_i : \)

\[
\sum_{i=1}^{3} \left( \frac{\partial}{\partial \theta_i} Q \right) \theta_i = Q \cdot W, \quad \theta_i(t_N) = 0
\]

(5.3.15)

\[
Q(\theta_1, \theta_2, \theta_3) = \begin{bmatrix}
(c \theta_1 c \theta_3) & (c \theta_1 s \theta_3 + s \theta_1 s \theta_2 c \theta_3) & (s \theta_1 s \theta_3 - c \theta_1 s \theta_2 c \theta_3) \\
(-c \theta_2 s \theta_3) & (c \theta_1 c \theta_3 - s \theta_1 s \theta_2 s \theta_3) & (s \theta_1 c \theta_3 + c \theta_1 s \theta_2 s \theta_3) \\
(s \theta_2) & (-s \theta_1 c \theta_2) & (c \theta_1 c \theta_2)
\end{bmatrix}
\]

\(^{58}\)Complete details are given by Conte and de Boor [43].

\(^{59}\)Rubinstein and Atluri have also studied this problem.
R W. Reed, S.N. Atluri, Large quasistatic deformations of inelastic bodies

The local error of such schemes is usually expressed in the form

\[ |\theta(t_0) - \theta_k| \leq ch^{N+1} \theta^{(N+1)}_k, \]

(5.3.16)

where \( \theta^{(N+1)}_k \) is the \((N + 1)\)th time derivative of \( \theta \). The scheme whose error term is of the form (5.3.16) we call ‘Nth order’. If the \( \theta \) are polynomials in time of degree less than or equal to \( N \), then the angles, and hence the rotation increment, may be integrated precisely. The set of all time dependent rotations whose angles are of degree less than or equal to \( N \) we denote by \( r(N) \). If an approximate rotation increment (whose angles were found by an \( N \)th order scheme) is used in place of the precise rotation increment then we describe the modified scheme as ‘objective relative to \( r(N) \)’. For example, the classical Euler’s method is objective relative to \( r(0) \) (since it is absolutely objective when the \( \theta \) are constant). If Euler’s method is used to find approximations for the angles \( \theta \), and the resulting approximate rotation increment is used in place of \( Q \) in (5.3.7), then the modified scheme is objective relative to \( r(1) \). In an \( N \)th order method is used to approximate the angles, then the modification of (5.3.7) which results is objective relative to \( r(N) \).

The principal objection to the method above for approximation of rotation increments is that it entails extra numerical work. We now discuss explicit approximation of the rotation increments. Following Gantmacher \[59, p. 155\], each interval \((t_{n-1}, t_n)\) is divided up into \( K \) subintervals \((\tau^k, \tau^{k+1})\), such that \( t^0 = t_{n-1} \) and \( t^K = t_n \), and from each subinterval we pick an intermediate time \( \tau \). We define the subincremental rotation as

\[ \Omega^{*,-1} = Q(t^*) \cdot Q^i(t^*-1). \]

(5.3.17)

Then it follows that the rotation increment \( Q \) may be represented as

\[ Q(t_{N-1}) = \Omega^K_{K-1} \cdot \Omega^{K-1}_{K-2} \cdots \Omega^2_{1} \cdot \Omega^1_{0}. \]

(5.3.18)

Now we approximate \( \Omega^{*,-1} \) as

\[ \Omega^{*,-1} = [\exp(\Delta t^* W(\tau^*))]^{\tau}, \quad \Delta t^* = (t^* - t^{*-1}). \]

(5.3.19)

This is exact when \( W(t) = W(\tau^*) \), a constant. The matrix function \( \exp(Wt) \) has the representation \[54, p. 238; 53\]

\[ \exp(Wt) = I + \left( \sin \frac{wt}{w} \right) W + \left( \frac{1 - \cos \frac{wt}{w}}{w^2} \right) W^2, \quad w^2 = \frac{1}{2} W : W \]

(5.3.20)

for any constant skew-symmetric \( W \). Hence, we can get an explicit approximation for the rotation increment by use of (5.3.18), (5.3.19) and (5.3.20). This approximation converges to

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60Euler’s angles are less convenient since the precession and spin are not always well-defined; see [58, p. 143].

