ON A STUDY OF THE $(\Delta T)_c$ AND C$^*$ INTEGRALS
FOR FRACTURE ANALYSIS UNDER NON-STEADY CREEP†

R. R. STONESIFER and S. N. ATLURI
Center for the Advancement of Computational Mechanics, School of Civil Engineering, Georgia Institute of
Technology, Atlanta, GA 30332, U.S.A.

Abstract—The validity of the parameter $(\Delta T)_c$, (which was previously introduced by one of the authors), its
meaning, and its calculation, in the analysis of cracked bodies under non-steady creep conditions, are
studied. Comparisons are made with the widely used parameter $C^*$, which is valid for steady-state creep.
An efficient finite element method for general, small-strain, elastic-viscoplastic analyses is described and the
results of an example calculation for a standard compact specimen are presented and discussed.

INTRODUCTION

Methods for improved accuracy of life predictions of components subjected to low-cycle creep-fatigue
are currently in great demand. Present procedures too often require a component to be resigned from
service at a time when a significant portion of its useful life still remains. A contributing factor to this
state of affairs is the lack of a parameter which correlates well with a wide range of crack growth data.

The most common parameters which have been studied are the elastic stress intensity factor $K$, the net
section stress $\sigma_{net}$, and in the case of steady-state creep, a path-independent integral $C^*$ (see[1] for a
review). Whereas, $K$, and $C^*$ are parameters applicable to geometries with macroscopic cracks, $\sigma_{net}$
seems mostly applicable to geometries where large portions of the body are subjected to stress levels of
a magnitude to induce void growth and the formation of micro-cracks.

The parameters $K$, and $C^*$ are clearly related to opposite extremes of the creep cracking phenomena.
That is, very localized creep behavior and/or fast propagation rates would imply that the crack-tip region
is controlled by an elastic field which is characterized by $K$, whereas widespread creep and very slow
propagation speeds would imply that the crack-tip field is essentially at steady-state and therefore
characterized by $C^*$. If the crack behavior lies between these extremes, then it is doubtful if either
parameter is applicable.

In the present study we consider a new parameter $(\Delta T)_c$ which has recently been developed by
Atluri[2]. This parameter which is a path-independent integral, vector quantity has the attractive feature
that it characterizes the crack-tip field for both the extreme cases discussed above, as well as all
behavior between. In addition to its role as a crack-tip field characterizing parameter, $(\Delta T)_c$, also has an
energy interpretation[2].

In as much as $(\Delta T)_c$, is a new parameter, the present study is primarily intended to further explore the
parameter, its meaning, its calculation, and finally to give the results of an example calculation for a
standard compact specimen. In the process of doing this, an efficient finite element method for general
elastic-viscoplastic analysis is also described.

PRELIMINARIES

We shall consider problems of creep wherein

$$\dot{\epsilon}_{ij} = \epsilon_{ii}' + \epsilon_{ij}' = L_{ijkl} \sigma_{kl}^{ij} + (3/2) \gamma (\sigma_{eq})^{n-1} \sigma_{ij}.'$$

(1)

If we denote $u_i$ as the rate of displacement from the current configuration, then $\epsilon_{ii}$ is the symmetric part
of the rate of displacement gradient $\epsilon_{ii} = (\nabla u_i)^T = (\partial u_i/\partial y_j)$ and $\epsilon_{ij}' = \epsilon_{ij} + \omega_{ij}$. The gradient operator $\nabla$, is with
respect to the current coordinates $y$, (as opposed to the coordinates of the underformed body $X$). $L_{ijkl}$
the tensor of instantaneous elastic moduli. We let $\sigma_{ii}$ denote the corotational rate (or "Zaremba–
Jaumann rate") of the Kirchhoff stress $\sigma_{ij}$ where $\sigma_{ij}$ is related to the Cauchy stress $\tau_{ij}$ by $\sigma_{ij} =
J_{ijkl}(\partial y_i^m/\partial X_j)$ (Zaremba–Jaumann rate) of the Kirchhoff stress $\sigma_{eq}$ is related to the deviatoric Kirchhoff stress

†This paper is based on a part of a Ph.D. thesis to be submitted by the first author.
‡Since $C^*$ does not have a similar energy interpretation (even at steady-state), it seems that experimental efforts to measure $C^*$ are
most likely resulting in the measurement of $(\Delta T)_c$, instead.
\[ \sigma_{ij}'(= \sigma_{ij} - 1/3 \sigma_{kk} \delta_{ij}) \] by \( \sigma_{eq} = (3/2)(\sigma_{ij}' \sigma_{ij}')^{1/2} \). The parameters \( \gamma \) and \( n \) are those of the familiar Norton's law

\[ \dot{\varepsilon}_{eq} = \gamma (\sigma_{eq})^n \]

where

\[ \dot{\varepsilon}_{eq} = [(2/3)\varepsilon_{ij} \varepsilon_{ij}]^{1/2} \]

In the following, we use the notation: \((,\) denotes a second order tensor; \((\_\_\) implies a vector; \(g = B \cdot \varepsilon \) implies \( a_i = B_{ij}\varepsilon_j \); \( A = B \cdot C \) implies \( A_{ij} = B_{ij}C_{ij} \); \( A : B = A_{ij}B_{ij} \).

### A Conservation Law and \((\Delta T)\)

A conservation integral relation given by Atluri][2], for a closed volume \( V \), at the current time, \( t \), which is free from singularities and any other defects (which would preclude the application of the divergence theorem), for the special case of material behavior characterized by \((\_\_)\), is:

\[ 0 = \int_V \{ \nabla \Delta W - \{ \nabla \tau \cdot [ (\tau + \Delta \tau) \cdot \Delta \varepsilon] 
\]

\[ - \rho(t - g) \cdot \Delta \varepsilon \, dV + \int_{S} [ \bar{g}_t \cdot (\tau + \Delta \tau) - \bar{f}] \cdot \Delta \varepsilon \, dS \]

\[ + \int_{S} \bar{n}_t \cdot (\tau + \Delta \tau) \cdot (\Delta \varepsilon - \Delta \bar{\varepsilon}) \, dS \quad (2) \]

In \((2)\), \( \Delta \tau \) is the incremental first-Piola-Kirchhoff (nonsymmetric) stress \((\Delta \tau = [\Delta \sigma - \Delta \varepsilon \cdot g] / \rho \), where \( \Delta \sigma \) is the material increment of Kirchhoff stress. The current mass density is denoted \( \rho \), and \( \bar{f} \) and \( g \) are the body force and acceleration vectors at time \( t + \Delta t \), respectively. The current outward normal to \( V \) is \( \bar{n}_t \).

