STABILITY ANALYSIS OF STRUCTURES VIA A NEW COMPLEMENTARY ENERGY METHOD

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Abstract A new procedure for the analyses of finite deformations and stability of structures, based on a complementary energy principle and an associated hybrid-mixed finite element method, is presented. In this procedure, the description of kinematics is based on the polar-decomposition of the displacement gradient into pure stretch and rigid rotation. The details of the procedure are illustrated through the problems of (i) post-buckling of a column, (ii) the elastica, and (iii) finite-displacements of a transversely loaded beam.

INTRODUCTION

As discussed by Atluri and Murakawa [1], the most consistent and easily applicable development of a complementary energy principle for finite deformations is due to the later work of Fraeijds de Veubeke [2]. Such a principle, involving both the first Piola-Kirchhoff stress tensor as well as the point-wise rigid rotation tensor as variables, has been stated in [2] so as to govern the finite deformations of a compressible nonlinear elastic solid. Also discussed in [1] are contributions to the subject of complementary energy principles for finite elasticity due to Zubov, Koiter, Christoffersen, and others. Since the appearance of [1], the authors became aware of the work by Ogden [3] who discussed more critically the key element in the works of Zubov and Koiter, namely, the invertibility of the relation between the first Piola-Kirchhoff stress tensor and the displacement-gradient tensor. Ogden [3] demonstrates clearly the non-unique nature of this inverse relation.

The concepts of discretizing the equations of angular momentum balance through a complementary energy principle involving rigid rotations also as variables have been explored by the authors in their studies related to incremental (rate) analyses of finite strain problems involving nonlinear elastic solids (compressible as well as incompressible), as well as plastic-plastic solids [4-9]. All of the studies in [4-9] were limited to problems of solids in plane stress/plane strain or of axisymmetric strain. In this paper we explore the concepts outlined in [1, 4-9] as they may be applied in the analysis of finite deformations and stability of structural members such as beams, plates and shells wherein certain plausible deformation hypotheses of the well-known "Kirchhoff-Love" type are invoked.

The case for the possible advantages of using a complementary energy approach to structural stability problems has been succinctly presented by Masur and Popelar [10], and Koiter [11]. The analyses presented in [10, 11] are, however, limited to the cases of bifurcation instability of beams/columns with irrotational fundamental states (linear prebuckling states). In the present paper we consider, as an example, the general problem of finite deformation of a "one-dimensional" structural member undergoing arbitrarily large rotations but only moderate stretching. The material is considered to be isotropic and semilinear, i.e., exhibiting a linear relation between the stretch (or engineering strain) tensor and the Jaumann stress (or equivalently, in the case of isotropy, the Lure of Biot stress) tensor. While the procedures presented herein may be directly extended to the cases of plates and shells, such extensions are not included here.

We present here detailed results, and their discussion, for the problems of (i) post-buckling of a column, (ii) the elastica, and (iii) large displacements of a transversely loaded beam.

In the following, we present, as a starting point, a general variational principle for finite elasticity, analogous to the well-known Hu-Washizu principle of linear elasticity, involving the displacements, stretches, and the first Piola-Kirchhoff stresses, as variables. By incorporating appropriate "plausible" assumptions for a structural member, such as a beam, the above principle is specialized to the case of the respective structural member. From this general principle an appropriate complementary energy principle, and an associated "hybrid-mixed" finite element method, are developed for the case of a beam.

1. PRELIMINARIES AND A GENERAL VARIATIONAL PRINCIPLE

We use a fixed rectangular Cartesian Coordinate system. We adopt the notation: Bold denotes a vector; bold italic denotes a second-order tensor: a = A · b implies that ai = Ai bi; A · B denotes a product such that (A · B)i j = Aik Bij; (A · B) = Aij Bij; and u · t = uit. The position vector of a particle in the undeformed body is x = (xe, e, e, e) where e, are unit Cartesian bases, and the gradient operator V in the initial configuration is V = ed/ax. The position vector of the same particle in the deformed body is y = (xe, e, e, e) where x, are unit Cartesian bases, and the corresponding gradient operator V(y) = ed/ax. The deformation gradient tensor F is given by F = (y)y; F = ye, = 0y/0x. For non-singular F the polar-decomposition, F = u (I + h), exists, where (I + h) is a symmetric positive definite tensor called the stretch tensor (with h often being called the en-
engineering strain tensor, \( I \) an identity tensor; and \( \alpha \) an orthogonal rotation tensor such that \( \alpha^T = \alpha^{-1} \). The deformation tensor \( \alpha \) is defined by \( \alpha = F^{-1} \cdot \gamma = [1 + h \cdot \gamma] \). The Green–Lagrange strain tensor is defined by \( g = \frac{1}{2} (G - l) = \frac{1}{2} (h^2 + 2k) = 1/2 \left( \frac{e^T e + e^T e}{} \right) \) where \( e \) is the gradient of the displacement vector \( \mathbf{w} = y - x \), i.e. \( e = (\mathbf{w}) \). For our present purposes we introduce the stress measures: (i) the true (Cauchy) stress tensor, \( \tau \); (ii) the Piola–Lagrange or the first Piola Kirchhoff stress tensor, \( \sigma \); (iii) and the Jaumann stress tensor \( (o) \) or what is also at times referred to as the symmetrized Lure stress tensor or the symmetrized Biot stress tensor \( r \). As discussed in [1], and elsewhere, the relations between \( \tau, r, \) and \( \sigma \) become coaxial, and the relation (2) becomes:

\[
\tau = \frac{1}{2} (\sigma + \sigma^T) \cdot \tau
\]

where \( F^{-1} \) is the inverse of \( F \), and \( J \) is the determinant of the Jacobian \( \mathbf{y} \).

As discussed in detail in [1], a functional \( \Pi (u, h, \alpha, \tau) \) whose stationary conditions are the field equations governing the finite deformation of a nonlinear elastic body can be stated as:

\[
\pi_{uu}(u, h, \alpha, \tau) = \int_{V_0} \left\{ \mathcal{W}(h) - \rho \cdot \frac{\partial}{\partial \tau} (\tau \cdot \alpha \cdot \gamma \cdot \tau) \right\} \cdot dV - \int_{S_{ho}} \cdot dS - \int_{S_{ho}} \frac{\partial}{\partial \tau} (\tau \cdot u) \cdot dS \quad (4)
\]

The stretch \( h \) is required to be symmetric a priori, the rotation \( \alpha \) is required to be orthogonal, a priori, and the first Piola Kirchhoff stresses \( \sigma \) must be allowed to be unsymmetric, a priori. In eqn (3), \( V_0 \) is the volume of the space occupied by the undeformed body; \( S_{ho} \) and \( S_{ho} \) are, respectively, the surfaces where displacements and traction are prescribed; \( \mathcal{W}(h) \) is the strain energy density, per unit initial volume, as a function of the symmetric engineering strain tensor \( \alpha \); \( g \) are body forces/unit mass; \( \rho \) is the mass density/unit initial volume; \( t = n \cdot t \) are surface tractions, and superposed \( \mathbf{b} \mathbf{s} \) is a prescribed quantity. The first variation of the functional in eqn (4) can be shown [1] to be:

\[
\delta \pi_{uu}(u, h, \alpha, \tau) = \int_{V_0} \left\{ \frac{\delta \mathcal{W}}{\delta \tau} \cdot \delta u - \frac{1}{2} (\tau \cdot \alpha \cdot \gamma \cdot \tau) \cdot \delta \tau \right\} \cdot dV - \int_{S_{ho}} \cdot dS - \int_{S_{ho}} \frac{\partial}{\partial \tau} (\tau \cdot u) \cdot dS = 0 \quad (5)
\]

The vanishing of the above first variation leads to the usual Euler–Lagrange equations from the usual arguments of calculus of variations: (i) the constitutive equation (corresponding to \( \delta \tau \)); (ii) the linear momentum balance condition for \( t \) (corresponding to \( \delta u \)); (iii) the angular momentum balance condition for \( \tau \), viz. that \( (I) \cdot \mathbf{h} = \mathbf{w} \cdot \mathbf{s} \)-symmetric (due to the skew-symmetric nature of \( x^T \cdot \mathbf{w} \cdot \mathbf{s} \)), since \( \alpha \) is required to be orthogonal \( a \) priori, such that \( \alpha^T \cdot \alpha = I \); (iv) the compatibility condition between \( u, \alpha \), and \( h \) (corresponding to \( \delta t \)); (v) the traction at \( S_{ho} \) viz. \( t = n \cdot t \) (corresponding to \( \delta u \) at \( S_{ho} \)); and (vi) the displacement boundary condition at \( S_{ho} \) corresponding to \( \delta \tau \) at \( S_{ho} \).

In the technical theory of beams, or plates, and shells, certain "plausible" approximations are introduced to reduce these problems, respectively, to one or two dimensional in nature from what may rigorously be classified as three-dimensional problems. It is well-known that variational principles often provide a convenient way of deriving the field equations and consistent boundary conditions for these problems. One may systematically introduce "plausible" approximations for the field variables in a functional, then the stationary conditions of the functional yield the relevant field equations and boundary conditions for the considered structural member. Thus, the "modus operandi" of the present procedure is to introduce certain approximations to \( u, h, \alpha \), and \( t \) appearing in eqn (4) so that the relevant field equations for the structural problems of beams, plates, and shells may be derived from the stationary condition of the thus approximated functional \( \pi_{uu} \). While this approach can be systematically extended to the cases of plates and shells along the same general lines as indicated here, we present in the following the details of only the case of arbitrarily large deformations (characterized by large rotations and perhaps moderate stretches) of a beam.