61The rate of the angle between two vectors is an absolute scalar, i.e. has the same value in all frames, so standard numerical integration schemes may be applied to obtain an ‘objective’ approximation to the angle itself.
the precise rotation increment if we let $K$, the number of subintervals, approach infinity in such a way that the largest $|\Delta t^j| \to 0$ [59]. Only a partial ordering of these approximate rotation increments appears to be possible.\textsuperscript{62}

The discussions above apply to integration schemes for all orders of tensors. Instead of mechanically deriving a plethora of formulas, we shall examine one certain scheme for integration of rate-type constitutive equations. The Jaumann integral for the true stress is given by

\begin{equation}
\tau(t) = J_t^{-1}(t) \tau'(t) \cdot J(t) + \int_t^\tau J^{-1}(\xi) \tau'(\xi) \cdot J(\xi) \cdot \dot{\boldsymbol{\sigma}}^*(\xi) \cdot \tau(\xi) d\xi. \tag{5.3.21}
\end{equation}

Here $\dot{\boldsymbol{Q}}(t)$ is defined as the solution of

\begin{equation}
\dot{\boldsymbol{Q}}(t) = -\boldsymbol{Q}(t) \cdot \omega(t), \quad \boldsymbol{Q}(\tau) = \mathbf{I}, \tag{5.3.22}
\end{equation}

and may be thought of as an 'integrating factor'.\textsuperscript{63} Frames which spin with the principal axes of stretching play the role of defining frames for numerical schemes based on (5.3.21). By inspection we write the objective generalized midpoint rule as

\begin{equation}
\tau_{N+1} = J^{-1}(t_{N+1}) \tau_{N+1} \cdot \tau_N \cdot \tau(t_{N+1}) + h J^{-1}(t_{N+1}) \tau'(t_{N+1}) \cdot \tau(t_{N+1}) \cdot \dot{\boldsymbol{\sigma}}^* \cdot \tau(t_{N+1}) \cdot \tau(t_{N+1}). \tag{5.3.23}
\end{equation}

Now suppose that in this formula we use $[\mathbf{V} : \mathbf{E} + \mathbf{\Sigma}]$ in place of $\dot{\boldsymbol{\sigma}}^*$. Equation (5.3.23) then reduces, in essence, to the formulas of Key et al. [52] and Rubinstein and Atluri [53] for $\theta = \frac{1}{2}$. The formula of Hughes and Winget [51] cannot be recovered since they use midpoint constitutive evaluation but fail to properly rotate the resulting stress increment. An analogous formula has been given by Pinsky et al. [61]. Those workers [52, 53, 61] did not give any schemes other than midpoint schemes, or indicate how other schemes could be obtained.\textsuperscript{64}

Moreover, their discussions revolve around integration of (rate-type) constitutive equations; this could cause confusion of general physical objectivity with the Principle of Objectivity of Material Properties. Finally, no distinction is made between relative objectivity and absolute objectivity.

The present choice of defining frames, those frames in which the material spin $\omega$ vanishes, is outwardly as arbitrary now as it was before; we could just as well have chosen those frames whose absolute angular velocity $\omega$ vanishes as defining frames. Intuitively, one would expect the local error to be smallest in the former frames,\textsuperscript{65} however, an easy proof is not apparent. This is the only motive we can imagine for choosing corotational frames for defining frames.

\textsuperscript{62}Approximate rotation increments constructed over different sets of subintervals appear not to be comparable, in general.

\textsuperscript{63}This idea has been exploited by Petrie [60]. His integral may be used as a basis for more general numerical schemes, giving as special cases all of the schemes below (including that of Pinsky et al. [61]).

\textsuperscript{64}Pinsky et al. [61] do introduce defining frames, so it is apparent that their results may be extended. Hughes and Winget [51] remark that their algorithm may be generalized to a subincremental one, but omit details.

\textsuperscript{65}Especially when the deformation consists of a rigid motion.
6. Examples: finite homogeneous deformations

6.1. Introduction

Through the study of homogeneous deformations various important aspects of the performance of the finite element algorithm can be identified and studied. Since closed form solutions to problems of homogeneous deformation are widely available, questions of the accuracy of the finite element solutions can be resolved quickly and absolutely. If we immediately engaged problems complicated by inhomogeneous deformation, the accuracy of any solution we obtained would be no more than a subject for speculation. Homogeneous deformations also make convenient subjects for testing of time step bounds, and in the present case, for study of the effect on accuracy of the forward gradient scheme. Finally, the results serve to underscore the fact that the material models themselves are too idealized to be used in problems of technological interest when strains are very large.

