HYBRID STRESS FINITE ELEMENT ANALYSIS OF BENDING OF A PLATE WITH A THROUGH FLAW

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SUMMARY
A hybrid stress finite element procedure for the solution of bending stress intensity factors of a plate with a through-the-thickness crack is presented. Reissner's sixth-order plate theory including the effects of transverse shear deformation is used. The dominant singular crack tip stress field is embedded in the crack tip singular elements and only regular polynomial functions are assumed in the far field elements. The stress intensity factors can be calculated directly from the crack tip singular stress solution functions. The effects of the plate thickness, the ratio between the crack size and the inplane dimension of the plate, and the singular element size on the stress intensity factor solution are investigated.

The effects of the explicit enforcement of traction-free conditions along crack surfaces, which are the natural boundary conditions in the present hybrid stress finite element model, are also investigated. The numerical results of bending of a plate with a straight central crack compare favourably with analytical solutions. It is also found that the explicit enforcement of traction-free conditions along crack surfaces is mandatory to obtain meaningful results for the Mode I type of bending stress intensity factor.

INTRODUCTION
The nature of stresses near the tip of a through-the-thickness crack in a plate under bending loads has been analytically studied by several investigators. Williams obtained the bending stress singularity near the tip of a straight-line crack, by using the method of eigenfunction expansion based on Kirchhoff plate theory, and found this to be of \(1/\sqrt{r}\) (where \(r\) is the radial distance from the crack tip) type. The above-mentioned results of Williams were incomplete in that the magnitudes of local stresses were left undetermined. Later, Sih and Rice indicated a way of finding the coefficients in the eigenfunction expansions through the application of the complex variable theory. However, the above results based on Kirchhoff theory have some discrepancies. In the formulation based on Kirchhoff theory the three physically distinct boundary conditions on the crack surface (i.e. the vanishing of bending moment, twisting moment and shear) are reduced to two approximate boundary conditions through the Kirchhoff hypothesis. On account of this, the stress distribution near the crack edge in References 1 and 2 was found to be inaccurate. To overcome this difficulty, Knowles and Wang employed a sixth-order Reissner's plate bending theory, wherein the above-mentioned three boundary conditions can be satisfied distinctly, to treat the problem of an infinite plate containing a finite through crack under constant bending moment applied at infinity. The results in Reference 3 were good only for a vanishingly thin plate; and hence the work was later generalized by Hartranft and Sih to include the effect of plate thickness. Recently Wang, with a treatment
analogous to that in Reference 3, solved the problem of a cracked plate subjected to remote external twisting moment, and also included the effects of plate thickness.

From the results of the aforementioned works, it is found that the stress distributions near the crack front caused by bending as well as twisting in a plate are identical to those associated with the general opening, sliding and tearing modes of crack extension. However, it is noted that the solutions obtained by the above-mentioned methods are based on some approximations to a fully three-dimensional theory, namely the theory of bending of a plate, whether it is a fourth-order classical or a sixth-order advanced theory. Thus, through the above methods the nonlinear disturbances near crack edges and plate surfaces for cracks in very thick plates cannot be accounted for; for these cases, the use of a fully three-dimensional theory may be necessary. In the study of the effect of plate thickness on the stress-intensity factors for a crack in a plate in bending, Sih obtained qualitative features of exact solutions by using a three-dimensional asymptotic expansion of stresses and displacements.

The available solutions of bending stress intensity factors by analytical and semi-analytical methods as reviewed above are limited to simple and idealized cases. For practical purposes, numerical methods may be used efficiently for the analysis of the problems. As is now well known, the finite element technique, if suitably formulated, can be used to solve for the stress intensity factors for arbitrary shaped cracks in structural elements of arbitrary geometry. It is also known that the most efficient finite element formulation for this class of problems is that of incorporating the asymptotic (singular) solutions for stresses/strains/displacements, in finite elements in the vicinity of the crack front.