The quantity \( \Delta W \), discussed in detail in [2], is the incremental stress-working density in time \( \Delta t \), and is given by:

\[ \Delta W = \tau : \Delta \varepsilon + \frac{1}{2} \Delta \tau : \Delta \varepsilon = \tau : \Delta \varepsilon + \Delta U \quad (3) \]

where

\[ \Delta U = \frac{1}{2} \Delta \tau : \Delta \varepsilon, \text{ such that } \Delta \tau : \Delta \varepsilon = \frac{\partial \Delta U}{\partial \Delta \varepsilon} \quad (4) \]

The validity of \((2)\) is readily verified through the two identities [2]:

\[ \nabla \Delta W = \nabla \tau : \Delta \varepsilon + \nabla \Delta \sigma = \nabla \tau : \Delta \varepsilon + [\tau : (\Delta \varepsilon) ; mg_{nm}] \]

\[ + [\Delta \tau : (\Delta \varepsilon) ; mg_{nm}] \quad (5) \]

and

\[ \nabla \cdot [(\tau + \Delta \tau) \cdot \Delta \varepsilon] = [\nabla \cdot (\tau + \Delta \tau)] \cdot \Delta \varepsilon + [(\tau + \Delta \tau) : (\Delta \varepsilon) ; mg_{nm}] \]

\[ \text{the satisfaction of linear momentum balance in } V: \]

\[ \nabla \cdot [(\tau + \Delta \tau) + \rho(t - g)] = 0 \quad (7) \]

and the satisfaction of the boundary conditions:

\[ \bar{n}_t \cdot [\tau + \Delta \tau] = \bar{f} \text{ on } S_e \quad (8) \]

\[ \Delta \varepsilon = \Delta \bar{\varepsilon} \text{ on } S_e \quad (9) \]

\[ \text{The validity of (2) does not require } S_1 + S_e = \partial V, \text{ where } \partial V \text{ denotes the surface bounding } V. \]
Note that identity (5) assumes that $\tau$ (the initial stress for the increment) is an explicit function of its position in $V$. The existence of $\Delta U$ is discussed in the work of Atluri[3, 4].

The conservation integral (2) is used [2] to obtain a "path-independent integral" which is applicable to the analysis of cracks by considering a volume $V_{1} - V_{s}$ such as illustrated in Fig. 1. (Note that a two-dimensional case is illustrated for simplicity.) The use of the divergence theorem for the region depicted in Fig. 1 results in (2) being rewritten

$$
\int_{\Gamma_{234}} \{\eta_{i} \Delta W - \eta_{j} \cdot [(\tau + \Delta \tau) \cdot \Delta \varepsilon] \} dS + \int_{V_{1} - V_{s}} [( - \nabla_{s} \tau_{s}) : \Delta \varepsilon - \rho_{i} (f - q) \cdot \Delta \varepsilon] dV
$$

$$
+ \int_{\Gamma_{12}} \eta_{i} \Delta W dS + \int_{\Gamma_{45}} \eta_{i} \Delta W dS - \int_{S_{j}} \tilde{t} \cdot \Delta \varepsilon dS - \int_{S_{j}} \eta_{i} \cdot [(\tau + \Delta \tau) \cdot \Delta \varepsilon] dS
$$

$$
= \int_{\Gamma_{45}} \{\eta_{i} \Delta W - \eta_{i} \cdot [(\tau + \Delta \tau) \cdot \Delta \varepsilon] \} dS = (\Delta T),'.
$$

In writing (10) it has been assumed that $S_{s} + S_{t} = \Gamma_{12} + \Gamma_{45}$ (so as to have a well posed boundary value problem) but that $S_{s}$ and $S_{t}$ do not include any of the contour $\Gamma_{234}$. Equation (10) is used to define the quantity $(\Delta T)$, where it can be assumed without loss of generality that $\Gamma_{s} = \Gamma_{65}$ is a circle of radius $\epsilon$ centered at the crack-tip. Clearly, $(\Delta T)$, will depend on $\epsilon$ but not on the selection of $\Gamma_{234}$.

Following the reasoning of Atluri[2], we define $(\Delta T)$, as the limit of $(\Delta T)$, as $\epsilon$ goes to zero.†

$$(\Delta T)_{e} = \lim_{\epsilon \to 0} \int_{\Gamma_{12}} \{\eta_{i} \Delta W - \eta_{i} \cdot [(\tau + \Delta \tau) \cdot \Delta \varepsilon] \} dS \equiv \int_{\Gamma_{234}} \{\eta_{i} \Delta W - \eta_{i} \cdot [(\tau + \Delta \tau) \cdot \Delta \varepsilon] \} dS
$$

$$
+ \lim_{\epsilon \to 0} \left\{ \int_{V_{1} - V_{s}} (- \nabla_{s} \tau_{s}) : \Delta \varepsilon - \rho_{i} (f - q) \cdot \Delta \varepsilon] dS
$$

$$
+ \int_{\Gamma_{12}} \eta_{i} \Delta W dS + \int_{\Gamma_{45}} \eta_{i} \Delta W dS - \int_{S_{j}} \tilde{t} \cdot \Delta \varepsilon dS - \int_{S_{j}} \eta_{i} \cdot [(\tau + \Delta \tau) \cdot \Delta \varepsilon] dS \right\}.
$$

It is clear that $(\Delta T)$, as defined by (11):

(i) Characterizes the crack-tip field.

(ii) Can be evaluated via the arbitrary contour integral $\Gamma_{234}$ and the volume integral over the region bounded by $\Gamma_{234}$.

For symmetrical deformation about the $x_{1}$ axis and cracks oriented along the $x_{1}$ axis with traction free

†The existence of the limit is discussed in Appendix A.
crack surfaces, no body forces and negligible inertial effects, the first component of \( \Delta T \) is:

\[
(\Delta T)_c = \int_{\Gamma_{ext}} (n_i \Delta W - n_i (\tau_{ij} + \Delta \tau_{ij}) \Delta e_{ij}) \, dS
\]

Note that the limit of the volume integral has been written in its explicit form as a result of the existence arguments of Appendix A.

It has been shown by Atluri [2], that the vector \( \Delta T \) has the following physical meaning. Let two cracked bodies be identical except for the second body having an additional, arbitrarily directed, infinitesimal increment in crack length characterized by the vector \( dc \). It is assumed that both bodies experience identical load histories. Define total potential energy increments corresponding to the time increment \( \Delta t \) as

\[
AE_1 = \Delta W - \Delta K \tag{13}
\]

\[
AE_2 = \Delta W - \Delta K \tag{14}
\]

for the first and second bodies, respectively. In (13), \( -\Delta \psi \) is the incremental work of external forces, \( \Delta W \) is the incremental stress-work and \( \Delta K \) is the increment in the kinetic energy. (It should be noted that \( \Delta W \) includes the inelastically dissipated energy.) Then

\[
(\Delta T)_c \, dc = AE_2 - AE_1 \tag{15}
\]

If one is only interested in self-similar crack extension in the \( x_i \)-direction, then \( dc = dq = 0 \) and

\[
(\Delta T)_c = \frac{AE_2 - AE_1}{dc} \tag{16}
\]

Therefore, \( \Delta T \) can be related to the incremental total potential energy difference between two bodies which are identical except for an incremental crack length difference \( dc \).