6.2. Finite plane extension

We begin our study of homogeneous deformations by considering finite plane extension of (i) a hypoelastic material and (ii) an elastico-viscous fluid. The geometry of the specimen is given in Fig. 1. In these examples we focus on the relative efficiencies (accuracy/effort) of the Euler, Runge-Kutta second order (RK2) and Runge-Kutta fourth-order (RK4) integration schemes, and the effect on accuracy of the forward gradient scheme.

Consider plane extension of the hypoelastic material defined by

\[ \dot{\sigma}^* = 2\mu \dot{\varepsilon} + \lambda (I : \dot{\varepsilon}) I. \]

(6.2.1)

The closed form solution for the stress is given by

\[ s^{11}(L) = 0, \quad s^{22}(L) = \nu s^{33}(L), \quad s^{33}(L) = \left( \frac{1}{1 - 2\nu} \right) (1 - L^{-\tilde{\nu}}) \]

(6.2.2)

where \( \nu = \tau / 2 \mu \) and \( \tilde{\nu} = (1 - 2\nu)/(1 - \nu) \). In Figs. 2, 3 and 4 the stresses found by application of

Fig. 1. Plane extension specimen.
Fig. 2. Stress accompanying plane extension of a hypoelastic material—Euler integration.

Fig. 3. Stress accompanying plane extension of a hypoelastic material—RK2 integration.
the finite element algorithm are compared to the closed form solution for $\nu = \frac{1}{2}$. The Euler scheme (Fig. 2) leads to an underestimation of the strain-softening, but the RK2 and RK4 schemes (Figs. 3 and 4) give results virtually indistinguishable from the closed form solution. Since the time steps used gave stretch increments of (0.01), (0.02) and (0.04), respectively, the computational effort was the same for each of these three cases. In view of the differences in accuracy attained, we rank the RK4 scheme as most efficient, followed by RK2 and Euler.

Here we pause to remind the reader that 'incremental' finite element algorithms are predisposed to integration by the relatively inefficient Euler scheme.

Now consider plane extension of the elastico-viscous fluid defined by

$$
\varepsilon = \dot{s} - \left(\frac{\nu}{1+\nu}\right)(I : \dot{s})I + \frac{1}{T} s', \quad \dot{s}' = \dot{\sigma}'/2\mu, \quad s' = \tau'/2\mu.
$$

The closed form solution for the stress, when $\nu = 0.5$, is given by

$$
\begin{align*}
 s^{11}(t) &= 0, \\
 s^{22}(t) &= \frac{1}{2}s^{33}(t), \\
 s^{33}(t) &= s^{33}(0) e^{-\nu \tau} + 2 \int_0^t e^{33}(\xi) e^{-\nu (\xi - \tau)} d\xi.
\end{align*}
$$

[60, Appendix A]. For the stretch history $L(t) = 1 + Vt$, the stretching is given by $\varepsilon^{33}(t) =$

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For an incompressible fluid we recover from (6.2.3) the model of Zaremba (1903), Fromm (1947, 1948), and Dewitt (1955), according to Bird et al. [62]. The constant $T$ is the relaxation time.
V/(1 + Vi), and \( s^{33} \) may be expressed as a function of the stretch \( L \) as

\[
s^{33}(L) = s^{33}(1) e^{-(L-1)/VT} + 2e^{-L/VT}\left\{Ei\left(\frac{L}{VT}\right) - Ei\left(\frac{1}{VT}\right)\right\}.
\]

In the two cases following we have taken \( V = 10^{-14} \text{ sec}^{-1} \), \( \nu = 0.4 \), and \( T = \frac{1}{2} \cdot \frac{28}{39} \times 10^{12} \text{ sec} \). Cormeau's [37] time step bound is \( h < h_c = 2T \).