Applications of the finite element method with singular element concept for bending of plates with cracks has been studied by several authors. Barsoum has proposed a quarter point singular element. In this element, the appropriate crack tip singularity can be achieved by locating the mid-side nodes near the crack tip of a three-dimensional 20-node isoparametric element at the quarter point. Recently, Yagawa and Nishioka have solved cylindrical bending of a cracked plate problem by a superposition finite element method. In this method, analytical solution functions which can represent crack tip displacement field are superposed to the regular finite element modes. The analytical displacement functions can be derived by the combination of the Taylor series expansion of displacements variation functions along the thickness direction and Muskhelishvili's complex stress functions which depend on the boundary conditions of a problem.

The purpose of this paper is to present a hybrid stress finite element approach to the problem. In the present element, the exact form of the crack tip singularities of stress and displacements which are consistent to plate theory and asymptotic crack tip displacement field are embedded in the finite element in the vicinity of the crack front. In elements far away from the crack front, arbitrary polynomial variation in stresses and displacements can be assumed. However, the conditions of interelement displacement compatibility can be satisfied at the common boundaries of elements throughout the entire model.

In the present approach, the stress intensity factors can be solved directly from the formulation as unknown parameters, while in the other two approaches, stress intensity factors are solved indirectly from nodal displacements or Taylor series coefficients of displacement variation functions along the thickness direction.

ASYMPTOTIC SOLUTIONS FOR STRESSES NEAR CRACK TIP OF A PLATE UNDER BENDING

As previously discussed, the functional forms of the crack tip fields should be known since in
the present formulation the exact forms of asymptotic stresses and displacements which are consistent to plate theory are embedded in a crack tip singular finite element.

The structures of asymptotic solutions for stresses near a crack tip of a plate under a bending deformation are expressed as the following when Reissner's sixth-order plate theory is used:  

\[ M_{xx} = \frac{K_1}{\sqrt{(2r)}} \left( \cos \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{(2r)}} \left( 7 \sin \frac{\theta}{2} + \sin \frac{5\theta}{2} \right) - 2\sqrt{(2r)}K_{III} \sin \frac{\theta}{2} + 0(r^n) \]  

(1a)

\[ M_{yy} = \frac{K_1}{\sqrt{(2r)}} \left( \cos \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{(2r)}} \left( \sin \frac{\theta}{2} \sin \frac{5\theta}{2} \right) + 0(r^n) \]  

(1b)

\[ M_{xy} = \frac{K_1}{\sqrt{(2r)}} \left( \frac{1}{2} \sin \theta \cos \frac{3\theta}{2} \right) - \frac{K_{II}}{\sqrt{(2r)}} \left( 3 \cos \frac{\theta}{2} + \cos \frac{5\theta}{2} \right) - \sqrt{(2r)}K_{III} \cos \frac{\theta}{2} + 0(r^n) \]  

(1c)

\[ Q_x = \frac{K_{III}}{\sqrt{(2r)}} \sin \frac{\theta}{2} + 0(r^n), \quad Q_y = -\frac{K_{III}}{\sqrt{(2r)}} \cos \frac{\theta}{2} + 0(r^n) \]  

(1d, 1e)

where \( n > -\frac{1}{2} \).

In the above equations, \( M_{ij} \) and \( Q_i \) \((i, j = x, y)\) are moment and shear stress components in the appropriate direction respectively, \( x \) and \( y \) are Cartesian co-ordinates, \( r \) and \( \theta \) are polar co-ordinates originating from the crack tip as shown in Figure 1, and \( K_1, K_{II} \) and \( K_{III} \) are opening, sliding and tearing mode bending stress intensity factors, respectively. It is noted that terms with \( \sqrt{r} \) in \( M_{xx} \) and \( M_{xy} \) of the above were introduced deliberately to make the asymptotic stress field to satisfy the plate equilibrium equations

\[ \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_x = 0 \]  

(2a)

\[ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + Q_y = 0 \]  

(2b)

It is further noted that, in the present case, the functional form of only the basic singular part of the analytical solution is readily available in the literature; \(^{3,6,8}\) not so readily available is the asymptotic solution for the displacement field. Discussions on the displacement field near the crack tip will be given later.