It is worth emphasizing, as noted in [2], that the conservation law (2), and the attendant path-independent integral (10) or (11), are valid: (i) even when both elastic as well as creep strains are present, (ii) under steady as well as non-steady creep conditions, and (iii) when finite deformations are accounted for.

We now consider the special case of steady-state, pure creep behavior.

STEADY STATE CREEP AND \( C^* \)

It has been shown that \( \Delta T \) characterizes the crack-tip field for materials which exhibit creep behavior such as in (1). It is known that under certain conditions of applied loading, the constitutive relation (1) can (after long times) result in a steady-state. This steady-state is primarily characterized by the time independence of the stresses (i.e. \( \Delta U = \Delta \tau_{ij} = 0 \)). Specializing (11) and (12) to steady-state conditions, we define the steady-state value of \( \Delta T \).

\[
(\Delta T)_c, ss = \int_{\Gamma_{ext}} (n_i \tau_{ij} \Delta e_{ij} - n_i \tau_{ij} \Delta e_{ij}) \, dS
\]

Because (1) results in a power-law relation at steady-state, which is analogous to the power law deformation-theory plasticity (or essentially nonlinear elasticity), Goldman and Hutchinson [5] have suggested a path-independent \( C^* \) integral,

\[
C^* = \int \left[ n_i \Delta W - n_i \tau_{ij} \Delta e_{ij} \right] \, dS \tag{18}
\]
where

\[ W^* = \int_0^t \tau_{ij} \, d\dot{e}_{ij} \]  

(19)

such that

\[ \tau'_{mn} = \partial W^*/\partial \dot{\varepsilon}_{mn}. \]  

(19b)

The question of how \( C^\dagger \) and \((\Delta T)_{\text{cs}}\) are related, is a natural one. Before obtaining an equation relating \( C^\dagger \) to \((\Delta T)_{\text{cs}}\), however, the conservation integral (2) will be used to derive a generalized vector integral \( C^* \).

In specializing (2) to steady-state we note that now stress is a function of the strain-rate, and that stress-increments are zero. Thus, \( \Delta W = \tau \cdot \Delta e \). Also we may write:

\[ \int_{V_i} \{ \nabla, \Delta W - (\nabla, \tau) : \Delta e \} \, dV = \int_{V_i} \tau : \nabla, \Delta e \, dV = \int_{V_i} [\tau : (\Delta e); \, m g^{mn}] \, dV. \]  

(20)

Thus, at steady-state, we may write (2) as:

\[ 0 = \int_{V_i} \{ \tau : (\nabla, \Delta e) - \nabla, \tau \cdot [\tau \cdot \Delta e] - \rho_s (f - g) \cdot \Delta e \} \, dV \]

\[ + \int_{S_i} [\tau \cdot (\tau - \bar{f})] : \Delta e \, dS + \int_{S_i} [\tau \cdot (\tau - \bar{f}) \cdot (\Delta e - \Delta \hat{e})] \, dS \]  

(21)

or equivalently, in rate form,

\[ 0 = \int_{V_i} \{ \tau : (\nabla, \dot{e}) - \nabla, \tau \cdot (\dot{e} - \bar{f}) - \rho_s (f - g) \cdot \dot{e} \} \, dV \]

\[ + \int_{S_i} [\tau \cdot (\tau - \bar{f})] \cdot \dot{e} \, dS + \int_{S_i} [\tau \cdot (\tau - \bar{f}) \cdot (\dot{e} - \dot{\hat{e}})] \, dS. \]  

(22)

Since, at steady-state, stress is a single-valued function of the strain-rate, we note that (using the symmetry of \( \tau \)):

\[ \tau : \nabla, \dot{e} = \tau : [\{\dot{e}; \, m g^{mn}\}] = \tau : [\frac{1}{2} (\dot{e} + \dot{e}^T); \, m g^{mn}] \]

\[ = \tau : \nabla, \dot{W}^*. \]  

(23)

We can then observe that the mathematical potential \( W^* \), under steady-state conditions, has the form:

\[ W^* = \int_0^{\varepsilon_{mn}} \tau_{ij} \, d\dot{e}_{ij} \]  

(24)

such that

\[ \frac{\partial W^*}{\partial y_k} = \frac{\partial W^*}{\partial \dot{\varepsilon}_{mn}} \frac{\partial \dot{\varepsilon}_{mn}}{\partial y_k} = \tau'_{mn} \dot{\varepsilon}_{mn,k} \]  

(25)

or

\[ \nabla, W^* = \tau : \nabla, \dot{e}. \]  

(26)
In particular, we note that for steady-state creep law of type $\dot{\varepsilon}_{eq} = \gamma (\sigma_{eq})^n$, (24) becomes

$$W^* = \frac{n}{n + 1} \left( \frac{1}{\gamma} \right)^{1/n} (\dot{\varepsilon}_{eq})^{(n+1)/n}.$$  

Applying (22) to $V_t - V_s$ and using the divergence theorem, we define the vector quantity $(C^*)^\dagger$.

$$\int_{\Gamma_{24}} \{ n_i W^* - n_t \cdot (\tau \cdot \hat{e}) \} \, dS - \int_{V_t - V_s} \rho_0 (f - g) \cdot \hat{e} \, dV$$

$$+ \int_{\Gamma_{12}} n_i W^* \, dS + \int_{\Gamma_{45}} n_i W^* \, dS - \int_{S_1} \bar{t} \cdot \hat{e} \, dS - \int_{S_2} n_t \cdot \tau \cdot \hat{e} \, dS$$

$$= \int_{\Gamma_t} [ n_i W^* - n_t \cdot (\tau \cdot \hat{e}) ] \, dS = (C^*)^\dagger. \quad (27)$$

If we define the limit of $(C^*)^\dagger$ as $\epsilon \to 0$ to be $C^*$, we have a quantity which characterizes the crack-tip field and is independent of the selection of $1234$. Restricting our attention to problems involving symmetric deformations about the $x_1$ axis and cracks oriented along the $x_1$ axis, with traction free crack-faces, no body forces and negligible inertia effects, we find that

$$C^\dagger = \lim_{\epsilon \to 0} \int_{\Gamma_t} \left[ n_i W^* - n_t \tau_{ij} \frac{\partial u_i}{\partial y_j} \right] \, dS = \int_{\Gamma_{234}} \left[ n_i W^* - n_t \tau_{ij} \frac{\partial u_i}{\partial y_j} \right] \, dS. \quad (28)$$

Now, we will relate $C^\dagger$ of (28) to the steady-state value of $(\Delta T)_c$ of (17). First, we may rewrite (17) in rate form as

$$(\dot{T}_t)_{css} = \lim_{\epsilon \to 0} \int_{\Gamma_t} (n_t \tau_{ij} \dot{e}_{ij} - n_t \tau_{ij} \dot{e}_{ij}) \, dS = \int_{\Gamma_{234}} [ n_t \dot{r}_{ij} \dot{e}_{ij} - n_t \tau_{ij} \dot{e}_{ij} ] \, dS$$

$$- \int_{V_t} \frac{\partial n_t}{\partial y_j} \dot{e}_{ij} \, dV \quad (29)$$

$$= \int_{\Gamma_{234}} (n_t W - n_t \tau_{ij} \dot{e}_{ij}) \, dS - \int_{V_t} \frac{\partial n_t}{\partial y_j} \dot{e}_{ij} \, dV. \quad (29b)$$

