DYNAMIC CRACK PROPAGATION ANALYSIS USING A NEW
PATH-INDEPENDENT INTEGRAL AND MOVING ISOPARAMETRIC ELEMENTS

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Abstract

The objective of the paper is to present a procedure based on moving (or shifting) isoparametric elements for the analysis of dynamic crack propagation. The fracture parameters such as the stress intensity factors or energy release rates for dynamically propagating cracks are evaluated by far-field path independent integrals which were recently derived by the authors. The combined use of the moving isoparametric elements and the path independent integrals gives excellent results as compared to those obtained by other techniques. The utility of other types of path independent integrals which do not have the meaning of energy release rates for a dynamically propagating crack is also discussed. It is found that the use of the path independent integral which has the meaning of energy release rate is essential to a correct evaluation of the stress intensity factors.

Introduction

In Refs. [1,2] the authors have presented a "moving singular element" procedure for dynamic analysis of fast crack-propagation in finite bodies. In the moving singular element, a large number of analytical eigen-functions corresponding to a steadily propagating crack are used as basis functions for displacements. The moving singular element may move by an arbitrary amount in each time-increment of the numerical time-integration procedure. As demonstrated in Refs. [1,2], the moving singular element formulation leads to a direct evaluation of the dynamic stress intensity factors, in as much as they are the parameters in the assumed basis functions for the singular element.

On the other hand, Atluri [3] has recently derived many types of path independent integrals for cracks in elastic and inelastic solids, on the basis of very general conservation laws. For an elasto-dynamically propagating crack it was found that the path independent integral \( J \) in Ref. [3] has the meaning of the rate of change of Lagrangean per unit crack extension in a solid in dynamic equilibrium. Later the present authors [4] have, through a simple modification of the integral in Ref. [3], derived a new path independent integral \( J' \) which has the meaning of energy release rate for dynamic crack propagation. Using the general solutions of the asymptotic fields near the tip of an elasto-dynamically propagating crack under combined Mode I and bending conditions, the relations of the path independent integrals to the stress intensity factors were established in Ref. [4]. The practicality of the above path independent integrals was demonstrated in Ref. [5] for the analysis of dynamic crack propagation using the above mentioned "moving singular element" procedure [1,2]. It is noted that the path independent integral \( J \) derived by Kishimoto et al. [6,7] which is somewhat different from \( J \) and \( J' \), was also investigated in Ref. [5].

Although the moving singular element procedure was successfully used in the fracture simulation analyses [8,9,10], this procedure may be difficult to be applied by general users of the finite element method because of its sophistication. One can naturally expect to use less sophisticated finite elements such as the isoparametric elements, in conjunction with the path independent integrals, since it is well known in the static analysis of crack problems that the numerical results obtained through far-field path independent integrals are insensitive to inaccuracies in modeling of the stress/strain singularity at the crack-tip.

In the present paper, a procedure of moving (or shifting) isoparametric elements for the analysis of dynamic crack propagation is presented. Both nonsingular isoparametric elements as well as distorted isoparametric elements which induce an appropriate \((r)^{-2}\) type singularity in strains are used near the crack-tip. The path independent integrals [4,5] are used to determine the fracture parameters such as the stress intensity factor and the energy release rate.

Finite Element Formulation

The present authors [1,2] have previously developed a moving singular finite element procedure which leads to a direct evaluation of the dynamic stress intensity factors. Whereas the special principle of virtual work for a dynamically propagating crack was used in Refs. [1,2], later in Refs. [5, 11], the following standard principle of virtual work was used:

\[
\int_0^L [\sigma_{ij} \delta \epsilon_{ij} + p \delta u_{ij}] dV - \int_0^L \sigma_{ij} \delta u_{ij} ds = 0
\] (1)

where \( \sigma_{ij}, \epsilon_{ij}, \) and \( u_{ij} \) are the stress, strain, and displacement components, respectively. \( V \) denotes the volume and \( S_{ij} \) is the boundary of the body subjected to the prescribed tractions \( F_{ij} \). A superposed dot denotes a total derivative with respect to time \( t \).