2. FINITE DEFORMATIONS OF A BEAM

We consider, without loss of generality, an initially straight rectangular beam (of a symmetrical cross section) as shown in Fig. 1, with material coordinates \( x_1, x_2 \). Coordinate \( x_1 \) is along the length of the beam, and \( x_2 \) is in the depth-direction of the beam \( (x_2 = 0 \) being the mid-plane). We consider the beam to be of a unit width and consider deformation of the beam only in the \( x_2 \) plane.

The position vector of a particle on the mid-plane of the beam is denoted by \( x = x_1, e_1 \). Upon deformation, the same particle is located by the position vector, \( y = x + u \cdot e_1 + u \cdot e_2 \) (see Fig. 1). The position of an arbitrary point in the undeformed beam is denoted by the vector, \( x - x_2 \). We invoke, in the present work, the Kirchhoff–Love hypothesis for the deformation of the beam, viz. the \( x_2 \) lines of the undeformed beam remain normal to the deformed mid-plane and, moreover, remain unstretched. Thus, the position vector of an arbitrary particle in the deformed beam is given by:

\[
y = y = x + x_2 N \quad (6)
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\[
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\]

The base vectors at an arbitrary point in the deformed beam are given by:

\[
\frac{\partial y}{\partial x_1} \equiv y_{11} = G_1 = (\delta_{11} + u_1 e_1 + N_1 x_1) \quad (8)
\]

\[
\frac{\partial y}{\partial x_2} \equiv y_{22} = G_2 = N \quad (9)
\]
Thus, in eqn (8), and throughout the remainder of the paper, the notation, \((\mathbf{1}_1 = \mathbf{1})/\mathbf{1}x_1\), is used.

The deformation gradient tensor \(F\) can be represented as:

\[
F = (\nabla \mathbf{y})^T = G_{11} \mathbf{e}_1 + N_{\mathbf{e}_2}. \tag{10}
\]

We consider here a class of problems characterized by large rotation, but moderate stretches. For all deformations given by eqn (7) the material coordinates coincide with the principal axes of stretch, so the deformation of the beam may be decomposed into a pure stretch along the \(e_1\) direction, followed by a rigid rotation. The stretch tensor \(h\) is thus,

\[
h = h_{11}(e_1, e_1). \tag{11}
\]

From eqns (10) and (11) it is seen that,

\[
N = F \cdot e_2 = [\mathbf{a} \cdot (I + h)] \cdot e_2 = \mathbf{a} \cdot e_2. \tag{12}
\]

Using eqn (12), eqn (7) can be written as:

\[
u = (u^e e_1 + u^e e_2) + (a - I) \cdot e_2 x_2 \tag{13a}
\]

where

\[
u^e = u^e(x_1)(a = 1, 2). \tag{13b}
\]

In the present case of moderate stretches, we assume that \(h(x_1, x_2)\) can be represented as:

\[
h_{11}(x_1, x_2) = (h + \chi x_2) \tag{14}
\]

where \(h = h(x_1)\) is the midplane stretch, and \(\chi = \chi(x_1)\) is the curvature strain. These are the well-known engineering measures of strain. Further, we assume plane-stress conditions in the beam and assume the first Piola–Kirchhoff stress tensor (i.e. stress measured/unit area in the undeformed configuration) to be represented by:

\[
t = t_{1n} \mathbf{e}_n \mathbf{e}_n (a, \beta = 1, 2) \tag{15}
\]

where

\[
t_{1n} = t_{1n}(x_1, x_2). \tag{16}
\]

Likewise, we assume the rigid rotation tensor to be represented by:

\[
\alpha = \alpha_{1n} \mathbf{e}_n \mathbf{e}_n, \tag{17}
\]

where, under the present deformation assumptions,

\[
\alpha_{1n} = \alpha_{1n}(x_1) \text{ only}. \tag{18}
\]

The orthogonal tensor \(\alpha_{1n}\) is represented conveniently in terms of the angle \(\theta\) (Fig. 1) as:

\[
\alpha_{1n} = \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix} \tag{19}
\]

We assume the beam to be of a semi-linear isotropic material, such that the relation between the stretch \(h_{11}\) and its conjugate stress measure, the Jaumann stress \(r_{11}\), is given by

\[
r_{11} = F r_{11}. \tag{20}
\]

We assume the following system of external tractions on the beam in general:

\[
at x_1 = 0 \text{ or } L: t_{11} = T_{11}(x_2); t_{12} = T_{12}(x_2). \tag{21}
\]