Thus, $\dot{W}$ in (29) is the rate of stress-working density, while $\dot{W}$ of (28) is just a mathematical potential for $\tau_{ij}$. In particular,

$$\dot{W} = \tau_{ij} \dot{e}_{ij} = \gamma \frac{1}{2} \dot{e}_{ij} = \gamma \sigma_{eq} \dot{e}_{eq} = \gamma \sigma_{eq}^{n+1} = \left( \frac{1}{\gamma} \right)^{1/n} (\dot{\varepsilon}_{eq})^{(n+1)/n} \quad (30)$$

as contrasted with $W^*$:

$$W^* = \left( \frac{n}{n + 1} \right) \left( \frac{1}{\gamma} \right)^{1/n} (\dot{\varepsilon}_{eq})^{(n+1)/n}.$$

Comparing the left equalities of (28) and (29), it is seen that $(\dot{T}_t)_{css}$ and $C^\dagger$ are related by:

$$(\dot{T}_t)_{css} = C^\dagger + \lim_{\epsilon \to 0} \int_{\Gamma_t} n_t (\dot{W} - W^*) \, dS \quad (31a)$$

$$= C^\dagger + \frac{\gamma}{n + 1} \lim_{\epsilon \to 0} \int_{\Gamma_t} n_t (\sigma_{eq})^{n+1} \, dS. \quad (31b)$$

Appendix B gives several numerical examples of relation (31) for two rather extreme values of $n$. 
We now give the HRR field in terms of \((\Delta T)_c\). Whereas similar relations have been written in terms of \(C^*\) for steady-state creep [6], the relations in terms of \((\Delta T)_c\) will be valid for non-steady creep as well as steady-state creep. The HRR field as given in [5] but modified for creep by replacing \(e_0\) and \(\dot{u}_i\) by \(\dot{e}_0\) and \(\ddot{u}_i\) respectively, is:

\[
\begin{align*}
[\tau_{ij}, \sigma_{cq}] &= K_c r^{-\frac{n}{n+1}} [\dot{\sigma}_{ij}(\theta), \dot{\sigma}_{cq}(\theta)] \\
\dot{e}_{ii} &= \gamma K_c r^{-\frac{n}{n+1}} \dot{e}_i(\theta) \\
\ddot{u}_i &= \gamma K_c r^{-\frac{n}{n+1}} \ddot{u}_i(\theta)
\end{align*}
\]

where \(\dot{\sigma}_{cq}(\theta)\) has a maximum value of unity and

\[
K_c = (K_m)^n.
\]

Substituting (32a–c) into the left equality of (12), using (32d) and rearranging, gives:

\[
K_c = \left(\frac{\Delta T_c}{\gamma l^* \Delta t}\right)^{\frac{1}{n+1}} = \left(\frac{T_c}{\gamma l^*}\right)^{\frac{1}{n+1}}
\]

where \(I^*\) is analogous to \(I\) defined by eqn (24) of [7] except for the factor \(n/(n + 1)\) multiplying the energy density term. To be explicit,

\[
I^* = I + \frac{1}{n + 1} \int_0^1 [\dot{\sigma}_{cq}(\theta)]^{n+1} \cos \theta \, d\theta.
\]

To summarize, we have presented a path-independent integral which is valid for non-steady creep as well as steady-state creep. The parameter \((\Delta T)_c\) defined by the integral therefore characterizes the crack-tip field for all time. Also, the \(C^*\) parameter has been derived from the general conservation law (2), and has been related to the parameter \((\Delta T)_c\) for self-similar crack growth under mode \(I\) conditions.

**A FINITE ELEMENT MODEL FOR ELASTIC/VisCOPlastic MATERIALS**

We now describe a finite element model, which is applicable to problems of elastic/viscoplasticity as defined by Perzyna [8]. This model accommodates the material’s nonlinear behavior through a step-wise time integration procedure. In the following, it is assumed that strains are infinitesimal and displacements are small. The finite element model is based on the principle of virtual work:

\[
\int_V \tau_{ij} \delta e_{ij} \, dV - \int_{S_{nn}} \bar{t}_i \delta u_i \, dS = 0
\]

where \(\tau_{ij}\) are current stresses, \(\bar{t}_i\) are current prescribed tractions on the surface \(S_{nn}\), and \(\delta u_i[\delta e_{ij} = 1/2(\delta u_{ij} + \delta u_{ji})]\) are arbitrary compatible virtual displacements. Following customary procedures we introduce the element displacement shape functions which relate element displacements \(u_i\) to element nodal displacements \(\{\delta q\}\)

\[
u_i - \{u\} - [N]\{\delta q\}; \delta u_i = \{\delta u\} = [N]\{\delta q\}.
\]

We also use the customary notation wherein strain (and stress) components are placed in one-dimensional arrays

\[
\{\varepsilon\} - \{D\}\{\delta q\}; \{\delta \varepsilon\} - \{B\}\{\delta q\}.
\]

Substituting (36) and (37) into (35) and applying conventional element summation procedures we have

\[
\sum_{\text{elements}} \left\{ \int_V \{\tau\}^T \{B\} \, dV - \int_{S_{nn}} \{\bar{t}\}^T [N] \, dS \right\} x \sum_{\text{elements}} \{\delta q\} = \{F\}^T \{\delta Q\} = 0.
\]
Since \{\delta Q\} are arbitrary virtual nodal displacements, it follows that

\[
\sum_{\text{elements}} \left\{ \int_{V_e} \{\delta r\}^T \{B\} \, dV - \int_{S_e} \{\delta \tilde{r}\}^T \{N\} \, dS \right\} = \{F\}^T = \{0\}^T. \tag{38}
\]

We now write the current stress array in the form

\[
[t]_I = [t]_{I-1} + \{\Delta t\}_I, \tag{39}
\]

where the subscript "I" indicates the current time step (or increment) and I - 1 indicates the previous time step. Application of the incremental elastic constitutive law results in

\[
\{\Delta \epsilon\}_I = \{\Delta \epsilon\}_{I-1} + [E][\Delta \epsilon\}_I - [E][\Delta \epsilon_{\text{ep}}]_I \tag{40}
\]

where \{\Delta \epsilon_{\text{ep}}\}_I are the incremental viscoplastic strains and \([E]\) is the matrix of elastic constants. Substituting (40) into (38), taking the transpose, and placing the known terms on the right hand side we have the final form of the finite element equations:

\[
[K][\Delta Q]_I = \{T\}_I + \{S_{\text{ep}}\}_I - \{R\}_{I-1} \tag{41}
\]

where

\[
[K] = \sum_{\text{elements}} \int_{V_e} \{B\}^T [E][B] \, dV \tag{42}
\]

\[
[T]_I = \sum_{\text{elements}} \int_{S_e} \{N\}^T \{\tilde{r}\}_I \, dS \tag{43}
\]

\[
[S_{\text{ep}}]_I = \sum_{\text{elements}} \int_{V_e} \{B\}^T [E][\Delta \epsilon_{\text{ep}}]_I \, dV \tag{44}
\]

\[
[R]_{I-1} = \sum_{\text{elements}} \int_{V_e} \{B\}^T [\tau]_{I-1} \, dV. \tag{45}
\]