HYBRID STRESS FINITE ELEMENT FORMULATIONS

As noted in equations (1), only the basic singular part of the asymptotic solution for moments and shear near the crack tip is readily available, and this will be embedded in elements adjoining the crack tip. However, to keep the size of the crack tip elements reasonably large, a regular polynomial variation of the stress field will also be assumed in the crack tip element, in addition to the basic singular terms. Compatibility of displacements and equilibrium of the tractions between two neighbouring elements will be maintained through a Lagrange multiplier method, based on the hybrid stress finite element model.\(^{14}\)

For the formulation of the hybrid stress finite element model, two solution functions must be assumed in an element. The two solution functions are: stresses which satisfy the equilibrium conditions \( \sigma_{ij} + \vec{F}_i = 0 \) in \( V_m \), where \( \sigma_{ij} \) are the derivatives of stresses \( \sigma_{ij} \) with respect to co-ordinates \( x_i, \vec{F}_i \) are body forces and \( V_m \) is the \( m \)th finite element; and continuous boundary displacements \( \vec{u}_i \) along boundary, \( \partial V_m \) of \( V_m \) (and \( \vec{u}_i = \vec{u}_i \) on \( S_{um} \) where \( S_{um} \) is portion of \( \partial V_m \) with prescribed displacements \( \vec{u}_i \)).
The vanishing of the first variation of the functional

\[
\pi_{HS}(\sigma_{ij}, \mathbf{u}) = \sum_{m=1}^{N} \left\{ \int_{V_m} B(\sigma_{ij}) \, dV - \int_{\partial V_m} T_i \mathbf{u}_i \, dS + \int_{S_{in}} \mathbf{T}_i \mathbf{u}_i \, dS \right\}
\]

leads to the Euler equations: (i) \( \varepsilon_{ij} = \frac{1}{2}(f_{ij} + f_{ij}) \) in \( V_m \), (ii) \( f_i = \mathbf{u}_i \) on \( S_{on} \), (ii) \( f_i = \tilde{\mathbf{u}}_i \) on \( S_{on} \), (iv) \( f^T_i = f^T_i = \tilde{\mathbf{u}}_i \) on \( \rho_m \), (v) \( T_i^+ + T_i^- = 0 \) on \( \rho_m \), and (vi) \( T_i = \tilde{T}_i \) on \( S_{on} \). In the above, \( N \) is the total number of elements in a model, \( B \) is the complementary energy density, \( T_i \) and \( \mathbf{u}_i \) are continuous functions with continuous first derivatives, \( \rho_m \) are prescribed tractions, \( \mathbf{u}_i \) are continuous boundary displacements, \( \rho_m \) are prescribed boundary moment and shear stresses, \( V_m \) is now the area of \( m \)th plate element on the reference surface with \( dV_m \), \( S_{on} \), etc., being defined correspondingly to the new definition of \( V_m \). Also, \( D \) is the compliance property matrix for the plate (assumed to be isotropic), \( D_1 = [12(1 + \nu)]/Eh \) and \( D_2 = 12\nu/5Eh \), with \( E, \nu \) and \( h \) being Young’s modulus, Poisson’s ratio and the plate thickness, respectively.

In each singular element, stresses are assumed as follows:

\[
\mathbf{M} = \mathbf{N}_{\mathbf{O}} \mathbf{\beta} + \mathbf{N}_{\mathbf{P}} \mathbf{\beta} + \mathbf{N}_{\mathbf{P}} \mathbf{\beta} + \mathbf{N}_{\mathbf{P}}
\]

and

\[
\mathbf{Q} = \mathbf{N}_{\mathbf{O}} \mathbf{\beta} + \mathbf{N}_{\mathbf{O}} \mathbf{\beta} + \mathbf{Q}_{\mathbf{P}}
\]

where the functions \( \mathbf{N} \) and \( \mathbf{N}_{\mathbf{O}} \) with undetermined parameters \( \mathbf{\beta} \) correspond to a self-equilibrated moment and shear fields, respectively, and are obtained from two stress functions, using a static geometric analogy. Further, \( \mathbf{\beta}^T = [K_1 K_{12}] \), \( \mathbf{\beta}^2 = \mathbf{K}_{iii} \), and thus \( \mathbf{N}_\mathbf{O}, \mathbf{N}_\mathbf{P}, \) and \( \mathbf{N}_{\mathbf{O}} \) are the corresponding asymptotically correct singular functions as given in equation (1). Finally \( \mathbf{N}_{\mathbf{P}} \) and \( \mathbf{Q}_{\mathbf{P}} \) are particular solution functions corresponding to the applied external forces.