The external forces distributed along the beam are assumed, without much loss of generality, to be specified per unit length \(x_1\) along the midaxis of the beam to be \(\mathbf{g}_n = \mathbf{g}_n(x_1)\). Finally, the specified displacements at the ends of the beam are assumed to be:

\[
at x_1 = 0 \text{ or } L: \mathbf{u}_1 = \mathbf{u}_1(x_2); \mathbf{u}_2 = \mathbf{u}_2(x_2). \tag{22}
\]

We assume that \(\mathbf{u}_1\) and \(\mathbf{u}_2\) are compatible with the presently invoked hypotheses, such that:

\[
at x_1 = 0 \text{ or } L: \mathbf{u}_1(x_2) = \mathbf{u}_1^* + \mathbf{u}_1^* x_2; \quad \mathbf{u}_2 = \mathbf{u}_2^* + (\mathbf{u}_2 - 1) x_2. \tag{23}
\]

Even though the boundary conditions (21) and (23) are given in their general form, it is to be understood that at either end of the beam, either both the traction components \(t_{11}\) and \(t_{12}\), or both displacements \(\mathbf{u}_1\) and \(\mathbf{u}_2\), or one component of traction and a complementary component of displacement, such as \(t_{11}\) and \(\mathbf{u}_1^*\), are assumed to be given.

Substituting eqns (13)-(18) into eqn (4), we obtain, after some straightforward manipulations, that for a beam under the above discussed assumptions,

\[
\Pi_{HW}(\mathbf{u}^*, \alpha_{1n}, h_{11}, t_{1n}) = \Pi_{HW}(\mathbf{u}^*, \mathbf{u}^*, h, \chi, \theta, t_{1n}) = \int_{x_1}^L \left\{ \frac{1}{2} E^2 h^2 (x_1, x_2)^2 + t_{11}[(1 + \mathbf{u}^* \cdot \mathbf{u}^*) - (1 + h) \cos \theta] + \chi x_2 \cos \theta(\theta_2 - x_1) \right\} dx_1 dx_2 + t_{12}[(1 + h) \sin \theta - \chi x_2 \sin \theta(\theta_2 - x_1)] dx_1 dx_2
\]

(21)

It is of interest to note that only \(t_{11}\) and \(t_{12}\) enter the above energy expression, due to the nature of the presently assumed deformation pattern. We now define first Piola–Kirchhoff stress-resultants (\(T_{1n}\)) and stress-couples (\(M_{1n}\)) such that:

\[
T_{11} = \int_{x_2}^L t_{11} dx_2; \quad T_{12} = \int_{x_2}^L t_{12} dx_2
\]

\[
M_{11} = \int_{x_2}^L t_{11} x_2 dx_2; \quad M_{12} = \int_{x_2}^L t_{12} x_2 dx_2. \tag{22}
\]

Accordingly, we define the prescribed equivalent stress-resultants and stress-couples, at \(x_1 = 0\) or \(L\), as

\[
\mathcal{T}_{11} = \int_{x_2}^L \mathbf{T}_{11} dx_2; \quad \mathcal{T}_{12} = \int_{x_2}^L \mathbf{T}_{12} dx_2
\]

\[
\mathcal{M}_{11} = \int_{x_2}^L \mathbf{M}_{11} x_2 dx_2; \quad \mathcal{M}_{12} = \int_{x_2}^L \mathbf{M}_{12} x_2 dx_2. \tag{23}
\]

With the definitions of eqns (22) and (23), eqn (21) can be written as:

\[
\Pi_{HW}(\mathbf{u}^*, \mathbf{u}^*, h, \chi, \theta, \mathcal{T}_{11}, \mathcal{T}_{12}, \mathcal{M}_{11}, \mathcal{M}_{12})
\]

(24)

\[
- \int_0^L \left\{ \frac{1}{2} E^2 h^2 (x_1, x_2)^2 + \frac{1}{2} E^2 \chi^2 \mathbf{u}^* \cdot \mathbf{u}^* + \mathbf{T}_{11}[(1 + \mathbf{u}^* \cdot \mathbf{u}^*) - (1 + h) \cos \theta] + \mathbf{T}_{12}(\mathbf{u}^* \cdot \mathbf{u}^* + (1 + h) \sin \theta) \right\} dx_2
\]

(25)
\[(M_{11} \cos \theta - M_{12} \sin \theta) dx_1 - \{ T_{11}(u_1^* - u_1^*) + T_{12}(u_2^* - u_2^*) + M_{11}(\sin \theta - \sin \theta) \} dx_1 + M_{12}(\cos \theta - \cos \theta) \right]_0^L (or)
\]
\[-T_{11}u_1^* + T_{12}u_2^* + M_{11} \sin \theta + M_{12}(\cos \theta - 1) \right]_0^L (or).
\]

where

\[I = \int_{x_1}^{x_2} dx_2.\]