The field variables used within the singular element (see Fig. 1) are assumed as [1,2]:

\[
y(x_0, t) = \psi(x_0, t) \phi(t)
\] (2a)

\[
\dot{y} = \psi_{\dot{x}} - C_\psi \dot{\psi}
\] (2b)

\[
\ddot{y} = \psi_{\ddot{x}} - 2C_\psi \dot{\psi} + C_\psi^2 \psi_{\dot{x}}
\] (2c)

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where \( U \) is the matrix of eigen-functions for an elasto-dynamically propagating crack including appropriate rigid body modes, \( \beta \) is the vector of undetermined coefficients, \( C \) is the crack propagation speed and \((\xi, X)\) are the coordinates relative to the moving crack-tip \((\xi = x_1 - Ct)\).

The eigen functions \(U_{ij}(t = 1, 2)\) lead to the familiar \((1/\sqrt{r})\) singularity in stresses and strains. Thus the coefficient \(\beta_j(t)\) is indeed the Mode I stress intensity factor, \(K_I(t)\). Since the convection term of the total velocity has the \((1/\sqrt{r})\) singularity (see Eq. (2b)), the kinetic energy density \(T(=1/2\dot{u}^T u)\) also has the singularity of order \(r^{-1}\). The displacement compatibility between the singular element and the surrounding regular elements is satisfied through a least-square technique [1]. The final form of finite element equations for the singular element can be written as [1]:

\[
P^* \ddot{g}_s + \frac{1}{r} \ddot{r}_s + \frac{r}{ks} \dot{g}_s = 0
\]

where \( g_s \) is the vector of displacements at the nodes surrounding the singular element. \(k^*, D^*, \text{ and } M^*\) are the generalized stiffness, damping, and mass matrices for the singular element, and these were defined in Ref. [1]. It is noted that \(k^*\) and \(D^*\) are unsymmetric, and the apparent "damping" matrix \(D^*\) is attributed to the effect of the convection terms in velocity and acceleration as shown in Eq. (2).

In the present study, the region of the singular element (denoted by \(A\) in Fig. 1) is replaced by a group of the isoparametric elements \((A', A''\text{ in Fig. }1)\). In the type \(A\) model, the \((1/\sqrt{r})\) singularity in stress and strain is incorporated by shifting the midside nodes, on element edges emanating from the crack-tip node, to the quarter-points of the element sides as illustrated in Fig. 1. However, the type \(A\) model has no singularity in stress and strain since only the regular isoparametric elements are used. The field variables for the models \(A'\) and \(A''\), which are the same as those in the types \(B\) and \(C\) elements, are given by:

\[
y = N q, \quad \dot{u} = \dot{N} q, \quad \ddot{u} = \ddot{N} q
\]

where \(N\) is the matrix of shape functions, and \(q\) is the vector of nodal displacements. Through the usual finite element formulation for dynamic analysis, we obtain:

\[
E \ddot{q} + M \dot{q} = 0
\]

where \(E\) and \(M\) are the stiffness and mass matrices for the isoparametric elements. Contrary to the previously explained moving singular element procedure (model \(A\)), in the models \(A'\) and \(A''\), there is no singularity in the kinetic energy density since the regular shape function is used for the total velocity (see Eq. (4)).

To simulate the crack propagation, the finite element mesh pattern is shifted and readjusted as shown in Fig. 1. The field variables, such as the displacements, velocities, and accelerations at the newly created nodes after shifting or readjusting the mesh pattern, are obtained through the interpolation of the field variables at the old nodes.

Path Independent Integrals and Fracture Parameters

Consider a contour integral path including the bounded volume as shown in Fig. 2. The radius \(\varepsilon\) is considered to be very small and shrunk to zero in the limiting process. Thus, \(\Gamma_0\) may be referred to as the limiting path or near-field path while \(\Gamma\) may be called the far-field path.