It should be noted that \([K]\) is just the elastic stiffness and therefore only needs to be formed and decomposed once. This results in significant savings in the number of computations per time step as compared to methods using stiffness matrices which must be reformed at each step (i.e. tangent stiffness methods). It should also be noted that the term \([S_{\text{ep}}]_I\) is computed from incremental viscoplastic strains \{\Delta \epsilon_{\text{ep}}\} which are estimated using \{\tau\}_{I-1} in conjunction with the material constitutive law (i.e. (1) for the present case of creep). Only for the special situation when the stresses do not change with time will this estimate be exact. Having obtained the incremental nodal displacements \{\Delta Q\}_I by solving (41), one can easily find the total incremental strains \{\Delta \epsilon\}_I via the incremental analogue of (37). We now describe two procedures for obtaining \{\tau\}_I.

The first and simpler method to obtain \{\tau\}_I is to substitute the estimated \{\Delta \epsilon_{\text{ep}}\} used in solving for \{\Delta Q\}_I into (40). If one does this, then it happens that

\[
\{\tau\}_I = \{T\}_I \tag{46}
\]

and therefore (41) becomes for the next step:

\[
[K][\Delta Q]_{I+1} = \{T\}_{I+1} + \{S_{\text{ep}}\}_{I+1} - \{T\}_I = \{\Delta T\}_{I+1} + \{S_{\text{ep}}\}_{I+1}. \tag{47}
\]

This method was compared to the following method and was found to require smaller time steps to achieve similar results.

\[\text{The eqns } (41) \text{ are solved in the current work by the decomposition } [K] = [L][D][L]^T, \text{ see, e.g. [9].}\]

\[\text{This procedure results in the current model reducing to that of Zienkiewicz and Cormeau [11].}\]
Rather than using the estimated values of \( \{\Delta \epsilon_{vp}\} \) and the constitutive relation (i.e. (1) for the creep problem), the constitutive relation is integrated over the current time step at each Gaussian quadrature point with the condition that the total strain \( \{\epsilon\} \) varies linearly with respect to time from \( \{\epsilon\}_{-1} \) to \( \{\epsilon\}_r \). (While any number of integration schemes could be adopted for this purpose, the present study uses an Eulerian scheme with each time step being divided into five subincrements.) The result of this procedure is better adherence to the postulated constitutive law at the expense of introducing a somewhat unequilibrated stress state. The amount of disequilibrium depends on the accuracy of the original estimate for the incremental viscoplastic strains and thus on the time step size.

At this point one has two alternatives. The first is to use the viscoplastic strain increments obtained through the time integration procedure as an improved estimate and to re-solve (41) for the current time step. This procedure would, after several iterations, result in a stress state which is equilibrated to within some small user specified tolerance. With this type of procedure the time steps could be as large as those used with tangent stiffness methods. Further, it appears reasonable to expect the solution to be at least as accurate as if a tangent stiffness method were used.†

The second alternative is to go immediately to the next time step with the understanding that the terms \( \{R\}_{-1} \) in (41) results in the disequilibrium from the \( I-1 \) step being corrected in the \( I \)th step. This feature is the result of the virtual work statement (35), being written in terms of total stress and tractions rather than incremental quantities. Owing to this corrective nature and to the diminishing returns one obtains from additional iterations, the second alternative is used in the present study.

Non-Steady Creep Calculations for a Compact Specimen

We now will discuss the application of the model to the analysis of a plane strain compact specimen. The dimensions of the ASTM standard specimen as well as the material properties and applied loading were chosen to coincide with those used by Ehlers and Riedel[12]. These are illustrated in Fig. 2.

Several finite element meshes have been used in the analysis. All of these meshes employ two-dimensional, eight-noded, isoparametric elements. The element integrations are accomplished with 2 x 2 Gauss quadrature and therefore only elements with straight sides are employed. The pin-loading-hole is not modeled. In all models the horizontal placement of the point load corresponds with the load line of the ASTM standard geometry (\( x = 25.0 \) mm). The vertical position is \( y = 32.5 \) mm. A sensitivity

![Fig. 2. Compact specimen geometry, loading condition, and material properties which are used in the calculations.](image-url)
study showed that shifting the load to \( y = 40 \) mm had virtually no effect on the pertinent aspects of the solution.

Most of the meshes contain collapsed quadrilateral elements at the crack tip as illustrated in Fig. 3. In several calculations, the midside nodes of these crack-tip elements were shifted to their quarter-points so as to produce a singular \( (r^{-1/2}) \) strain field at the crack tip. Table 1 identifies the meshes for which calculations have been made and also gives the load point displacement and \( J \) for the elastic solution. These \( J \) values are compared to those based on the expression given by Srawley [13] and are seen to be in good agreement.

These elastic solutions are assumed to exist at time \( t = 0 \) so that the creep analyses then proceed from this initial elastic state. The creep calculations use a variable time step size which is automatically regulated by the finite element program based on two criteria. The first criterion is the maximum percent difference between the incremental equivalent estimated creep strain and the incremental equivalent integrated creep strain for all the Gauss points in the mesh:

\[
C_1 = \text{Max} \left| \frac{\Delta \varepsilon_{\text{EST}} - \Delta \varepsilon_{\text{INT}}}{\Delta \varepsilon_{\text{INT}}} \right|
\]

(48)

![Fig. 3. Finite element meshes used for modeling the compact specimen.](image-url)
On a study of the \(\Delta T\), and \(C^*\) integrals

Table 1. Summary of computational aspects and comparison with results from the literature

<table>
<thead>
<tr>
<th>Mesh Description</th>
<th>Elastic Solution</th>
<th>Creep Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Load Point</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Displacement</td>
<td>Time*</td>
</tr>
<tr>
<td></td>
<td>(mm)</td>
<td>(sec)</td>
</tr>
<tr>
<td>55 elements</td>
<td>0.224</td>
<td>22.8 (-5.8)</td>
</tr>
<tr>
<td>(non-singular)</td>
<td>0.234</td>
<td>24.2 (0.0)</td>
</tr>
<tr>
<td>120 elements</td>
<td>0.231</td>
<td>24.0(-0.8)</td>
</tr>
<tr>
<td>(non-singular)</td>
<td>0.232</td>
<td>24.0(-0.8)</td>
</tr>
<tr>
<td>57 elements</td>
<td>0.233</td>
<td>24.3 (0.4)</td>
</tr>
<tr>
<td>(1/4-point singular)</td>
<td>0.233</td>
<td>24.1(-0.4)</td>
</tr>
</tbody>
</table>