For convenience, we define new variable \(M\) and \(W\) as:

\[M = M_{11} \cos \theta - M_{12} \sin \theta \]
\[W = M_{11} \sin \theta + M_{12} \cos \theta.\]

The thus transformed functional can be written as:

\[\pi_w(u_1^*, u_2^*, h, \chi, \theta, T_{11}, T_{12}, M, W) = \int_0^L \left\{ \frac{1}{2} E (u_1^* + u_2^*)^2 - \bar{g} u_1^* + T_{11}(1 + u_1^*, \cos (1 + h)) + T_{12}(u_2^* - u_2^*) + M \sin \theta \right\} dx_1 + \left\{ \frac{1}{2} E (u_1^* + u_2^*)^2 - \bar{g} u_1^* + T_{11}(1 + u_1^*, \cos (1 + h)) + T_{12}(u_2^* - u_2^*) + M \sin \theta \right\} dx_1 + M_{11}(\cos \theta - \cos \theta) \right]_0^L (or)
\]
\[-T_{11}u_1^* + T_{12}u_2^* + M_{11} \sin \theta + M_{12}(\cos \theta - 1) \right]_0^L (or).
\]

(24)

It is of interest to note that the Jaumann stress-couple \(W\) does not appear in eqn (29), due, primarily, to the nature of the present deformation assumptions. The Euler–Lagrange equations and natural boundary-conditions from the stationarity of the functional in eqn (29) can be seen to be:

\[E A h = T_{11} \cos \theta - T_{12} \sin \theta \equiv R_{11}\]
\[E I x = M \]
\[1 + u_1^* = (1 + h) \cos \theta \]
\[u_1^* = -(1 + h) \sin \theta \]
\[\theta_1 = \chi \]
\[M_{11} - (1 + h) T_{11} \sin \theta + T_{12} \cos \theta \]
\[\equiv M_{11} - (1 + h) R_{12} = 0 \]
\[T_{11,1} + \theta = 0 \]
\[T_{12,1} + \theta = 0 \]
\[u_1^* = u_1^* \]
\[u_2^* = u_2^* \]
\[\theta = \theta \] at \(x_1 = 0\) or \(L\)
\[T_{11} - T_{12} - T_{112} - M \equiv R_{11} \cos \theta - R_{12} \sin \theta \]
\[\theta = \theta \] at \(x_1 = 0\) or \(L\).

(28a, b)

Again, it is seen that \(R_{11}\) and \(R_{12}\) defined in eqns (30a) and (30b) respectively, can be identified as the Jaumann stress-resultants:

\[R_{11} = \int_{x_1}^{x_2} dx_2; \quad R_{12} = \int_{x_1}^{x_2} dx_1.\]

(31)

Equations (30a, b) are constitutive relations, \((-h-e)\) are compatibility conditions, \((f-g)\) are equilibrium equations, and \((i,j)\) are boundary conditions, in terms of the presently defined beam variables.

By satisfying, a priori, eqns (30a, b) one may eliminate \(h\) and \(\chi\) as variables from eqn (29); and satisfying, a priori, eqns (30g, h) one may eliminate \(u_2^*\) as variables within the integral in eqn (29). Thus one derives a modified complementary energy functional, as:

\[\pi_{NW}(u_1^*, u_2^*, h, \chi, \theta, T_{11}, T_{12}, M) = \int_0^L \left\{ \frac{1}{2} E (u_1^* + u_2^*)^2 - \bar{g} u_1^* + T_{11}(1 + u_1^*, -\cos (1 + h)) + T_{12}(u_2^* + \sin \theta (1 + h) + M (\theta_1 - \chi)) \right\} dx_1 + \left\{ \frac{1}{2} E (u_1^* + u_2^*)^2 - \bar{g} u_1^* + T_{11}(1 + u_1^*, -\cos (1 + h)) + T_{12}(u_2^* + \sin \theta (1 + h) + M (\theta_1 - \chi)) \right\} dx_1 + M_{11}(\cos \theta - \cos \theta) \right]_0^L (or)
\]
\[-T_{11}u_1^* + T_{12}u_2^* + M_{11} \sin \theta + M_{12}(\cos \theta - 1) \right]_0^L (or).
\]

(29)

(32)
ary conditions corresponding to \( \delta \pi_c = 0 \), with \( \pi_c \) as in eqn (32), can easily be seen to be: (i) the compatibility conditions, eqn (30c-e); (ii) the moment balance condition, eqn (30f); (iii) the displacement boundary conditions, eqn (30g); and (iv) the traction and moment boundary conditions, eqn (30h).