The boundary tractions which are derived from \( \mathbf{M} \) and \( \mathbf{Q} \) of equation (5) can be given as

\[
\mathbf{T} = \mathbf{R}_{\mathbf{O}} \mathbf{\beta} + \mathbf{R}_{\mathbf{P}} \mathbf{\beta} + \mathbf{R}_{\mathbf{P}} \mathbf{\beta} + \mathbf{R}_{\mathbf{P}}
\]

where \( \mathbf{R}, \mathbf{R}_{\mathbf{P}}, \mathbf{R}_{\mathbf{P}}, \) and \( \mathbf{R}_{\mathbf{P}} \) are defined correspondingly to the appropriate coefficient matrices of equation (5).

For a regular element far from the crack tip the same type of regular and particular solution functions, as in a singular element, can be assumed. Therefore, the first and last terms of
equations (5) and (6) can be used as assumed solution functions for each of regular elements.

Finally, the continuous boundary displacements can be assumed by interpolating generalized nodal displacements uniquely along a boundary segment with appropriate boundary coordinates as

\[ u = Lq \]  

(7)

where \( L \) is a matrix of interpolation functions which interpolates generalized nodal displacements \( q \) uniquely.

Substituting equations (5)–(7) and the regular element equivalent to them into equation (4) would yield

\[ \pi_{HS} = \sum_{m=1}^{r} \pi_{HS,m} + \sum_{n=r+1}^{N} \pi_{HS,n} \]  

(8a)

where \( r \) is the number of singular elements in a model and the singular and regular energy functionals, \( \pi_{HS}^{s} \) and \( \pi_{HS} \) are, respectively, defined as

\[ \pi_{HS}^{s} = \frac{1}{2} \beta^{T} (H + U) \beta + \beta^{T} H_{p} \beta_{p} + \beta^{T} (H_{p} + U_{p} - V) \\
+ \frac{1}{2} \beta^{T} H_{b} \beta_{b} + \beta^{T} (H_{b} - V_{b}) + \frac{1}{2} (H_{n} + U_{n}) \beta_{n}^{2} \\
+ (H_{p} + U_{p} - V_{p}) \beta_{p} - \beta^{T} Gq - \beta^{T} G_{p} q - G_{p}^{T} q + S^{T} q + \text{constant} \]  

(8b)

and

\[ \pi_{HS} = \frac{1}{2} \beta^{T} (H + U) \beta + \beta^{T} (H_{p} + U_{p}) - \beta^{T} Gq - G_{p}^{T} q + S^{T} q + \text{constant} \]  

(8c)

with

\[ H_{ij} = \int_{V_{m}} N_{i}^{T} D N_{j} \ dV (i, j = o, s, t, p), \]

\[ U_{ij} = \int_{V_{m}} D_{1} N_{o}^{T} N_{o} \ dV (i, j = o, t), \]

\[ U_{ip} = \int_{V_{m}} D_{1} N_{o}^{T} \begin{pmatrix} p_{x} \\ 0 \end{pmatrix} \ dV (i = o, t), \]

\[ G_{i} = \int_{V_{m}} R_{i}^{T} L \ dS (i = o, s, t), \]

\[ V_{i} = \frac{1}{2} \int_{V_{m}} D_{2} N_{i}^{T} P \ dV (i = o, s, t), \]

and

\[ S = \int_{S_{m}} \bar{T}^{T} L \ dS. \]

In the above, a quantity with \( o \) represents the quantity without index.