\(^{a}\) Control Data CYBER 74
\(^{b}\) quasi steady-state criteria and calculatin is stopped at t=400 hours

The second criterion is the maximum ratio of incremental equivalent integrated creep strain to the equivalent elastic strain:

\[ C = \text{Max} \left[ \frac{\Delta \varepsilon_{\text{eq}} (E)}{\varepsilon} \right]. \]  

The user specified, maximum permissible values for \(C_1\) and \(C_2\) are \(\tilde{C}_1\) and \(\tilde{C}_2\), respectively. The size of the next step is then obtained from

\[ \Delta t_{n+1} = \Delta t_n \times \min \left[ \frac{\tilde{C}_1}{\tilde{C}_1}, \frac{C_2}{\tilde{C}_2} \right]. \]  

In the present study, the values of \(\tilde{C}_1\) and \(\tilde{C}_2\) are 0.2 and 1.0, respectively. With these values, it has been found that the initial time steps are controlled by \(\tilde{C}_1\) while later time steps are controlled by \(\tilde{C}_2\). The values of \(C_1\) and \(C_2\) are strongly affected by the mesh refinement since a finer mesh results in Gauss points being closer to the crack tip and therefore having larger stresses and strain rates. The initial time increment cannot be determined from \(\tilde{C}_1\) and must be specified by the user so as to satisfy the two step size criteria. Table 1 gives approximate initial time step sizes which satisfy \(C_1 \leq 0.2\) and quasi steady-state time step sizes which satisfy \(C_2 \leq 1.0\). The initial time step size used by Ehlers and Riedel [12] with their tangent stiffness method was \(10^{-3}\) hr. For times approaching steady-state, they note that this step size was increased by a factor of 100.

To determine the sensitivity of the solution to the selection of \(\tilde{C}_1\) and \(\tilde{C}_2\), a calculation was done using the 57 element model (quarter-point singularity with \(\tilde{C}_1\) and \(\tilde{C}_2\) being halved (i.e. \(\tilde{C}_1 = 0.1\) and \(\tilde{C}_2 = 0.5\)). It was found that the load point displacement differed by less than 0.5\% for all time and that the steady-state solutions were essentially identical in terms of contour integral evaluations. It therefore appears that \(\tilde{C}_1\) and \(\tilde{C}_2\) are small enough to ensure that the solutions to be discussed do not depend on these step size criteria.

**NUMERICAL EVALUATION OF CONTOUR INTEGRALS \((T_1)\) AND \(C^*\)**

We now describe the evaluation of the previously defined contour integrals. The \(C_i\) integral is evaluated using (18 or 28) with \(\tilde{W} = (n/n + 1) \gamma_{\text{eq}}^{*+1}. \) Despite \(\tilde{C}_i\) being well defined and path-independent only for steady-state conditions, a quantity we will designate \((C_i)^*\) was evaluated during all stages of creep. The \(\epsilon\) superscript designates the particular contour which is used, with \(\epsilon\) being the nondimensional distance from the crack tip to the point where the contour crosses the crack plane. Therefore, \(\epsilon\) is zero at the crack tip and has a maximum value of unity when the contour is at the boundary of the specimen. The stress used in the evaluation are the stresses at the end of the time step. The
displacement gradient rates are approximated by

$$\frac{\partial u_i}{\partial y_1} \approx \frac{1}{\Delta t} \frac{\partial \Delta u_i}{\partial y_1}.$$

The contours, which are indicated in Fig. 3 by dashed lines, pass through the elements as opposed to along their boundaries so as to benefit from the presumably more accurate solution within the element. Each element contour is divided into two segments with the integration on each segment being accomplished by two point Gaussian quadrature. Stresses within the element are interpolated to the required quadrature points by using the $2 \times 2$ element Gauss point values and bilinear interpolation.

The evaluation of the $(\Delta T)$, related integrals will now be described. In (10), we defined the vector quantity $(\Delta T)_e^*$ where $e$ is the same non-dimensional parameter as in $(C)_e^*$. The parameter $(\Delta T)_e^*$ describes the crack tip field during non-steady creep and therefore is the parameter of primary interest. Based on previous discussions we have

$$(\bar{T}_i)_e = \lim_{\epsilon \to 0} (\bar{T}_i)_e^* = \lim_{\epsilon \to 0} \frac{(\Delta T)_e^*}{\Delta t}$$

where for infinitesimal strains and small deformations

$$(\Delta T)_e^* = \int_{G} \left[ n_i \Delta W - n_i (\tau_{ij} + \Delta \tau_{ij}) \frac{\partial \Delta u_i}{\partial x_1} \right] \mathrm{d}S$$

$$= \int_{G} \left[ n_i \Delta W - n_i (\tau_{ij} + \Delta \tau_{ij}) \frac{\partial \Delta u_i}{\partial x_1} \right] \mathrm{d}S - \int_{V_t - V} \frac{\partial \tau_{ij}}{\partial x_1} \Delta \epsilon_{ij} \mathrm{d}V$$

and

$$\Delta W = \left( \tau_{ij} + \frac{1}{2} \Delta \tau_{ij} \right) \Delta \epsilon_{ij}$$

In the evaluation of (52)–(54), it is to be understood that $\tau_{ij}$ is the stress at the beginning of the current step. The contour integrals are evaluated in a similar manner to the $(C)_e^*$ contour integrals already described. The stress derivative appearing in the area integral is evaluated based on the $2 \times 2$ element Gauss point values and the assumption that the stressed are distributed bilinearly with respect to element local coordinates. Elements which are entirely within $V_t - V$ are integrated with the usual $2 \times 2$ Gauss quadrature. Elements which are only partially within $V_t - V$ have each applicable quadrant integrated by one point Gauss quadrature.

**DISCUSSION OF RESULTS**

The path-dependence of $(C)_e^*$ during non-steady creep is illustrated in Fig. 4 using results from the 300 element mesh. Values of $(C)_e^*$ are plotted as a function of time for nine values of $e$ ranging from 0.03 to 0.92. It is seen that $(C)_e^*$ is largest for contours close to the crack tip (small $e$) and that as steady-state is approached, the values from all contours converge to $(C)_t$. The solution has essentially reached steady-state at 300 hr. After 300 hr, the values of $(C)_e^*$ for all nine contours are within 1.5% of their average value. The value of $(C)_t$, as well as values from calculations with the other meshes, is given in Table 1.