In the first case we assign the initial stress as if nearly-steady conditions existed from the outset: \( s(1) = 0 \). From (6.2.3) we thus obtain

\[
s^{11}(1) = 0, \quad s^{22}(1) = \frac{1}{2} s^{33}(1) = VT.
\]

We take time steps \( h = \frac{45}{28}h_c, h = 2 \cdot \frac{45}{28}h_c \) and \( h = 4 \cdot \frac{45}{28}h_c \) (corresponding to stretch increments of 0.01, 0.02 and 0.04), applying Euler, RK2 and RK4 schemes, respectively. Since the time steps exceed the critical time step \( h_c \), the forward gradient scheme is used, with \( \theta \) set to \( \frac{1}{2} \). The stresses found by application of the finite element algorithm (for \( \nu = 0.4 \)) are compared to the closed form solution (for \( \nu = 0.5 \)) in Figs. 5, 6 and 7.

It is apparent from these figures that the numerically integrated stress is slightly greater than the actual stress in the incompressible material, which is surprising, for one would expect a lower stress in the more compliant material when the deformation was prescribed. We note

\[\text{Fig. 5. Stress accompanying plane extension of a elastico-viscous fluid—Euler integration.}\]

\[\text{\footnotesize{\textsuperscript{67}Here Ei is the exponential integral Ei(x) = }\int_{-\infty}^{x} (e^z/z)dz, \text{ whose values are tabulated [63].}}\]
Fig. 6. Stress accompanying plane extension of an elasto-viscous fluid—RK2 integration.

Fig. 7. Stress accompanying plane extension of an elasto-viscous fluid—RK4 integration.
that using the forward gradient scheme is essentially the same as replacing the constitutive equation (6.2.3) by

$$\varepsilon = \left( 1 + \frac{\partial h}{T} \right) \dot{s}^* - \left( \frac{\nu + \frac{1}{3}}{1 + \nu + \frac{1}{3} \frac{\partial h}{T}} \right) (I : \dot{s}^*) + \frac{1}{T} \dot{s}'. $$

Defining \( z = (1 + \partial h/T) s, \) \( T_\theta = T + \partial h, \) and setting \( \nu = 0.5 \) gives

$$\varepsilon = \dot{s}^* - \frac{1}{T} (I : \dot{s}^*) I + (1/T_\theta) z'. $$(6.2.6)

But this equation is of the same form as (6.2.3), which has already been integrated for plane extension; distinguishing the closed form solution of (6.2.6) from that of (6.2.3) with a subscript \( \theta, \) we write

$$s_{\theta}^{31}(t) = 0, \quad s_{\theta}^{22}(t) = \frac{1}{2} s_{\theta}^{33}(t),$$

$$s_{\theta}^{33}(t) = s_0^{33} e^{-\omega T_\theta} + \frac{2T}{T_\theta} \int_0^t \varepsilon^{33}(\zeta) e^{-(\omega - \omega T_\theta)} d\zeta. $$

Now, to compare \( s \) and \( s_{\theta}, \) we consider the case that \( \varepsilon^{33}(t) = \varepsilon_0^{33}, \) a constant. From (6.2.4) and (6.2.7) we easily obtain

$$s^{33}(t) = s_0^{33} e^{-\omega T} - 2T \varepsilon_0^{33} (1 - e^{-\omega T})$$

and

$$s^{33}(t) = s_0^{33} e^{-\omega T} + 2T \varepsilon_0^{33} (1 - e^{-\omega T}).$$

At times much later than \( T \) and \( T_\theta, \) both \( s^{33} \) and \( s_{\theta}^{33} \) achieve the steady value \( 2T \varepsilon_0^{33}. \) The important difference is the rate at which the stresses approach their steady state value. It is clear that \( s_{\theta}^{33}(t) \) is related to \( s^{33}(t) \) as

$$s_{\theta}^{33}(t) = s^{33}(t - \theta h t/T_\theta)$$

so the stress \( s_{\theta}^{33} \) lags the actual stress \( s^{33}. \) This is exactly what is shown in Figs. 5, 6 and 7. Moreover, that lag becomes more and more apparent as the order of the integration scheme is increased, since the numerical solution tends towards \( s_{\theta}^{33} \) (not shown), rather than the actual stress \( s^{33}. \)