Eliminating \( \beta \) from equation (8) by extremizing \( \pi_{HS} \) with respect to \( \beta \) and solving several matrix equations will yield

\[ \pi_{HS} = \sum_{m=1}^{r} \left( \frac{1}{2} [q^{T} \beta_{p}], \beta_{i} \right) \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} \begin{bmatrix} q \\ \beta_{p} \end{bmatrix} \\
- [Q^{T} Q_{s} Q_{p}] \begin{bmatrix} q \end{bmatrix} + \sum_{n=r+1}^{N} \left( \frac{1}{2} q^{T} K_{11} q - Q^{T} q \right) \]  

(9)
where
\[ K_{11} = G^T \tilde{H} G, \quad K_{12} = G_s^T - G^T \tilde{H} s, \]
\[ K_{13} = G_i^T - G^T \tilde{H} i, \quad K_{22} = H_s^T \tilde{H} s - H_{ss}, \]
\[ K_{23} = -H_{ii} + H_s^T \tilde{H} i, \quad K_{33} = H_i^T \tilde{H} i - \tilde{U}_{ts}, \]
\[ Q = G^T \tilde{H} \tilde{U}_p - G_p + S, \quad Q_s = H_{sp} - V_s - H_i^T \tilde{H} \tilde{V}_p. \]

and
\[ Q_t = H_{tp} + U_{tp} - V_i - H_i^T \tilde{H} \tilde{U}_p \quad \text{with} \quad \tilde{H} = (H + U)^{-1}, \quad \tilde{U} = H + U. \]

Following the usual finite element solution procedure, the stress intensity factors \( \beta_s \) and \( \beta_i \) can be evaluated directly with \( q \) at the same time. However, for the convenience of assembly, \( \beta_s \) and \( \beta_i \) can also be eliminated by the similar way as for \( \beta \) from equation (9). Later, they can be evaluated from \( q \) indirectly.

The procedure to obtain, and the form of these algebraic equations, is analogous to that in Pian, Tong and Luk\(^\text{16}\) who treat the plane stress crack problem, and in Luk\(^\text{17}\) who also formulates a crack element based on Reissner type plate theory but without implementation (it is also noted that, in Reference 17, boundary displacements along singular sides which will be discussed in detail later in this paper are not consistent to the plate theory used and could lead to erroneous results).

**NUMERICAL EXAMPLE**

Even though the formulations presented are valid for general mixed-mode loading conditions, example problems are presented here pertaining to Mode I loadings only (specifically for a uniformly distributed \( M_{yy} \) and \( M_{xx} \) in a double symmetric manner all around a square plate shown in Figure 1). Due to symmetry, only a quarter of the plate needs to be analysed, as shown in Figure 2, with two crack tip square singular elements A and B.

![Figure 1. Nomenclature of cracked plate](image)
The regular stress solution functions for each of the finite elements are given in Appendix I. For elements which share the crack surface as part of their boundary, one more set of regular stress solution functions which can satisfy crack surface traction-free conditions inherently as well as equilibrium conditions is used. Thus, the identical problem is solved twice with two different regular assumed solution functions; one is with and the other is without satisfaction of traction free boundary conditions in the appropriate elements a priori.

The purpose of the extra analysis is to study the effect of the explicit enforcement traction boundary conditions which are natural boundary conditions in the formulation. The second set of regular stress system can be obtained by dropping the non-vanishing terms from the first set when they are evaluated along the appropriate boundary. The stress and boundary tractions which are reduced from the first system are given in Appendix I. The singular stress functions of equation (1) satisfy crack surface traction-free conditions inherently.

The continuous boundary displacements for regular sides can be easily generated from appropriate one-dimensional interpolation functions. However, along the boundaries which contact with the crack tip, these boundary displacements must be able to represent asymptotic crack tip displacement behaviour as well as be consistent to plate theory used in the formulation.

Since the asymptotic solution for the displacement field near the crack tip is not readily available, the boundary displacement for the singular element is assumed based on considerations of the qualitative features of the asymptotic solution in a fully three-dimensional case. In the three-dimensional case, all the displacements vary as $\sqrt{r}$ functions near the crack front. But in a plate bending problem, components of inplane displacements, $u_\alpha$, are linearly proportional to the total rotation $\phi_\alpha$. Further, $\phi_\alpha = \partial W/\partial x_\alpha + g_\alpha$, where $W$ is normal displacement and $g_\alpha$ are shear strains. Thus, in parallel to the three-dimensional situation, the displacement $W$ at the boundary for the present element is assumed to vary as $r^{3/2}$ and the total rotation to vary as $r^{1/2}$ near the crack front. Thus, along the line (12) for instance (see Figure 2), where 2 is the crack tip, the displacement field is assumed as, (with $S = (c + x)/c$),