Now we consider the evaluation of $(\bar{T}_i)_e$ as given by (51). Since we are interested in the limit of $(\bar{T}_i)_e^*$ as $e$ goes to zero, we have plotted $(\bar{T}_i)_e^*$ as a function of $e$ for several times (see Fig. 5). The open points are the values of $(\bar{T}_i)_e^*$, as computed by (52), for nine contours in the 300 element model. The value of the crack-tip parameter $(\bar{T}_i)_e$ is given by the intersection of each respective curve with the $e = 0$ axis. Due to the large gradient in $(\bar{T}_i)_e^*$ for small $e$ (except near steady-state) it is seen that the accuracy of any extrapolation based solely on the evaluation of (52) (i.e. open points) would be of questionable accuracy.
On a study of the $(\Delta T)_c$ and $C^*$ integrals

Fig. 4. $(\bar{T}_c)$ and $(C^*)^*$ as a function of time for several paths (results from 300 element mesh).

At this point the advantage of the alternative equation (53) for evaluating $(T_c)_c$ can be more fully appreciated. By taking the limit of (53) as $\epsilon$ goes to zero, we obtain an explicit formula for $(\Delta T)_c$, which does not involve $\epsilon$:

$$
(\Delta T)_c = \int_{G_{234}} n_1 \Delta W - n_1 (\tau_{ii} + \Delta \tau_{ii}) \frac{\partial \Delta u_i}{\partial x_i} \, dS - \int_{V} \Delta \tau_{ii} \Delta \epsilon_{ii} \, dV.
$$

The solid points at $\epsilon = 0$ in Fig. 5 have been obtained using (55). It is seen that these values of $(\bar{T}_c)_c$ appear to be reasonable extrapolations of the curves of $(T_c)_c^*$ computed through (52) thus giving some degree of confidence in their accuracy.

Based on arguments put forth in earlier portions of this paper, the value of $(\bar{T}_c)_c$ obtained through (55) should be independent of the path which is used in its computation. This path-independence is illustrated in Fig. 6a. We have plotted $(\bar{T}_c)_c$ as a function of the nondimensional distance of $\Gamma_{234}$ from the crack-tip, $\xi$, for several times. Generally, the path-independence is seen to be quite good. The largest deviation from path-independence in this figure is for the intermediate time of 10.8 hr with the difference between the extreme contour values being less than 3%. To further emphasize this path-independence, we have plotted $(\bar{T}_c)_c$ as a function of time in Fig. 4. As a result of its path-independence, $(\bar{T}_c)_c$ is represented by a single curve. Interestingly, this curve is a straight line for times before approx. 10 hr.

Riedel and Rice [6] have arrived at the following approximation for $K_c$ [which they call $A(t)$] based on the assumed approximate path-independence of $J_t$ during the initial portion of non-steady creep:

$$
K_c \approx \left[ \frac{K_{0}^2 (1 - \nu^2)}{n + 1} \right]^{1/2}.
$$

Comparing (56) with (33) one concludes that $(\bar{T}_c)_c$ should behave like $1/t$ for times when (56) is valid. In a log-log plot of $(\bar{T}_c)_c$ versus time this would result in a straight line with a slope of $-1$. The straight line shown in Fig. 4 is inclined from the horizontal by $40^\circ$ and therefore has a slope of $-0.84$. The current work has resulted in some evidence that $J_t$ is approximately path-independent during initial non-steady creep but that its value tends to increase with time. The tendency for $J_t$ to increase with time could explain the rather significant departure of the current results from the behavior of (56).
We next consider the results of computations using the 57 and 102 element meshes with quarter-point singularities. The purpose of considering these coarse meshes is to determine if the expense and effort in using the 300 element model is necessary for obtaining accurate results. Table 1 summarizes the results of these coarser meshes for the limiting cases of purely elastic behavior and steady-state creep behavior. For the elastic problem it is seen that the results from the coarser meshes agree with the 300 element mesh results to within 1 percent. At steady-state the 102 element model still agrees with the 300 element mesh (in terms of $t_\infty$) to within 1 percent while the 57 element model now differs by approximately 8%.

The contours used for the 57 and 102 element mesh are indicated in Fig. 3. It is seen that the 57 element mesh has four contours while the 102 element mesh has eight. The path independence of $(T_i)_c$, as computed from (55), is illustrated for these two meshes in Fig. 6(b), 6(c). It is seen that the degree of path-independence in both is similar to that observed for the 300 element mesh. In as much as we have evidence that the 57 element mesh is less accurate than the other meshes at steady-state, it appears that the high quality of the path-independence cannot be interpreted as meaning the solution is accurate. Put more precisely, it seems that while poor path independence of $(T_i)_c$ would imply the solution is inaccurate, the converse is not generally true.

To determine the adequacy of the 57 and 102 element meshes for the non-steady creep problem we now compare their $(T_i)_c$ histories with that obtained with the 300 element mesh (see Fig. 7). The curve appearing in this figure has been placed through computed points from the 300 element mesh. The results of the 102 element mesh agree almost perfectly with this curve for times between 0.2 and 16 hr. Prior to this period and after this period the results fall below the curve by as much as 20%. While little can be said about the absolute accuracy of the calculations for the early portions of non-steady creep, we know (based on appendix B) that $(T_i)_c$ should agree numerically with $C_1$ at steady-state to within a few
On a study of the $(\Delta T)_e$ and $C^*$ integrals

![Graph A](image1)

![Graph B](image2)

![Graph C](image3)

Fig. 6. Path-independence of $(\Delta T)_e$ for several times during non-steady creep ($\xi$ is the non-dimensional size of $T_{2an}$).

percent. Therefore it can be said that the 102 element results are significantly in error at steady-state. The 57 element results do not compare favorably with the curve of Fig. 7 for any significant portion of the solution. For most times the values of $(\dot{T}_1)_e$ fall below the curve with the percent difference ranging from 50% at $t = 0.02$ hr to 15% at steady state.

Based on the discrepancy of $C^*$ indicated in Table 1 and in the generally bad comparison of $(\Delta T)_e$ in Fig. 7, it appears that the 57 element mesh with quarter-point singularity is not sufficiently refined for accurate creep calculations. The conclusion is perhaps a bit unexpected considering the degree of accuracy which this mesh displayed for the elastic problem (see Table 1). The reason for this drastic change of accuracy in going from elastic to creep behavior may be the result of the crack tip strain singularity being appropriate for the elastic problem (i.e. $r^{-\frac{1}{2}}$) but inappropriate for the $r^{-\frac{2}{n+1}}$ type singularity which is expected to exist during creep.

In considering the suitability of the 102 element mesh with quarter-point singularity for modeling the present creep problem it seems that the apparent discrepancy for times less than 0.2 hr should be less of a concern than the discrepancy as steady-state is approached. This is due in part to the general inaccuracy of the Norton type creep law during the primary stage of creep and in part to the experimental evidence that creep crack growth occurs at rates which would make the later portion of this curve more important. Recalling that this model gave a steady-state value of $C^*$ which agreed quite
well with the 300 element mesh results (see Table 1) it is perhaps surprising that such a significant
difference in \((\dddot{T}_1)_{C}\) can exist. To better understand the results of this model we therefore plot \((\dddot{T}_1)_{C}^{*}\) as a function of \(\epsilon\) in Fig. 8. It is noted from this figure that the value of \((\dddot{T}_1)_{C}\) based on (55) (i.e. the solid points) appear to be reasonable extrapolations for times when the results are in agreement with the 300 element mesh results. However, as steady-state is approached, it is seen that these solid points no longer appear reasonable. If one crudely extrapolates the values of \((\dddot{T}_1)_{C}^{*}\) to \(\epsilon = 0\) for the bottom two curves of Fig. 8, it is found that these values of \((\dddot{T}_1)_{C}\) are in much better agreement with the 300 element mesh results.