In the second case of plane extension of the elastico-viscous fluid defined by (6.2.3) we set the initial stress to zero. The stress in this case differs from that of the first case solely because of the different initial condition, and that difference attenuates like \( e^{-\omega T}. \) At times much later than \( t = T, \) the stress in this case is indistinguishable from that of the first case. We use Euler's method exclusively, taking time steps of \( h = \frac{1}{6} T_c, h = \frac{45}{26} h_c \) and \( h = 2.45 h_c, \) for stretch increments of 0.005, 0.01 and 0.02, respectively. The parameter \( \theta \) in the forward gradient scheme is set to unity, \( \theta = 1, \) giving \( T_\theta/T = 2.61, T_{\theta_0}/T = 4.21 \) and \( T_{\theta_0}/T = 7.43, \) respectively. In Figs. 8, 9 and 10 the stress found by application of the finite element algorithm (for \( \nu = 0.4 \)) is plotted alongside the actual stress \( s^{33} \) and the stress \( s_{\theta}^{33} \) (for \( \nu = 0.5 \)).
Fig. 8. Stress accompanying plane extension of a elastico-viscous fluid—\( (T_0/T) = 2.61 \).

Fig. 9. Stress accompanying plane extension of a elastico-viscous fluid—\( (T_0/T) = 4.21 \).
It is clear that even for \((T_0/T) = 2.61\) only a qualitative estimate of the transient response is given by the finite element algorithm, and that estimate is degraded as \((T_0/T)\) is increased. More important, though, is the fact that use of a higher order integration scheme cannot improve the accuracy of the numerical solution, since that solution would be drawn towards \(s^{33}_0\) not the actual stress \(s^{33}\). It is therefore senseless to use integration schemes other than Euler’s in conjunction with the forward gradient scheme.

### 6.3. Finite plane shear

We conclude our study of homogeneous deformations by considering finite plane shear of (i) the hypoelastic material defined by (6.2.1), (ii) a second hypoelastic material which resembles an elastic-perfectly plastic material and (iii) the elastico-viscous fluid defined by (6.2.3). The geometry of the specimen is given in Fig. 11. These examples serve not only to further demonstrate the performance of the finite element algorithm, but also to portray aspects of the finite deformation behavior of materials themselves. In the third case, involving the elastico-viscous fluid, only time steps smaller than the critical time step are taken, making it possible to compare the numerical results obtained through use of the forward gradient scheme to those obtained without it.

Throughout this section only the second-order Runge–Kutta integration scheme is used.

Consider plane shearing from a stress-free state of the hypoelastic material (6.2.1). The closed form solution for the stress is given by

\[
  s^{33}(e) = -s^{11}(e) = \frac{1}{2}(1 - \cos(e)) , \quad s^{13}(e) = \frac{1}{2}\sin(e).
\]  

(6.3.1)
The closed form and finite element solutions are plotted in Fig. 12. Though the finite element solution does agree with the closed form solution, it is evident that the predicted large strain behavior is not representative of any real material. The fault, of course, lies with the over-simple constitutive equation (6.2.1). This might seem to be of little consequence if one is interested only in metals such as those used in structures because some mechanism of inelasticity always becomes active long before such large shear strains as \( e = \frac{1}{2} \pi \) are reached.

\(^6\)Time steps corresponding to nominal shear strain increments \( (e_{N+1} - e_N) = 0.16 \) were used in the finite element algorithm.
However, since the constitutive equation (6.2.1) is frequently used as a component in ‘superposition’ models, it is relevant to ask if similar periodic behavior might be predicted by those models.

Consider plane shearing from a stress-free state of a hypoelastic material of grade two, defined by

\[
\dot{s} = \epsilon + \left( \frac{\nu}{1-2\nu} \right) (I : \epsilon) I - \frac{1}{K^2} (s' : \epsilon) s'.
\] (6.3.2)

When \( K^2 \) is identified as \( \frac{3}{8}(\tau_y/2\mu)^2 \), (6.3.2) is the same as the constitutive equation for an elastic-perfectly plastic material; only the yield surface has been removed. The closed form solution for the stress has been given [22, p. 423] as

\[
s^{33}(\phi) = -s^{11}(\phi) = \frac{1}{2} c^2 (1 - \cos(\phi)), \quad s^{13}(\phi) = \frac{1}{2} c \sin(\phi)
\] (6.3.3)

where, for \( K^2 < \frac{1}{2} \),

\[
c^2 = K^2/(\frac{1}{2} + K^2), \quad d^2 = K^2/(\frac{1}{2} - K^2), \quad \phi = 2 \tan^{-1}(c/d) \tanh(c/d).
\]

The closed form and finite element solutions, for \( K = (0.10) \), are plotted in Fig. 13. The shear stress \( s^{13} \) reaches its maximum value at a nominal shear strain \( e_c \) (for \( \phi = \frac{1}{2}\pi \))

\[
e_c = d \ln \left( \frac{1}{2K^2} + \frac{1}{cd} \right) = 0.656431,
\]

and declines after that, so again the predicted large strain behavior is not representative of any real material.