$$W_{12} = (1 - S)^{3/2}W_1 + [1 - (1 - S)^{3/2}]W_2$$
$$\phi_{y12} = -\sqrt{(1 - S)}\phi_{y1} - (1 - \sqrt{(1 - S)})\phi_{y2}$$
$$\phi_{x12} = \sqrt{(1 - S)}\phi_{x1} + (1 - \sqrt{(1 - S)})\phi_{x2}$$
Similar assumptions are made for side (23), whereas at sides (14) and (54), which adjoin the surrounding regular elements, the displacement field is assumed such that \( W \) is a cubic and the normal slope is a linear function, in terms of the generalized nodal displacements at points 1, 4 and 5, respectively, as indicated in Figure 2. However, this implies the following approximation: the nodal displacements at node 1, for instance, are interpreted as \( W_1, \phi_{11}, \phi_{12} \), for purposes of interpolation along side (12), whereas they are interpreted as \( W_1, W_{111}, W_{112} \), for purposes of interpolation along side (14). This approximation was made with the aim of keeping the total number of nodal generalized displacements as small as possible, and yet still obtain meaningful engineering results. Finally, for all the regular elements which are Kirchhoff type elements, the boundary displacement functions are such that \( W \) is cubic and the normal slope is linear at any boundary segment. The boundary displacement functions for element 1 is given in Appendix I.

RESULTS AND DISCUSSIONS

It is natural that the computed solutions for \( K_1 \), using the hybrid stress model, should vary with the size of a singular element in the finite element grid, since the singular nature of moments and shear is predominated in a small but finite region near the crack tip. That an optimum size exists for singular elements based on the hybrid stress model and the reasons for it are discussed in Reference 18. It is also seen that the three-dimensional effects (and thus the transverse shear stresses) become more important as the ratio \( h/a \) (\( a \) is semi-crack length) increases. Since transverse shear effects (and the transverse shear singularities) are properly accounted for only in the singular elements in the present formulation, it can be heuristically seen that the optimum size–ratio \( c/a \) (\( c \) is characteristic size of singular element

![Figure 3. Bending stress intensity factor vs. plate thickness and singular element size](image-url)
as in Figure 2) for the singular element must increase as $h/a$ increases. This has been confirmed by the numerical results for computed $K_I$ values for various $h/a$ ratios, as shown in Figure 3. The size ratio $c/a$ which gives the peak value for $K_I$ for each $h/a$ ratio is taken to be the optimum for the particular $h/a$ ratio. In subsequent computations for different $h/a$ and $2a/L$ (crack length to a plate width ratio), the singular element size was taken to correspond to the above optimum size. Thus, for instance, Figure 4 shows the variation of $K_I$ for different $h/a$ ratios for the case when $(2a/L) = 0.1$. In this figure $K_I$ is the analytical solution for the

![Graph showing bending stress intensity factor vs. plate thickness (2a/L = 0.1)](image)

Figure 4. Bending stress intensity factor vs. plate thickness $(2a/L = 0.1)$

infinite domain obtained by SiH. $K_F$ is the computed solution for the finite domain when the stress free conditions on the crack face are satisfied a priori by using the second set of regular stress fields; and $K_{F1}$, is the computed solution for the finite domain when the stress free
conditions on the crack face are satisfied \textit{a posteriori} through the variational principle by using the first set of regular stress field. From Figure 4 it can be concluded that the explicit satisfaction of traction boundary condition on the crack face is mandatory to obtain meaningful results for $K_I$, based on the hybrid stress model. Finally, by using the optimum singular element size $c/a$, stress intensity factors $K_F$ are computed for various $2a/L$ ratios and $h/a$ ratios. These results are summarized in Figure 5, which shows the factor $K_F/K_I$ ($K_F$ and $K_I$ as defined above) for cracks in finite plates as the ratio of crack length to plate width varies. It is seen that the finite size correction factor $K_F/K_I$ varies significantly, as expected, with the $2a/L$ ratio; however, this variation with the $h/a$ ratio is not as significant for any particular $2a/L$ ratio. Further detailed numerical results are available in Reference 15.