We now consider formulations for a piecewise linear incremental solution procedure. Thus, let \( C^N \) denote a known deformed configuration of the beam, and \( C^{N+1} \) be a further deformed state of the beam which is to be solved for. We assume that \( C^{N+1} \) is sufficiently close to \( C^N \), such that \( C^{N+1} = C^N + \Delta C \) where \( \Delta C \) represents a "small" change in the variables, \( \theta, T_1, T_2, \) and \( M \).

We will use a total Lagrangean (TL) formulation, the details for which are elaborated in Atluri and Murakawa [1], in the following. Using this (TL) formulation, we can write:

\[
\Pi_c(C^{N+1}) = \Pi_c((\tilde{u}_1)^{N+1}, (\tilde{u}_2)^{N+1}, \theta^{N+1}, T_1^{N+1}, T_2^{N+1})
\]

Further, in the present complementary energy formulation, the incremental constitutive relations eqns (30a, b), and the incremental linear momentum balance conditions, eqns (30g, h) are assumed to be satisfied a priori, i.e.

\[
E\Delta \dot{u} = E\Delta \dot{M} = 0 \quad \Delta T_{1,1} + \Delta \theta_1 = 0 \quad \Delta T_{1,2} + \Delta \theta_2 = 0.
\]

Expanding \( \Pi_c(C^{N+1}) \), we find, through some straightforward algebra, that:

\[
\Delta^2 \Pi_c = \int_0^L \left\{ -\frac{1}{2EA} (T_{11}^N \cos \theta - T_{12}^N \sin \theta) \right\} \cos \theta \Delta \theta
\]

where

\[
R_{11} = (T_{11}^N \sin \theta + T_{12}^N \cos \theta); \quad R_{12} = (T_{11}^N \cos \theta - T_{12}^N \sin \theta)
\]

Recognizing that \( \Delta T_{11} \) and \( \Delta T_{12} \) are subject to the constraints, eqns (36) and (37), a priori, it can be shown that the condition of stationarity of the functional in eqn (39) leads to the following Euler equations and n.b.c.:

\[
\Delta u_{1,1} = -\frac{1}{EA} \cos \theta (\Delta T_{11}^c \cos \theta - \Delta T_{12}^c \sin \theta - R_{11}^c \Delta \theta)
\]

\[
\Delta u_{2,1} = -\frac{1}{EA} \cos \theta (\Delta T_{11}^c \cos \theta - \Delta T_{12}^c \sin \theta - R_{11}^c \Delta \theta)
\]

In view of the fact that the a priori conditions eqns (30a, b, g and h) hold at \( C^N \), and the incremental variables \( \Delta T_{11} \) and \( \Delta T_{12} \) are required to satisfy eqns (36) and (37) it is seen that \( \Delta^2 \Pi_c \) must be identically zero if the solution obtained, from the present complementary energy approach, at \( C^N \) satisfies the conditions, eqns (30c-f, i-k) exactly. However due to the inherent errors of the present piecewise linear procedure, this may not be so, i.e. \( \Delta^2 \Pi_c \neq 0 \). Thus, the term \( \Delta^2 \Pi_c \) is retained to devise an iterative corrective procedure to keep the solution path from straying from the true path, as discussed in detail in Atluri and Murakawa [1].

The incremental functional \( \Delta^2 \Pi_c \) is obtained to be:

\[
\Delta^2 \Pi_c = \int_0^L \left\{ -\frac{1}{2EA} (\Delta T_{11} \cos \theta - \Delta T_{12} \sin \theta - R_{11}^c \Delta \theta)^2 \right\} \cos \theta \Delta \theta
\]

We now consider a finite element implementation of the complementary energy method represented by the stationarity of the functional in eqn (39). We segment the beam into \( M \) elements \( i = 0, 1 \ldots M, \) each with end points denoted by \( x_1 = x_{1i} \) and \( x_1 = x_{1i+1} \) with \( x_{10} = 0 \) and \( x_{1M+1} = L \). To satisfy eqns (36) and (37), a priori, we assume within each element;
\[
\Delta T_{i1} = - \int_{x_i} \Delta \alpha_i + \Delta \alpha_i'; (i = 0, \ldots, M) \tag{48}
\]
\[
\Delta T_{i2} = - \int_{x_i} \Delta \alpha_i + \Delta \alpha_i'; (i = 0, \ldots, M) \tag{49}
\]
wherein \(\Delta \alpha_i\) and \(\Delta \alpha_i'\) are undetermined parameters which are taken, for simplicity, to be independent for each element, \(i = 0, \ldots, M\). In view of this, the traction reciprocity conditions, at the node \(x_i = x_{1(0)}\) at which the \((i-1)\)th and \(i\)th elements are connected, viz. \((\Delta T_{i1})^* = (\Delta T_{i1})^-\) and \((\Delta T_{i2})^- = (\Delta T_{i2})^*\) (where \(^*\) and \(^-\) denote, respectively, the left and right hand sides of \(x_{1(0)}\) in the limit that \(x_{1(0)}\) is approached), are enforced through a Lagrange-multiplier technique as described in Atluri and Murakawa \cite{[1]}.