In comparing the equations for evaluating \(C_{1}, (\dddot{T}_1)_{C}^{*}\) and \((\dddot{T}_1)_{C}\), it is seen that \((\dddot{T}_1)_{C}\) is the only one of the three which involves an integration over the crack-tip quarter-point elements. Based on this and the apparently good accuracy of \(C_{1}\) and \((\dddot{T}_1)_{C}^{*}\) it is believed that the solution within these elements is the major cause of discrepancy between the 102 and 300 element mesh results (at least for times approaching steady-state). Again, it appears that forcing the crack tip field to have a \(r^{-1/2}\) strain singularity when the natural singularity is \(r^{-\alpha+1}\), may be the cause of difficulty.

**CONCLUSION**

This study shows that \(\Delta T\), a general path-independent integral given by Atluri[2], is easily applied to problems of non-steady creep as well as steady-state creep. In as much as the time rate of the first term of this vector quantity, \((\dddot{T}_1)_{C}\), characterizes a mode I crack-tip field for non-steady as well as steady creep, has an energy interpretation and is still readily calculated within numerical models, it seems that it has some advantage over the more common \(C_{1}\).

A finite element model has been derived which is generally applicable to viscoplastic material models. This model uses an initial strain approach which reduces computation time spent in forming the decomposing stiffness matrices and also circumvents the problem of element incompressibility constraints. Through special features including a correction term in the finite element equation, it appears that this model allows time steps which approach in size those used in tangent stiffness methods.

Finally, the finite element model has been used to model creep in a compact specimen. The model resulted in an elastic solution and a steady-state solution which agrees quite well with other solutions reported in the literature. Based on calculations with several meshes (some including quarter-point...
On a study of the \((\Delta T)_c\) and \(C^*\) integrals

**References**


(Received 27 July 1981; received for publication 26 August 1981)

APPENDIX A

This appendix discusses the existence of the various limits which have been taken in defining $\Delta T$, $\Delta f$, and $\Delta$. In considering these limits, we make use of the generally accepted result (see, e.g.[7]) that strain energy density quantities $W$ and $\mathcal{W}$ as well as the quantity $W$ behave as $1/r$ in the vicinity of the crack tip. This is assumed to be valid for non-steady as well as steady-state creep and also for the elastic state existing at $t = 0$.

Based on the known asymptotic behavior at the crack tip (i.e. the HRR field) the limits of $\Gamma_0$, contour integrals for eqns (11), (12), (17) and (28) can be written in the following form provided one takes $\Gamma_0$ as being a circular contour centered at the crack tip.

$$\Gamma_0 = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) e^{i\theta} d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} f(\theta) d\theta .$$

(A1)

It is clear therefore that all the limits of this type exist.

We next consider the limits taken of $V_0 - V$, type integrals. Inspection of these integrals show that they can all be put into the form

$$C + \frac{1}{2\pi} \int_{0}^{2\pi} g(\theta) e^{i\theta} d\theta = C + \frac{1}{2\pi} \int_{0}^{2\pi} g(\theta) d\theta ,$$

(A2)

where $V_0$ is a small volume in the vicinity of the crack tip and $C$ is the integral over the region $V_0 - V$. A first inspection of (A2) results in the conclusion that the limit does not exist since the integrand has a non-integrable singularity. If, however, we look at the right equality of (11), it is seen that this conclusion results in a contradiction. That is, we have shown that the limit of the integral on $\Gamma_0$ does exist and therefore (11) requires that the limit of the integral over $V_0 - V$ must exist. A re-inspection of (A2) shows that the only way for this apparent contradiction to be resolved is if the $g(\theta)$ of (A2) has the following property

$$\int_{0}^{2\pi} g(\theta) d\theta = 0 .$$

(A3)

The function $g(\theta)$ is known explicitly for the linear elastic case and therefore (A3) can be directly verified. For the HRR field, $g(\theta)$ is not known explicitly and therefore (A3) can only be verified numerically.

For infinitesimal strain, nonlinear elasticity, the following relation provides an alternative to verifying (A3) directly.

$$\int_{V_0 - V} \frac{\partial \sigma_{ik}}{\partial x_i} \Delta e_{0j} dV = \int_{\partial V_0} \sigma_{ik} \frac{\partial \epsilon_{0j}}{\partial x_i} dS - \int_{\partial V} \sigma_{ik} \frac{\partial \epsilon_{0j}}{\partial x_i} dS .$$

(A4)

The relation (A4) (which assumes zero crack surface tractions and no body forces) illustrates that this volume integral of type (A2) can be expressed in terms of the contour integral of type (A1). The relation (A4) can be verified through the divergence theorem, the linear momentum balance condition and the following identities.

$$\frac{\partial \sigma_{ik}}{\partial x_i} = \frac{\partial}{\partial x_i} \left( \frac{\partial W}{\partial \epsilon_{0j}} \right) = \frac{\partial}{\partial x_i} \left( \sigma_{ik} \frac{\partial \epsilon_{0j}}{\partial x_i} \right) = \Delta \sigma_{ik} \frac{\partial \epsilon_{0j}}{\partial x_i} .$$

(A5)
The purpose of this appendix is to give some examples to illustrate the numerical difference between \((\bar{T})_{esx}\) and \(\bar{C}_1\) as given by (31). Using (33), (32a) and (34), we have

\[
\frac{(\bar{T})_{esx}}{\bar{C}_1} = \frac{I^n}{I - 1} \cdot \frac{1}{2(n + 1)} \int_{0}^{1} \sigma_{eq}(\theta) \cos \theta \, d\theta.
\]

The values tabulated in Table B1 were computed approximately from values of \(I\) and plots of \(\sigma_{eq}(\theta)\) given in [7] and should be viewed accordingly. It is seen that for the range of \(n\) commonly encountered, \((\bar{T})_{esx}\) and \(\bar{C}_1\) are numerically very similar for plane strain and only slightly less so for plane stress.

| Table B.1 Comparison of \((\bar{T})_{esx}\) and \(\bar{C}_1\) |
|-----------------|-----------------|-----------------|-----------------|
| \(\bar{T}_{esx}\) | Plane Strain | Plane Stress | Plane Strain | Plane Stress |
| \(\bar{C}_1\) | \(n = 3\) | \(n = 13\) | \(n = 3\) | \(n = 13\) |
| \(0.98\) | \(0.00\) | \(1.11\) | \(1.14\) |