Finally, consider plane shearing from a stress-free state of the elastico-viscous fluid (6.2.3). The closed form solution for the stress is given by [64, p. 363, 60, Appendix A]

\[
s^{33}(t) = -s^{11}(t) = \frac{1}{2} C \left\{ E - e^{-\nu T}[\sin(\dot{\epsilon}t) + E \cos(\dot{\epsilon}t)] \right\}, \quad s^{13}(t) = \frac{1}{2} C \left\{ 1 - e^{-\nu T}[\cos(\dot{\epsilon}t) - E \sin(\dot{\epsilon}t)] \right\}
\] (6.3.4)

where

\[
C = E/(1 + E^2), \quad E = \dot{\epsilon} T,
\]

\( E \), of course, is the nominal strain at the instant \( t = T \). Three cases are examined, differing only in the strain rate prescribed. We consider values of the strain rate \( \dot{\epsilon} = \frac{90}{28} \times 10^{-12} \text{ sec}^{-1}, \dot{\epsilon} = 2 \times \frac{90}{28} \times 10^{-12} \text{ sec}^{-1}, \) and \( \dot{\epsilon} = 4 \times \frac{90}{28} \times 10^{-12} \text{ sec}^{-1} \), giving \( E = 0.5, E = 1.0 \) and \( E = 2.0 \), respectively. The reader should note that these strain rates are indeed very slow. In Figs. 14, 15 and 16 the

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\(^{69}\) The term seems to have been introduced by Truesdell [21]. Superposition models are dominant in the solid mechanics engineering literature.

\(^{70}\) Time steps corresponding to nominal shear strain increments \( (e_{N+1} - e_N) - 0.06 \) were used in the finite element algorithm.
Fig. 13. Stress accompanying recilinear shear of a hypoelastic material of grade 2.

Fig. 14. Stress accompanying recilinear shear of a elastico-viscous fluid—$E = 0.5$. 
Fig. 15. Stress accompanying rectilinear shear of a elastico-viscous fluid—\( E = 1.0 \).

Fig. 16. Stress accompanying rectilinear shear of a elastico-viscous fluid—\( E = 2.0 \).
The shear stress reaches its first maximum value at a nominal shear strain $e_c = \frac{1}{2}\pi$. This predicted large strain behavior bears a qualitative resemblance to some data for transient stresses in polymer melts [65], but no resemblance at all to stresses observed in metals. Thus, the utility of the constitutive equation is extremely limited in the finite strain range.

Though it was not necessary to use the forward gradient scheme in the three cases above, we may still introduce it to the finite element algorithm just to see what error it causes. Just as for the plane extension examples, it is possible to integrate the modified constitutive equation in closed form. We obtain

\[
\begin{align*}
 s^3_\theta(t) &= -s_\theta^{11}(t) = \frac{1}{2}(T/T_\theta) C_\theta \{E_\theta - e^{-\nu T_\theta} \{\sin(\dot{e}t) + E_\theta \cos(\dot{e}t)\}\}, \\
 s^{13}_\theta(t) &= \frac{1}{2}(T/T_\theta) C_\theta \{1 - e^{-\nu T_\theta} \{\cos(\dot{e}t) - E_\theta \sin(\dot{e}t)\}\}
\end{align*}
\]

(6.3.5)

where

\[
C_\theta = E_\theta(1 + E_\theta^2), \quad E_\theta = \dot{e} T_\theta, \quad T_\theta = T + \theta h.
\]

The closed form solutions $s$ and $s_\theta$, and the finite element solution, for $E = 2.0$ and $\theta = 1$, are plotted in Fig. 17.