For a crack propagating at an angle \(\theta_c\) measured from the \(x_1\) axis (see Fig. 2) the energy release rate \(G\) can be written as [3]:

\[
G = G_1 \cos^2 \theta_c + G_2 \sin^2 \theta_c
\]

and

\[
G_k = \lim_{\varepsilon \to 0} \int_{r \to \infty} [(W + T)n_k - t_1 \dot{u}_1,k] ds
\]

where \(W\) and \(T\) are the strain and kinetic energy densities respectively, \(n_k\) the outward normal direction cosines, \(t_1\) the traction, and \((\ )_k\) denotes \((\ )/\partial x_k\). It is noted that, for a Mode I crack propagation, the same or a similar expression for energy release rate has been independently derived by several investigators [13,14,15].

In Ref. [4], the path independent integral \(J'\), which has the meaning of energy release rate for dynamic crack propagation, was derived through a simple modification of the integral given in Ref. [3]. Here we demonstrate to rederive \(J'\) integral in a more straightforward fashion. For the near-field integral, we keep the right-hand side of Eq. (7), since \(J'\) integral should have the same near-field expression as the energy release rate. The far-field integral can be expressed by an integral which is the same as the one in the near-field integral, but along the paths \(\Gamma + \Gamma_0\), plus an unknown (residual) term. Thus we write:

\[
J'_k = \lim_{\varepsilon \to 0} \int_{\Gamma} [(W + T)n_k - t_1 \dot{u}_1,k] ds
\]

where \(W\) and \(T\) are the strain and kinetic energy densities respectively, \(n_k\) the outward normal direction cosines, \(t_1\) the traction, and \((\ )_k\) denotes \((\ )/\partial x_k\). It is noted that, for a Mode I crack propagation, the same or a similar expression for energy release rate has been independently derived by several investigators [13,14,15].

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It should be noted that the above path independent integral is valid under general mixed-mode conditions for a crack propagating with a non-constant velocity under unsteady conditions. The present authors have also shown in Ref. [16] that this integral can be reduced to that in Ref. [13] which is valid for only steady-state crack propagation at constant velocity.

Although the existence of many path independent integrals was discussed in Ref. 5, the other two types of path independent integrals derived by Kishimoto et al. [6] and Atluri [3] are also shown here:

\[
\dot{J}_k = \lim_{\varepsilon \to 0} \int_\varepsilon \left[ W_{\varepsilon} - t_{11} u_{11k} \right] dS = \lim_{\varepsilon \to 0} \int_{t+\varepsilon} \left[ W_{\varepsilon} - t_{11} u_{11k} \right] dS + \int_{t-\varepsilon} \left[ \dot{W}_{\varepsilon} - t_{11} u_{11k} \right] dV
\]

and

\[
\dot{J}_k = \lim_{\varepsilon \to 0} \int_\varepsilon \left[ (W-T)n_{\varepsilon} - t_{11} u_{11k} \right] dS = \lim_{\varepsilon \to 0} \int_{t+\varepsilon} \left[ (W-T)n_{\varepsilon} - t_{11} u_{11k} \right] dS + \int_{t-\varepsilon} \left[ \dot{(W-T)}n_{\varepsilon} - \dot{t}_{11} u_{11k} \right] dV
\]

Thus, the above path independent integrals are related to the energy release rate by

\[
\dot{J}_k = C_k - \int_\varepsilon T_n dS
\]

and

\[
\dot{J}_k = C_k - 2 \int_\varepsilon T_n dS
\]

For a stationary crack, the above three integrals are equivalent to the energy release rate, since the limit of \( \int_\varepsilon T_n dS \) vanishes. However, for a propagating crack, the integral \( \int_\varepsilon T_n dS \) does not vanish and remains finite because of the singularity in the kinetic energy density. Using the asymptotic solutions [4] for an elastodynamically propagating crack, the relations between the path independent integrals and the instantaneous stress intensity factors can be expressed as [4], for \( \theta_c = 0 \):

\[
J_1^O = \frac{1}{2\mu} \left[ K_{11}^2 A_1(C) + K_{11}^2 A_{11}(C) + K_{11}^2 A_{111}(C) \right]
\]

\[
J_2^O = -\frac{K_{11} K_{111}}{\mu} A_{111}(C)
\]

\[
J_3^O = \frac{1}{2\mu} \left[ K_{11}^2 \dot{F}_1(C) + K_{11}^2 \dot{F}_{11}(C) + K_{11}^2 \dot{F}_{111}(C) \right]
\]

