AN ELASTIC-PLASTIC ANALYSIS OF FATIGUE CRACK CLOSURE IN MODES I AND II.

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Abstract

In this paper, an efficient elastic-plastic finite element procedure to analyze crack closure and, its effects on fatigue crack growth under general spectrum loading, is presented. A hybrid-displacement finite element procedure is used to properly treat the stress and strain singularities near the crack-tip; and crack-growth under cyclic loading is simulated by the translation of certain "core" elements, near the crack-tip, in which proper stress and strain singularities were embedded. Both pure Mode I and Mode II types of cyclic loading are considered. In the mode I case, four types of cyclic loading, viz., constant amplitude block loading, high-to-low block loading, low-to-high block, and a single overload in an otherwise constant amplitude block loading are considered; whereas in Mode II, only a constant amplitude block loading is considered. Detailed results are presented for crack-closure and opening-stresses, crack-surface deformation profiles, etc., in each case. Certain observations, based on the present numerical results, concerning various factors that cause crack growth acceleration or retardation under general spectrum loading, are presented and discussed.

Introduction:

As discussed recently in an expository article by Schijve [1], the mechanism of crack-closure, as first observed experimentally by Elber [2], is generally considered to be a predominant mechanism that contributes to "interaction effects", which cause crack-growth retardation or acceleration, under variable-amplitude fatigue loading. It is also generally understood [1] that the crack-closure phenomenon is caused by residual plastic deformations remaining in the wake of the advancing crack-tip, as initially postulated by Elber. Analytical models that lend theoretical support to the existence of crack-closure phenomenon in fatigue crack-growth, and provide some rationality for the adoption of an effective stress-intensity range, based on closure effects, for the correlation of fatigue crack-growth rates, have also been proposed by Budiansky and Hutchinson [3]. As for a more general analysis of extending cracks under general block cyclic loading, to obtain crack-closure stresses, crack opening stresses, details of crack-surface deformations, and residual stresses in the crack-tip region, etc., elastic-plastic finite element analysis were first performed by Newman and his colleagues [4,5]. Apart from these analyses, the authors are aware of similar attempts only by Ohji and his co-workers [6,7]. The studies in [4-7] considered the mode I case only. Also, since the crack-growth was simulated in [4-7] by shifting a finite element node (the current crack tip) to an immediately adjacent node, and since constant strain-triangle type finite elements were used to model the cracked structure, a very fine finite element mesh (with the smallest element often being of the order of 10^{-3} times the crack length) is necessary in the modeling procedures of [4-7]. Thus the finite element computations of the type given in [4-7] can be very expensive, especially when cyclic loading of arbitrary spectrum are considered.

In the present paper, an alternate cost-effective and accurate elastic-plastic finite element procedure to analyze fatigue crack closure, and its effects, under general spectrum loading, is presented. In the present procedure, the well-known Hutchinson-Rice-Rosengren [8,9] type strain and stress singularities, for strain-hardening materials, are embedded in specially developed elements near the crack-tip. This eliminates the need for a very fine mesh near the tip. For instance, the crack-tip elements in the present procedure are of the order of 10^{-1} of the crack-length, as compared to constant strain triangles of the order of 10^{-3} to 10^{-4} times the crack-length generally used in the procedures of [4-7]. A hybrid-displacement finite element method [10,11] is used in developing these special elements. Also in the present procedure, crack-growth is simulated by: (i) the translation of a core of the aforementioned special elements by an arbitrary amount in the desired direction, (ii) reinterpolation of requisite data in the new finite element mesh, and (iii) proportional relaxation of tractions in order to create a new crack surface. Since the aforementioned special elements near the crack-tip are of circular-sector shape, centered at the crack-tip, crack growth in an arbitrary direction, from the initial-crack axis, under general mixed mode cyclic loading, can be modeled. The presently considered cases, of crack-geometries and far-field applied loading, result in "small-scale" yielding conditions near the crack-tip. In these cases, a static-condensation procedure is employed, wherein, the plastic portion of the structure is isolated from the elastic; the stiffness of the former keeps changing whereas that of the latter remain fixed. This condensation procedure results in a considerable saving of computer time.

In the present paper, both Modes I and II type cyclic loadings are considered. In the mode I case, four different types of cyclic block loading, viz., constant-amplitude, high-to-low, low-to-high, and a single overload in an otherwise constant amplitude block, are considered. Detailed results are presented for crack-opening
deformation profiles, and effective stress-intensity factor ranges for fatigue crack growth, in each case of block loading. Also presented here- in are detailed discussions, based on the obtained numerical results, concerning various factors that cause crack growth acceleration or retardation and delay effects under different types of cyclic loading. Finally, the results of an investigation of a center-cracked panel under external pure shear (Mode II) cyclic loading, of constant amplitude, are presented.