---

71Time steps $h = (0.2) h_c, (0.1) h_c,$ and $(0.05) h_c$ were used (corresponding to nominal strain increments of 0.2), so it was unnecessary to use the forward gradient scheme.
As is apparent, the two solutions $s$ and $s_\theta$ do not coincide at late times; rather we get the ratios

$$s^{13}/s^{13}_\theta \rightarrow (1 + E^2)/(1 + E^2_\theta), \quad s^{11}/s^{11}_\theta \rightarrow (s^{13}/s^{13}_\theta)(E_\theta/E),$$

which both go to zero as $\theta t \rightarrow \infty$. The unconditional stability of the forward gradient scheme is indeed of dubious value!

The source of the error is not hard to find. Equation (6.2.6) may be set in the form

$$(T/T_\theta) \varepsilon = \dot{s}^* - \frac{1}{2}(I : \dot{s}^*)(I + (1/T_\theta)s').$$ (6.3.6)

If, when steady-state conditions are reached, the corotational stress rate vanishes, $\dot{s}^* = 0$, then both the original constitutive equation (6.2.3) and (6.3.6) reduce to

$$\varepsilon = (1/T)s',$$ (6.3.7)

and no discrepancy arises. This was the situation in plane extension. However, ‘steady-state’ generally does not imply that the corotational stress rate vanishes, and when it does not, $T_\theta$ cannot ‘cancel out’ of equation (6.3.6). Of course, if the constitutive equations (6.2.3) and (6.3.6) cannot be reconciled under steady-state conditions, then neither can the stresses. This was the situation in plane shearing.

Deformations (or flows) of technological interest usually are neither pure extension nor pure shear, but somewhere between those two extremes. Petrie [67, p. 2] has remarked that “for polymeric liquids there is strong evidence that in flows involving both shear and elongation, the elongation will have a dominant influence.” If we assume that the actual solution to some deformation/flow boundary value problem may be so characterized, then one might still be able to use the forward gradient scheme and get a reasonable approximation to that solution.

7. Conclusions

A new hybrid-stress finite element algorithm for the analysis of large, quasistatic, inelastic deformations has been presented. Principal variables in the formulation are the nominal stress rate and spin.

In contrast to ‘incremental’ finite element algorithms, whose solutions (displacement increments, stress increments) are predisposed to integration by the relatively inefficient Euler’s method, the present ‘rate’ algorithm permits an independent treatment of the initial value problem. Much of this work is focused on that initial value problem, treating the subjects of numerical stability and numerical objectivity in depth. A new approach to numerical objectivity was proposed, and it was shown that previous ‘objective’ algorithms were but special cases of new algorithms that could be obtained by this approach. Finally, it was shown that the

For the same reasons, Oldroyd [66] (note) concluded that “an elastico-viscous liquid will in general behave differently from any inelastic viscous liquid, even in steady-state experiments.”
forward gradient scheme, and by implication, two ‘finite approximation schemes’, could lead to significant error.

Appendix A. Special notation

The special notations used in this work are summarized in the formulas below. We write a second-order tensor $T$ and its transpose $T'$ as

$$
T = T^{ij} e_i e_j = T'_{ij} e_i e_j = T_{ij} e_i e_j, \\
T' = T'^{ij} e_i e_j = T'_{ij} e_i e_j = T_{ij} e_i e_j
$$

where the $e_i$ are the natural base vectors of the $x^k$ coordinates. A fourth-order tensor $E$ may be written out as

$$
E = E^{ijkl} e_i e_j e_k e_l.
$$

The scalar product of two second-order tensors $S$ and $T$ is

$$
S: T = S^{ij} T_{ij}.
$$

The product of a fourth-order tensor $E$ and a second-order tensor $T$ is

$$
E : T = E^{ijkl} T_{kl} e_i e_j, \\
T : E = T_{kl} E^{ijkl} e_i e_j.
$$

The product of two fourth-order tensors $D$ and $E$ is

$$
D : E = D^{ijkl} E^{mnkl} e_i e_j e_m e_n.
$$

Differentiation by a tensor is defined as

$$
\frac{\partial R}{\partial T} = \frac{\partial R}{\partial T^{ij}} e_i e_j, \\
\frac{\partial S}{\partial T} = \frac{\partial S^{ij}}{\partial T_{kl}} e_i e_j e_k e_l.
$$

