A STRESS - HYBRID FINITE ELEMENT METHOD
FOR STOKES' FLOW

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Introduction

Computational fluid mechanics based on the finite element method has been developed so far following two distinct approaches (a) using a stream function, or stream function-vorticity formulation [1,2], and (b) using a primitive variables formulation [3,4,5]. Within the first approach a stream function is defined such that the continuity equation is identically satisfied and the momentum equations are combined into a single equation expressed in terms of stream function, or stream function and vorticity. In the second approach velocities and pressure become the main variables of the working equation system. The finite element method is then implemented via a variational, quasi-variational approach or the method of weighted residuals [6,7,8,9].

The purpose of this paper is to present a new approach based on an assumed stress distribution and a variational formulation. Stress and pressure distributions are assumed such that the equilibrium equations are satisfied in the interior of each discrete element while boundary velocities are assumed such that the compatibility with the neighboring elements is maintained.

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A Variational Functional Formulation for Stokes' Flow

Let us consider a discrete domain formed by assembling a finite number of finite sized elements which have piecewise continuous boundaries at which neighboring elements adjoin. We denote the sub-domain of each element by \( V_n \) and its boundary by \( \partial V_n \); that portion of \( \partial V_n \) which represents the inter-element boundary by \( S_n \), the traction boundary region by \( S_{on} \), and the velocity boundary region by \( S_{un} \). With these above notations, it is obvious that the field variables taken into consideration must satisfy the following necessary conditions in order that the solution generated for the finite element assembly would approach that for an otherwise continuous domain.

\[
\sigma_{ij,\gamma} + \rho F_i = 0 \quad \text{in} \quad V_n
\]
\[
V_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in} \quad V_n
\]
\[
\sigma_{ij} = \frac{\partial A}{\partial V_{ij}} \quad \text{in} \quad V_n
\]
\[
u_i = \bar{u}_i \quad \text{on} \quad S_{un}
\]
\[
\sigma_{ij}n_i = \bar{T}_i \quad \text{on} \quad S_{on}
\]

In addition, at the inter-element boundary, one has to satisfy [10]: (a) the velocity continuity condition \( u_i^+ = u_i^- \) at \( S_n \), and (b) the traction continuity condition \( T_i^+ = T_i^- = 0 \) at \( S_n \). The following notations have been used so far: \( \sigma_{ij} \) for denoting stress, \( \rho \) - density, \( F_i \) - specific body forces, \( u_i \) - velocity, \( V_{ij} \) - rate of strain tensor, \( n_j \) - direction cosines of surface normal, \( T_i \) - surface traction, \( A \) - strain energy density function given by

\[
A(\rho, V_{ij}) = -\rho V_{kk} + \frac{1}{2}V_{ij}V_{ij}
\]

where \( \mu \) is viscosity. Then the total potential energy functional [11, 12] can be written as follows

\[
\pi_p = \frac{\pi}{n} \left\{ \int_V [A(\rho, V_{ij}) - \rho F_i u_i] dv - \int_{S_{on}} \bar{T}_i u_i ds \right\}
\]

This variational functional can be generalized by including the compatibility eq. (2) as a constraint via the Lagrange multipliers \( \sigma_{ij} \):

\[
\pi_G = \frac{\pi}{n} \left\{ \int_V [A(\rho, V_{ij}) - \sigma_{ij} V_{ij} - (\sigma_{ij,j} + \rho F_i) u_i] dv + \int_{S_{on}} \bar{T}_i u_i ds - \int_{S_{on}} \bar{T}_i u_i ds \right\}
\]

For practical applications this functional involves too many variables and too many constraints imposed upon the admissible functions. One way
of simplifying this formulation is to consider some of the conditions (1) through (5) satisfied a priori. The modified functional used in present research has been obtained making the following considerations:

(a) assume stress and pressure distributions within each discrete element \( V_n \) such that the equilibrium equation (1) is satisfied a priori;
(b) assume boundary velocities \( \bar{u}_1 \) such that the continuity requirement \( \bar{u}_1 = \bar{u}_1 \) at \( S_n \) and the velocity boundary condition (4) are satisfied a priori;
(c) assume that relations of eq. (3) are invertible for \( V_{ij} \) in terms of \( \sigma_{ij} \) [10], and that the contact transformations exist.

Then the modified variational formulation is given by

\[
\pi_M(\sigma_{ij}', \bar{u}_1) = \frac{1}{4\mu} \int_{V_n} B(\sigma_{ij}^*) \, dv + \int_{S_n} T_1 \bar{u}_1 \, ds - \int_{S_{on}} T_1 \bar{u}_1 \, ds \]

where \( B(\sigma_{ij}^*) \) is the complementary energy function

\[
B(\sigma_{ij}^*) = \frac{1}{4\mu} \sigma_{ij}^* \sigma_{ij}'
\]

A Finite Element Scheme.