Brief Discussion of Mathematical Details of Analysis Procedure:

In the present paper an incremental, updated Lagrangian finite element formulation, for finite-deformation elasto-plasticity, is used. The flow theory of plasticity that is used is characterized by the well-known Huber-Mises-Hencky initial yield criterion, and a Prager-Ziegler type kinematic hardening rule. In the present work, as mentioned earlier, circular-sector shaped singularity elements (in which, a displacement field that corresponds to the aforementioned Hutchinson-Rice Rosengren singularities in strains and stresses, for strain-hardening elasto-plastic materials, are embedded) are used near the crack-tip, as shown in Fig. 1. The above singular elements are surrounded by "regular" eight-noded isoparametric quadrilateral elements, as also shown in Fig. 1. Compatibility of displacements, and reciprocity of tractions, between these "singular" and "regular" elements are enforced through a Lagrangian Multiplier technique (thus leading to a 'hybrid' element formulation), as shown below.

Let \( y^1 \) be the current (say, in a state \( C^N \)) Cartesian Spatial coordinates of a particle to be used as a reference system for the current increment, i.e., from \( C^N \) to \( C^N+1 \). Let \( \tau^{ij} \) be the true (Cauchy) stress in \( C^N \). Let \( S^{ij} \)

\[ (\begin{bmatrix} S^{ij} \end{bmatrix} + \tau^{ij}) \]

be the rate of Second Piola-Kirchhoff (P-K) stress referred to \( C^N \). It is noted that \( S^{ij} \) is the Second-P-K Stress in State \( C^N+1 \) as referred to \( C^N \). Let \( \bar{u}^i \) represent the rate of deformation from \( C^N \), and let \( \bar{u}^{i,j} \equiv \partial \bar{u}^i / \partial y^j \).

Also, let elements \( m = 1, 2, \ldots, p \) be the sector-shaped singularity elements, and \( m = p+1 \ldots, m \) be the surrounding regular elements. It can be shown that the variational principle governing: (i) the conditions of equilibrium in each element (singular as well as regular), (ii) traction boundary conditions for each element, where such exist, and (iii) conditions of compatibility of displacements and reciprocity of tractions between singular and regular elements, can be stated as the stationary condition of the functional:

\[
F = F_S(u^1; \bar{v}^i; T^{ij}) + F_R(\bar{v}^i_{m})
\]

\[
F_S = \sum_{m=1}^{p} \left( \frac{1}{2} \bar{v}^{i,j} \right)^T \left[ \left( u^{i,j} \right) - \rho_0 \bar{u}^{i,j} \right] + \bar{v}^{i,j} \left( \sigma^{ij} \right)_{m}^N \left( \bar{u}^{i,j} \right)_{m}^N + (\bar{y}^{i,j})_{m}^N \left( \bar{u}^{i,j} \right)_{m}^N \left( \bar{u}^{i,j} \right)_{m}^N \right)
\]

\[
F_R = \sum_{m=p+1}^{m} \left( \frac{1}{2} \bar{v}^{i,j} \right)^T \left( \left( u^{i,j} \right) + \rho_0 \bar{u}^{i,j} \right) + (\bar{y}^{i,j})_{m}^N \left( \bar{u}^{i,j} \right)_{m}^N \left( \bar{u}^{i,j} \right)_{m}^N \right)
\]

In the above, \( F \) is the functional for the group of singular elements, while \( F_R \) is that for the group of regular elements. Thus, in the singular element one assumes: (i) an arbitrary displacement field \( \bar{v}^i \) within the element \( \Omega_1 \) to include the proper strain/stress singularities as well as non-singular polynomial variations; (ii) an independent displacement field \( \bar{v}^i \) at the singular-element boundary, \( \Omega_1 \), and (iii) Lagrange Multiplier \( T^{ij} \) at the boundary to enforce the constraint \( \bar{u}^i = \bar{w}^i \) at the boundaries of singular elements. Whereas, in the regular elements, developed through the usual compatible displacement method, one assumes a single compatible displacement field \( \bar{v}^i \). Thus, it is seen that if \( \bar{v}^i \) at the boundary of singular element is so chosen that it matches with \( \bar{v}^i \) of regular elements at the interfaces of singular and regular elements, total displacement compatibility is enforced throughout the structure.

Also, in Eq. (1), \( \bar{v} \) is the rate potential for the rate of second P-K stress \( S^{ij} \), such that:

\[
\Delta \bar{w}/\Delta t = \bar{S}^{ij} + \varepsilon^{ij} + \varepsilon^{ij}_{i,j} \quad \Rightarrow \quad \Delta \bar{v} = \Delta \bar{w}/\Delta t = \bar{S}^{ij} + \varepsilon^{ij} + \varepsilon^{ij}_{i,j}
\]

This rate potential has been consistent: derived from a postulated rate potential for the corotational (Jaumann) rate of Kirchhoff stress, using the well-known classical rate theories of (finite deformation), rate-independent elasto-plasticity, as discussed in detail elsewhere [12, 13]. These details are omitted here due to space reasons.