Further, within each element, the moment \(M\) is assumed as:
\[
M' = M_0(1 - \xi) + M_{i+1}(i = 0, \ldots, M) \tag{50a}
\]
where
\[
\xi = \frac{x_i - x_{1(0)}}{x_{1(i+1)} - x_{1(0)}} \tag{50b}
\]
where \(M_0\) and \(M_{i+1}\) are, respectively, the moments at \(x_{1(0)}\) and \(x_{1(i+1)}\). Thus, eqn (50) inherently satisfies the moment reciprocity condition. Finally, we assume the rigid rotation within each element to be a constant, i.e.
\[
\Delta \theta^r = \Delta \theta^r; \quad x_{1(0)} \quad x_i \quad x_{1(i+1)}(i = 0, \ldots, M) \tag{51}
\]
when the interelement traction-reciprocity conditions, viz.
\[(\Delta T_{i1})^* = (\Delta T_{i1})^- = (\Delta T_{i2})^- = (\Delta T_{i2})^*\]
at \(x_i = x_{1(i)}\) are introduced as subsidiary conditions through Lagrange-multipliers \((\Delta \alpha_i^*)\) at \(x_{1(i)}\) to the functional for the finite element system, which is a modification to eqn (39) as described in Atluri and Murakawa \cite{[1]}, can be written as:
\[
\Delta^2 \Pi_c = \sum_{i=0}^{M} (\Delta^2 \Pi_{c^i}) \tag{52}
\]
where \((\Delta^2 \Pi_{c^i})\) is defined simply by changing the limits of integrals occurring in eqn (39) as follows:
\[
\int_{x_{1(0)}}^{x_{1(i+1)}} (i \text{ or change to}) \quad \int_{x_{1(i)}}^{x_{1(i+1)}}, (i \text{ or change to}) \quad [x_{1(i+1)} \text{ or}] \tag{53a}
\]
When the assumptions in eqn (48)-(50) and (57) are introduced, the functional in eqn (52) can be written as:
\[
\Delta^2 \Pi_c = \left\{ \begin{array}{c}
\Delta \alpha_i' \\
\Delta \alpha_i^* \\
\Delta M_i \\
\Delta \beta^r
\end{array} \right\}^T 
\left[ \begin{array}{cccc}
\Delta \alpha_i' & 0 & K_{14} & 0 \\
0 & K_{12} & K_{13} & 0 \\
K_{21} & K_{22} & K_{23} & 0 \\
K_{31} & K_{32} & K_{33} & 0 \\
K_{41} & 0 & 0 & 0
\end{array} \right]
\left\{ \begin{array}{c}
\Delta \alpha_i' \\
\Delta \alpha_i^* \\
\Delta M_i \\
\Delta \beta^r
\end{array} \right\} + \left\{ \begin{array}{c}
Q_1 \Delta \alpha_i^* \\
Q_2 \Delta \alpha_i^* \\
Q_3 \Delta \alpha_i^* \\
Q_4 \Delta \beta^r
\end{array} \right\} \tag{55}
\]
In the above, the notations, \([\Delta \alpha_i'] = [\Delta \alpha_i', \Delta \alpha_i'^*], [\Delta M_i] = [\Delta M_i, \Delta M_i^*], \) and \([\Delta \alpha_i^*] = [\Delta \alpha_i^*, \Delta \alpha_i^{**}], \) \(\Delta \alpha_i(0, M)\) \(\Delta \alpha_{i+1}(0, M)\) \(\Delta \alpha_{i+1}(0, M)\) \(\Delta \alpha_{i+1}(0, M)\) \(\Delta \alpha_{i+1}(0, M)\) are used. Since \(\Delta \alpha_i\) and \(\Delta \beta^r\) are independent for each element, they may be eliminated as variables at the element level, from the conditions of stationarity of the functional in eqn (55) w.r.t. \(\Delta \alpha_i\) and \(\Delta \beta^r\). When this is done, it is seen that the functional \(\Delta^2 \Pi_c\) can be written as:
\[
\Delta^2 \Pi_c = \sum_{i=0}^{M} \frac{1}{2} \left[ \begin{array}{c}
\Delta M_i^T \\
\Delta \alpha_i^*
\end{array} \right] \left[ \begin{array}{cccc}
K_{11} & K_{12} & 0 & K_{14} \\
K_{21} & K_{22} & K_{23} & 0 \\
0 & K_{32} & K_{33} & 0 \\
K_{41} & 0 & 0 & 0
\end{array} \right] \left\{ \begin{array}{c}
\Delta M_i \\
\Delta \alpha_i^* \\
\Delta \alpha_i^* \\
\Delta \beta^r
\end{array} \right\} + \left\{ \begin{array}{c}
Q_3 \Delta \alpha_i^* \\
Q_4 \Delta \alpha_i^*
\end{array} \right\} \tag{56}
\]
By carrying out the element assembly, eqn (56) can be reduced to:
\[
\Delta^2 \Pi_c = \frac{1}{2} \left[ \begin{array}{c}
\Delta M_i \\
\Delta \alpha_i^*
\end{array} \right] \left[ \begin{array}{cccc}
A & \{F_i\} & \{F_i\} & \{\Delta M_i\} \\
\{F_i\} & \{\Delta M_i\} & \{F_i\} & \{\Delta \alpha_i^*\}
\end{array} \right]
\tag{57}
\]
where \(\{\Delta M_i\}\) represents a column vector of moments at all nodes, and \(\{\Delta \alpha_i^*\}\) represents a column vector of displacements (in \(x_1\) and \(x_2\) directions) at all nodes. Finally, setting \(\Delta^2 \Pi_c = 0\) w.r.t. \(\Delta M_i\) and \(\Delta \alpha_i^*\), we obtain the algebraic equation:
\[
\left[ \begin{array}{c}
\Delta M_i \\
\Delta \alpha_i^*
\end{array} \right] = \left\{ \begin{array}{c}
F_i \\
F_i
\end{array} \right\}, \tag{58}
\]
Thus, in the present method, the final algebraic equations can be solved for both the nodal moment resultants as well as the nodal displacements. For this reason, in accordance with the definitions given in Atluri \cite{[12]} and Atluri and Murakawa \cite{[1]}, the present method can be classified as a mixed method. Moreover, since the reciprocity conditions for \(T_{11}\) and \(T_{12}\) at the nodes are satisfied through Lagrange-multiplier technique, the present method is also a hybrid method \cite{[1]}.