Let us express stresses and boundary velocities in a matrix form:

\[
\{\sigma\}' = [P] \{\beta\}
\]

\[
\{\sigma\} = [R] \{\beta\} - \{\delta\} \alpha_1
\]

\[
\{\bar{u}_1\} = [Q] \{q\}
\]

Matrices [P], [R] and [Q] are expressed in terms of polynomials, and \( \{\beta\} \), \( \{q\} \), and \( \alpha_1 \) represent unknown parameters. The task now is to express the unknown parameters \( \beta \) in terms of \( q \)'s via the variational functional \( \pi_M \) of eq. (9), and to generate the working equation systems [13, 14]. Introducing eqs. (11) into eq. (9) and neglecting the body forces \( F_i \), one obtains

\[
\pi = \pi_D \left[ \{q\}^T \{A\} \{\beta\} - \{q\}^T \{B\} \alpha_1 - \{q\}^T \{S_o\} - \frac{1}{2\mu} \{\beta\}^T \{H\} \{\beta\} \right]
\]

where:

\[
[A] = \int_{V_n} [Q]^T [N] [R] \, ds, \quad [B] = \int_{V_n} [Q]^T [N] \{\delta\} \, ds
\]

\[
[S_o] = \int_{S_{on}} [Q]^T [N] \{\sigma_o\} \, ds, \quad [H] = \int_{V_n} \frac{1}{2\mu} [P]^T [C] [P] \, dv
\]

\[
[N] = \begin{bmatrix} n_1 & n_2 & 0 \\ 0 & n_1 & n_2 \end{bmatrix}, \quad [C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Taking the first variation of the functional \( \pi \) with respect to \( \{\beta\} \) and putting
\[ \delta n = 0, \text{ one obtains} \]
\[ \{ \beta \} = [H^{-1}]^T[A]^T \{ q \} \]
\[ \text{(14)} \]
which introduced into eq. (12) yields
\[ \pi = \sum_{h=1}^{n} \{ q \}^T[K]\{ q \} - \{ q \}^T[B]\{ \alpha_1 \} - \{ q \}^T[S_0] \]
\[ \text{(15)} \]
with \[[K] = [A][H^{-1}]^T[A]^T. \] Let us now expand all the element matrices of eq. (15) into system matrices and denote these new matrices by (*).
\[ \pi^* = \frac{1}{2}\{ q^* \}^T[K^*] \{ q^* \} - \{ q^* \}^T[B^*][\alpha_1^*] - \{ q^* \}^T[S_0^*] \]
\[ \text{(16)} \]
Taking the first variation of this above functional with respect to all \( q^* \)'s and \( \alpha_1^* \)'s, which are independent at the system level, one obtains finally the following system of equations
\[ [K^*][q^*] - [B^*][\alpha_1^*] - [S_0^*] = 0 \]
\[ [B^*][q^*] = 0 \]
\[ \text{(17)} \]
Solving for \( q^* \)'s and \( \alpha_1^* \)'s and using eqs. (11) and (14) one obtains the problem solution in terms of velocities, pressure, and stresses.

**Example**

To illustrate the present method the steady, laminar fluid flow through a parallel channel is considered (FIG. 1). Let us assume a stress distribution expressed by polynomials of second order, and a constant pressure distribution (i.e. \( p=\alpha_1 \)) within each element. Applying the equilibrium equations, one obtains the following final expressions for the deviatoric stress
\[ \sigma_{11} = \beta_{10} + \beta_{11}x + \beta_{12}y + \beta_{13}x^2 + \beta_{14}y^2 \]
\[ \sigma_{12} = \beta_{12} + \beta_{23}x - \beta_{24}y - 2\beta_{25}xy + \beta_{26}x^2 - \beta_{27}y^2 \]
\[ \sigma_{22} = \beta_{10} + \beta_{11}x - \beta_{12}y - 2\beta_{13}xy + \beta_{14}x^2 + \beta_{15}y^2 \]
\[ \text{(18)} \]
Velocity expressions are obtained from the conventional finite element method. For the present example isoparametric quadrilateral elements with eight nodes have been used. Introducing eqs. (18) and velocity expressions into the system equations (17) one obtains a system of linear equations which can be solved by using a standard solution procedure. Boundary conditions used in this example are shown in FIG. 1. Figures 2 and 3 present computed results in comparison with the analytical solutions. As it can be seen from
FIG. 1
Discrete domain and boundary conditions.

FIG. 2
Velocity solution at station x = 1.0

FIG. 3
Pressure solution
FIG. 2, the computed velocity solution is in good agreement with the analytical solution. Although the pressure distribution was considered constant within each element it is obvious from FIG. 3 that the computed solution yields correct values for pressure at the centroid of each element. Therefore, the system pressure distribution can be obtained fairly accurate by interpolation, using as data points pressure values at the center of each discrete element.

Conclusions

Fluid flow solutions based on the finite element method have been developed so far using a stream function or stream function/vorticity formulation, and a primitive variable approach. The present paper describes a new approach in solving viscous incompressible fluid flow problems based on a hybrid finite element method. Stress and pressure distributions are assumed such that the equilibrium equations are satisfied within each discrete element while boundary velocities are chosen so that the compatibility with neighboring elements is maintained. A modified variational functional is formulated and a finite element scheme developed. The method is illustrated by an example of a steady, laminar channel flow. Stress distribution is represented by polynomials of second order while pressure is assumed constant within each element. Results are in good agreement with the analytical solution. However, it is expected that improved solutions can be obtained by considering a linear or higher order pressure distributions. These results will be reported elsewhere.

References