where \( \mu \) is the shear modulus; I, II, and III denote the fracture modes; and C is the crack speed. The detailed expressions for the crack speed functions in Eqs. (16-18) are given in Ref. [4]. We rewrite the functions here only for the Mode I case

\[
J_1^O = \frac{1}{2\mu} \left[ K_{11}^2 F_1(C) + K_{11}^2 F_{11}(C) + K_{11}^2 F_{111}(C) \right]
\]

\[
J_2^O = -\frac{K_{11} K_{111}}{\mu} F_{111}(C)
\]

\[
J_3^O = \frac{1}{2\mu} \left[ K_{11}^2 \dot{F}_1(C) + K_{11}^2 \dot{F}_{11}(C) + K_{11}^2 \dot{F}_{111}(C) \right]
\]

Results and Discussion

The problem considered here is identical to that in Refs. [2,5]. Fig. 3 shows the finite element breakdown for the initial configuration \( r = 0 \). A time-independent tensile stress \( \sigma \) acts at the edge of specimen parallel to the crack-axis. Under the plane strain condition, the crack propagates symmetrically from the initial crack length \( a_0 = 0.2W \) with a constant velocity C. This problem may be considered to be similar to that treated by Broberg [12] except that Broberg treated an infinite body with a zero initial crack length. The material properties are \( \mu = 29.4 \text{ GPa}, \nu = 0.286, \sigma = 2450 \text{ kg/m}^3 \). The shear wave speed in this material is
\( C_s = 3464.1 \text{ m/sec.} \) Three contour paths are considered as shown in Fig. 3. In the numerical calculation of the far-field integrals, it is convenient to consider the contour paths in which the near-field paths shrink to zero. Thus, the following equations were numerically evaluated in the present analysis.

\[
\begin{align*}
J'_k &= \int_{\Gamma} \left[ \frac{1}{2} (W+n) n_k - t u_{i,k} \right] ds \\
&+ \int_{V} \left[ \frac{1}{2} (W+n) n_k - t u_{i,k} \right] dV \\
\end{align*}
\]

(26)

\[
\begin{align*}
\dot{J}_k &= \int_{\Gamma} \left[ \frac{1}{2} (W+n) n_k - t u_{i,k} \right] ds \\
&+ \int_{V} \left[ \frac{1}{2} (W+n) n_k - t u_{i,k} \right] dV \\
\end{align*}
\]

(27)

\[
\begin{align*}
J_k &= \int_{\Gamma} \left[ (W+n) n_k - t u_{i,k} \right] ds \\
&+ \int_{V} \left[ (W+n) n_k - t u_{i,k} \right] dV \\
\end{align*}
\]

(28)

First, in order to study the influence of the number of elements in the models \( A' \) and \( A'' \), the numerical results of the \( J' \) integral for a very high speed crack propagation, \( C = 0.6 C_s \), are shown in Fig. 4. Comparing the results obtained by the fine mesh (12 moving elements) and the coarse mesh (4 moving elements), it is seen that, for both the models, the finer mesh gives a larger decrease in the \( J' \) value immediately after the initiation of crack propagation. Fig. 4 also shows the influence of the different models near the propagating crack-tip. The isoparametric regular elements give better results in comparison with Broberg's analytical solution [12] for a zero initial crack length in an infinite body. The quarter-point singular elements give considerably lower results for the case of \( C = 0.6 C_s \). However, the differences in results obtained from models \( A, A', \) and \( A'' \) becomes smaller when the cases of slower crack propagation were analyzed. Thus the model \( A' \) may be applicable to slow speed crack propagation, i.e. \( C = 0 \rightarrow 0.3 C_s \).

Numerical results from the three types of path independent integrals as given by Eqs. (26-28) are shown in Table 1, for different instants of time and different contour paths. As indicated in Table 1, all the three types of integrals gave exactly the same results throughout the computation. This is due to the fact that the present model \( A'' \) (and also \( A' \)) does not have a singularity in the kinetic energy density while this singularity was correctly incorporated in the moving singular element procedure (model \( A \)). It is also seen in Table 1 that the results of the moving isoparametric elements correspond to the \( J' \) values obtained by the moving singular element procedure [5]. Thus all the present integral values can be interpreted as the energy release rate. Table 1 also indicates the path independence of the integral values.