In the following we present three illustrative examples.

**Examples**

(i) **Post-buckling of a column.** The details of the problem are given in Fig. 2, which shows a cantilever beam subject to a compressive axial force \(P\) at the free end. Post buckling behavior was initiated by a small axial force \(q = 10^{-5} P_E\) as shown in Fig. 2. The finite element solution is obtained by using 4 elements, each with (i) a constant rotation. (ii) linear displacement
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field of $u_x^1$, $u_y^1$, and (iii) a linear moment field. The deformed shapes of the column for axial load of $aP_1$, for various values of $a$, are shown in Fig. 2.

The transverse displacement of the free end of the cantilever beam-column is shown in Fig. 3 as a function of the axial load $aP_1$. It has been verified that the solid line shown in Fig. 3 matches exactly the analytical solution given by Timoshenko and Gere [13]. It is noted that a finite element solution for this problem was also presented by Horrigome [14]. In [14] the beam-column was modeled as a 2-dimensional plate-strip. The solution [14], which is based on an incremental potential energy formulation, was obtained by using 14 triangular elements and was also noted [14] to correlate well with that of [13].

(ii) Elastic: The problem, depicted in Fig. 4, is that of a simply supported beam with an axially movable hinge, and subject to concentrated moments at the ends. The problem was analyzed by using 4 elements each with the previously mentioned field assumptions. The deformed shapes of the beam for various values of applied $M$ are shown in Fig. 4. The variation of $\alpha$ (the projection of the deformed axis of the beam on the axis of the undeformed beam as in Fig. 4) with $M$ is shown in Fig. 5. This variation is seen to correlate excellently with the analytical solution [13].

(iii) Transversely loaded simply-supported beam: The problem, depicted in Fig. 6, is that of a transversely loaded simply-supported beam with axially-immovable hinges. The predominant nonlinearity in the problem is due to the mid-plane stretching of the beam. The problem was analysed by using 4 elements in a half of the beam. The analytical solution for a rectangular plate-strip was given by Timoshenko and Woinowsky-Krieger [15]. The solution in [15] would thus have a Poisson-ratio effect, whereas the present beam solution does not have a similar effect. The solution of [15] was numerically evaluated in [14] for $v=0.3$ and is reproduced here. It is seen from Fig. 6 that there is an excellent correlation between the present results and those of [15]. It is noted that numerical solution for the problem of a simply supported rectangular plate-strip, based on an incremental potential energy formulation, and using 15 elements (10 triangular and 5 rectangular) in a half of the plate, was also given in Horrigome [14].

The comparison of the present results with those of Horrigome [14] for identical problems appears to indicate the relative merits of the presently proposed complementary energy method, in terms of accuracy as well as computational economy.

CLOSURE

A new complementary energy method for the stress and stability analyses of structures, which undergo
large rotations but moderate stretches, has been indicated. The relative merits of the present procedure have been illustrated in some representative problems of beams. Further work along the present lines is underway and will be reported elsewhere.

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