CONCLUSION

A hybrid stress finite element analysis procedure for the problem of bending of a plate with a through-the-thickness crack under arbitrary loading and boundary conditions has been formulated. A plate with a mode I loading and boundary conditions has been analysed. The stress intensity solution agrees well with that of an analytical method. The effect of an explicit
enforcement of traction-free conditions along the crack surface on the accuracy of the solution is investigated. It is found that the stress intensity solution obtained without enforcing those conditions explicitly is not unreliable.

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APPENDIX I

Table I. Regular self-equilibrated stress function

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -xy & 0 & 0 & 0 & 0 & 0 & 0 & -xy & 0 & 0 & 1 & 0 & 2y & \\
0 & 1 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & -x & 0 & 0 & 1 & 0 & 2x & \\
0 & 0 & 0 & -y & 0 & 0 & 0 & 0 & 1 & 0 & x & y & 0 & 1 & 0 & 2x & 0
\end{bmatrix} \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_1 y
\end{bmatrix}
\]

Table II. Reduced regular stress functions and boundary tractions for elements along crack surface

\[
M = \begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
1 & x & y & x^2 & xy & y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\
0 & 0 & 0 & -xy & 0 & 0 & -xy & y & y^2
\end{bmatrix} \beta
\]

\[
\beta = [\beta_1 \beta_2 \ldots \beta_9]
\]

\[
T = R\beta
\]

\[
T^T = [-Q_{y2} -M_{y2} -M_{xy2} -Q_{x33} -M_{x23} -Q_{y34} -M_{y34} -M_{xy34} -Q_{x41} -M_{x41} -M_{xy41}]
\]

\[
R = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & a & y & 0 & -a & 1 & 2y \\
-1 & -a & -y & -a^2 & -ay & -y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & ay & 0 & 0 & ay & y & y^2 \\
0 & 0 & 0 & -b & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -b^2 & 0 & 0 \\
0 & 0 & 0 & -bx & 0 & 0 & -bx & b & b^2 \\
0 & -1 & 0 & 0 & -y & 0 & -1 & -2y & 0 \\
-1 & 0 & -y & 0 & 0 & -y^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -y & -y^2
\end{bmatrix}
\]
Table III. Boundary displacement matrix of ‘A’ type singular element with dimensions \((a \times b)\)

\[
\mathbf{u} = \mathbf{Lq}
\]

\[
\mathbf{u}^T = \begin{bmatrix} w_{12} & -\theta_{x12} & \theta_{x12} & w_{23} & \theta_{x23} & w_{34} & \theta_{y34} & w_{y34} & -w_{x44} & w_{4i} & -w_{y44} \end{bmatrix}
\]

\[
\mathbf{q}^T = \begin{bmatrix} w_{i1} & \theta_{x1} & \theta_{y1} & w_{j2} & \theta_{x2} & \theta_{y2} & w_{j3} & w_{y3} & w_{j4} & w_{x4} \end{bmatrix}
\]

\[
\mathbf{L} \text{ is as follows:}
\]

\[
\begin{bmatrix}
(1 - \xi^{1/2}) & 0 & 0 & 1 - (1 - \xi^{1/2}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -(1 - \xi^{1/2}) & 0 & 0 & -1 + (1 - \xi^{1/2}) & 0 & 0 & 0 & 0 & 0 \\
0 & (1 - \xi^{1/2}) & 0 & 0 & 1 - (1 - \xi^{3/2}) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 - \xi^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 - \xi^{3/2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\xi^3 + 3\xi^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -n(\xi^3 - \xi^2) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -a(\xi^3 - \xi) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -6(\xi^3 - \xi) & -3\xi^2 + 2\xi & 0 & 0 & 0 & 0 \\
2\xi^3 - 3\xi^2 + 1 & 0 & -b(\xi^3 - 2\xi^2 + \xi) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(1 - \xi) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi \\
6(\xi^2 - \xi) & 0 & -2\xi^2 + 4\xi - 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\xi
\end{bmatrix}
\]

where \(\xi = \frac{x}{a}\) and \(\zeta = \frac{y}{b}\)
REFERENCES