Table 2 shows the comparison of the stress intensity factors as obtained by converting the integral values using the three formulas given by Eqs. (16-21). The corresponding results [5] obtained by the moving singular element procedure were also shown in Table 2. For the model \( A'' \), \( J' \) vs. \( K \) relation gives the best \( K \) values in comparison with those directly evaluated in the moving singular element procedure [5], i.e. \( \beta_1 \). The other relations \( (J \text{ vs. } K) \) give considerably lower results for Model \( A'' \) while these relations give acceptable results for Model \( A \) in which the singularity in the kinetic energy density is incorporated. It is interesting that, even for the Model A, \( J' \) vs. \( K \) relation gives the best agreement with the direct solution \( \beta_1 \). From Tables 1 and 2, it is concluded that the use of \( J' \) integral leads to a consistent procedure for any finite element model.

Figs. 5, 6, and 7 show, respectively, the results for the cases of \( C = 0.2, 0.4 \), and \( 0.6 C_s \), using the model \( A'' \) (isoparametric elements without the singularity in stress and strain). In these figures the far-field \( J' \)-integral values were converted to stress intensity factors using the formula given by Eq. (16a). These values were denoted by the open circles. As seen from the figures, the present results, at the earlier stages of the crack propagation, agree well with the analytical solutions of Broberg [12] and later agree quite well with the numerical results from the singular element model A [2]. The arrows in Figs. 5, 6, and 7 indicate the readjustment of the finite element mesh to shift the moving isoparametric elements (Model \( A'' \)) continuously. As seen from the figures, the far-field integral calculations were not affected by the mesh readjustment procedure.

The discrepancies in the earlier stages of the crack propagation can be attributed to the fact that the apparent damping matrix in the finite element system of equations for the moving singular element approach, as shown in Eq. (3), may cause a slower response as compared to the dynamic response in the system given by Eq. (5). However, it should be noted that there is an uncertainty in the transient response immediately after the initiation of crack propagation since the exact solution for this problem is not known.

**Conclusions**

A procedure based on moving (or shifting) isoparametric elements for the analysis of dynamic crack propagation is presented. The path independent \( J' \) integral, which has the strict meaning of an energy release rate, is rederived in a straightforward fashion. The combined use of the path independent \( J' \) integral and the moving isoparametric element procedure has been shown to be an effective tool for the calculation of the crack-tip fracture parameters such as the stress intensity factors as well as energy release rates in elastodynamic crack propagation.

A special attention should be paid to the use of other path independent integrals which do not have the meaning of energy release rate. The use of \( J \) and \( J \) integrals in a finite element model, in which the singularity in the kinetic energy density for a propagating crack is not incorporated, leads to false values of stress intensity factor.
Acknowledgements

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References


### Table 1: Comparison of Path Independent Integrals and Their Path Independency (C=0.6C_s)

<table>
<thead>
<tr>
<th>Time</th>
<th>Integral [N/m]</th>
<th>Isoparametric Elements (Model A&quot;&quot;)</th>
<th>Singular Element [5] (Model A)</th>
<th>Difference (Model A&quot; - Model A)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Path 1</td>
<td>Path 2</td>
<td>Path 3</td>
</tr>
<tr>
<td>t=20Δt (a/w=0.3)</td>
<td>J_1'</td>
<td>19.35</td>
<td>20.25</td>
<td>20.07</td>
</tr>
<tr>
<td></td>
<td>J_1</td>
<td>19.35</td>
<td>20.25</td>
<td>20.07</td>
</tr>
<tr>
<td></td>
<td>J_1</td>
<td>19.35</td>
<td>20.25</td>
<td>20.07</td>
</tr>
<tr>
<td>t=40Δt (a/w=0.4)</td>
<td>J_1'</td>
<td>23.21</td>
<td>23.12</td>
<td>22.95</td>
</tr>
<tr>
<td></td>
<td>J_1</td>
<td>23.21</td>
<td>23.12</td>
<td>22.95</td>
</tr>
<tr>
<td></td>
<td>J_1</td>
<td>23.21</td>
<td>23.12</td>
<td>22.95</td>
</tr>
<tr>
<td>t=60Δt (a/w=0.5)</td>
<td>J_1'</td>
<td>28.05</td>
<td>27.99</td>
<td>27.98</td>
</tr>
<tr>
<td></td>
<td>J_1</td>
<td>28.05</td>
<td>27.99</td>
<td>27.98</td>
</tr>
<tr>
<td></td>
<td>J_1</td>
<td>28.05</td>
<td>27.99</td>
<td>27.98</td>
</tr>
</tbody>
</table>