The operators GRAD, DIV and CURL may be represented as the vector operators

$$
\text{GRAD } v = \nabla v, \quad \text{DIV } v = \nabla \cdot v, \quad \text{CURL } v = \nabla \times v
$$

where $\nabla$ is the symbolic gradient operator:

$$
\nabla = e^k \frac{\partial}{\partial x^k}.
$$

*In the text we write $\nabla_\tau$ for the ‘material gradient operator’, $\nabla_\tau = E^k \partial / \partial x^k$, $E_k$ being the natural base vectors of the $X^i$ coordinate system at time $t = \tau$.\]
In dyad notation (A.7) may be written out as
\[ \text{GRAD} \mathbf{v} = \nabla \mathbf{v} = (v_i')_i e_i e_j, \]
\[ \text{DIV} \mathbf{v} = \nabla \cdot \mathbf{v} = (v_i')_j, \]
\[ \text{CURL} \mathbf{v} = \nabla \times \mathbf{v} = (v_i')_j e'_{jk} e^k \]
where \((\quad)_i\) denotes the covariant derivative with respect to \(x^i\), and \(e'_{jk}\) is the alternating tensor, defined by
\[ e'_{jk} = (e^j \times e^k). \]
\[ (A.10) \]

A convenient summary of formulas of vector analysis is given by Spiegel [68].

**Appendix B. Shape functions for velocity, stress rate and spin**

*Shape functions for plane strain*
\[ x^1 = x, \quad x^3 = z. \]

*Velocity shape functions*
\[ N_i = N_{1,i} e_1 + N_{3,i} e_3. \]

*Four noded element*
\[ N_{1,i} = \begin{cases} \frac{1}{4}(1 + \xi_i)(1 + \eta_i), & i = 1, 2, 3, 4, \\ 0, & i = 5, 6, 7, 8, \end{cases} \]
\[ N_{3,i} = \begin{cases} 0, & i = 1, 2, 3, 4, \\ \frac{1}{4}(1 + \xi_{i-4})(1 + \eta_{i-4}), & i = 5, 6, 7, 8, \end{cases} \]
\[ |\xi| \leq 1, \quad |\eta| \leq 1, \]
\[ \xi_1 = -1, \quad \xi_2 = 1, \quad \xi_3 = 1, \quad \xi_4 = -1, \]
\[ \eta_1 = -1, \quad \eta_2 = -1, \quad \eta_3 = 1, \quad \eta_4 = 1. \]

*Shape functions for spin*
\[ QW_i = QW_{13,i} e_1 e_3 + QW_{31,i} e_3 e_1, \]
\[ QW_{13,1} = c_1, \quad QW_{31,2} = xc_2, \quad QW_{13,3} = zc_3, \]
\[ QW_{31,1} = -c_1, \quad QW_{31,2} = -xc_2, \quad QW_{31,3} = -zc_3. \]
The constants were used to improve the condition of $[H]$.

**Shape functions for stress rate**

$$\mathbf{Q}_I = \mathbf{Q}_{11,1} \epsilon_1 \epsilon_1 + 0 + \mathbf{Q}_{13,1} \epsilon_1 \epsilon_3 + 0$$

$$+ \mathbf{Q}_{22,1} \epsilon_2 \epsilon_2 + \mathbf{Q}_{31,1} \epsilon_3 \epsilon_1 + 0 + \mathbf{Q}_{33,1} \epsilon_3 \epsilon_3 ,$$

$$\mathbf{Q}_{11,1} = 1, \quad \mathbf{Q}_{31,2} = -1, \quad \mathbf{Q}_{22,3} = 1,$$

$$\mathbf{Q}_{13,4} = -1, \quad \mathbf{Q}_{33,5} = 1, \quad \mathbf{Q}_{11,6} = x ,$$

$$\mathbf{Q}_{31,6} = -z, \quad \mathbf{Q}_{31,7} = -x, \quad \mathbf{Q}_{22,8} = x,$$

$$\mathbf{Q}_{13,9} = -x, \quad \mathbf{Q}_{33,9} = z, \quad \mathbf{Q}_{33,10} = x,$$

$$\mathbf{Q}_{11,11} = z, \quad \mathbf{Q}_{13,12} = -z, \quad \mathbf{Q}_{22,13} = z.$$

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**References**