### Table 2: Converted Stress Intensity Factors Using Various Formulas (C=0.6C_s)

<table>
<thead>
<tr>
<th>Time</th>
<th>Isoparametric Elements (Model A&quot;&quot;)</th>
<th>Singular Element (Model A) [5]</th>
<th>Direct Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Converted S.I.F.</td>
<td>Converted S.I.F.</td>
<td>Converted S.I.F.</td>
</tr>
<tr>
<td></td>
<td>K_1(J') / \sqrt{\pi a}</td>
<td>K_1(\tilde{J}) / \sqrt{\pi a}</td>
<td>K_1(J) / \sqrt{\pi a}</td>
</tr>
<tr>
<td>t=20Δt (a/w=0.3)</td>
<td>0.5649 (-1.2 %)</td>
<td>0.5299 (-7.3 %)</td>
<td>0.5006 (-12.5 %)</td>
</tr>
<tr>
<td>t=40Δt (a/w=0.4)</td>
<td>0.5231 (-0.3 %)</td>
<td>0.4907 (-6.5 %)</td>
<td>0.4636 (-11.6 %)</td>
</tr>
<tr>
<td>t=60Δt (a/w=0.5)</td>
<td>0.5166 (0.0 %)</td>
<td>0.4846 (-6.2 %)</td>
<td>0.4579 (-11.4 %)</td>
</tr>
</tbody>
</table>

\[ K_1(J') = \sqrt{\frac{2\mu J_1'}{A_1(C)}}, \quad K_1(\tilde{J}) = \sqrt{\frac{2\mu J_1}{\tilde{F}_1(C)}}, \quad K_1(J) = \sqrt{\frac{2\mu J_1}{F_1(C)}} \]

(X): Difference from \( \tilde{F}_1 \)
TYPE A: Moving element(s)
TYPE B: Distorting regular elements
TYPE C: Non-Distorting regular elements

Model A

Eigen-Function
Singular Element

Model A'

Quarter-Point
Singular Elements

Model A''

Isoparametric
Regular Elements

Fig. 1 Moving element procedure with various finite element modelings around a crack-tip

EXAMPLE: $v = 1000 \text{ m/sec}$
$\Delta t = 0.2 \mu \text{sec}$
$\Delta \Sigma = 0.2 \text{ mm}$
Fig. 2 Definition of integral paths

\[ \Gamma_c = \Gamma_c^+ + \Gamma_c^- \]
\[ \delta(V - V_\epsilon) = \Gamma + \Gamma_c^- - \Gamma_\epsilon \]

Fig. 3 Initial configuration of finite element mesh and contour integral paths

\( t=0 \quad a_0 = 8 \text{ mm} \), \( W = 40 \text{ mm} \)
Fig. 4 Variation of $J'$ integral for high speed crack propagation ($C = 0.6C_s$)
- $K_I$ from Singular Element: $\beta_i$
- $K_I$ from Isoparametric Elements: $\sqrt{\frac{2\mu J'_i}{A_i(c)}}$

--- Broborg

$c = 0.2c_s$

Fig. 5 Convered stress intensity factors from $J'$ integral ($c = 0.2c_s$)

--- $K_I$ from Singular Element: $\beta_i$
- $K_I$ from Isoparametric Elements: $\sqrt{\frac{2\mu J'_i}{A_i(c)}}$

--- Broborg

$c = 0.4c_s$

Fig. 6 Converted stress intensity factors from $J'$ integral ($c = 0.4c_s$)
Fig. 7 Converted stress intensity factors
from $J'$ integral ($C = 0.6 c_s$)