The details of assumed variables \( \bar{u}^i, \bar{v}^i \), and \( T^{ij} \) for the singular elements and are omitted here. However, for the present purposes, it is worth noting that each sector-shaped singular element has 3 nodes along the circumference, and 4 nodes along each of the radial boundary lines. Along the circumferential boundary, the field \( \bar{v}^i \) is assumed in the form:

\[
\bar{v}^i = a_{1i} + a_{2i} \theta + a_{3i} \theta^2
\]

where \( \theta \) is the circumferential angle; and \( a_{1i}, \ldots, a_{3i} \) are expressed in terms of the respective displacements at the 3 nodes along the circumference. It can be seen that the above boundary displacement, \( \bar{v}^i \), is compatible with that of the surrounding 8 isoparametric quadrilaterals. Along the radial boundaries of the singular elements, \( \bar{v}^i \) is assumed as:

\[
\bar{v}^i = b_{1i} + b_{2i} \theta + b_{3i} \theta^2 + b_{4i} \left( \theta^3 / n+1 \right)
\]

where \( r \) is the radial distance from the crack-tip; \( n \) is the exponent in the material hardening law (i.e., from the uniaxial stress-strain curve, \( \varepsilon \sim \sigma^n \), \( \varepsilon \) the plastic-strain, and \( \sigma \) the stress); and the coefficients \( b_{1i}, \ldots, b_{4i} \) are expressed in terms of the respective nodal displacements at the aforementioned 4 nodes.

Finally, it is noted that the variational principle in Eq. (1) assumes that the current state \( C^N \) is exactly equilibrated. In general, such is not the case; and hence, equilibrium
 correction iterations of the Newton-Raphson type have to be performed at the end of each increment- 
rection. The details of these iterations have already been presented elsewhere [11,12].

Finite Element Modelling of Crack Growth

The steps in the finite element simulation of crack growth in the present procedure may be de- 
cribed as: (i) geometrical change in the crack surface boundary; (ii) translation of the crack-tip 
singularities to the advanced crack-tip; and (iii) release of surface tractions on the newly created 
crack surface.

The change in the crack surface boundary is made by translating the whole set of crack-tip core 
elements, as shown in Fig. 1, by arbitrary dis. 

placement \( \Delta a \) in the direction of intended crack extension; thus the new crack-tip node which is designated by 
the center of the sector-shaped core elements need not be coincident with any previously existing fi-
nite element node before extension. Thus even though the fixed boundary in the uncracked ligament of the structure is changed, the constraining condi-
tion of the nodes need not be altered. Elements immediately adjacent to the core must be realigned 
to fit to the translated core. This process of translating the core mesh also moves the embedded 
singularity in the elements to the new crack tip 
area, leaving no singularities but large deformations 
and strains in the wake of advanced crack-tip. 

All the 5x5 Gaussian data points in each of the 
translated core elements (and also the 5x5 points for the con-
ventional elements) may generally not coincide with those before translation, at which plastic 
history data such as current stresses, plastic 
strains, plastically dissipated work, yield surface 
translation, etc., are available. Therefore the data at points in the new mesh are estimated by 
linearly interpolating data on four Gaussian points in 
the old mesh that are nearest to the point under 
question in the new mesh. The simple but cumbersome 
mathematical details of the interpolation and 
smoothing process are omitted here for the sake of 
brevity. With the fitted plastic data and the new element geometry, element stiffness matrices are 
recalculated for the core elements as well as for the 
surrounding rearranged elements and the global 
stiffness is appropriately modified. Subsequent 
equilibrium check iterations using the new stiffness 
of the structure correct fitting errors, if any, in the plasticity data in the new mesh. At the same 
time, the tractions over the distance \( \Delta a \) as 
shown in Fig. 1) are incrementally removed, with 
equilibrium check iterations being used at each step, 
to create a new traction-free crack surface of 
length \( \Delta a \). The finite element simulation of crack 
extension by the desired amount, \( \Delta a \), is now complet-
ed.

Analysis of Fatigue Crack Growth Under 
Mode I Cyclic Loading Description of the Problem

Throughout the series of the present elastic-
plastic finite element analyses of fatigue crack 
growth under Mode I type cyclic loading, a thin 
rectangular plate with a central crack and under 
uniform tensile stresses, in a direction normal to 
the crack-axis, at the edge of the plate, is con-
sidered (See fig. 2). The dimensions of the plate 
are: half width \( w = 230 \text{mm} \); and half-crack length \( a = 27.3 \text{mm} \), respectively. The material is con-
sidered to be a 2024-T3 aluminum alloy, whose

Mechanical properties are characterized by: yield 
stress, \( \sigma_y = 350 \text{ MPa} \); and Young's modulus, \( E = 70,000 \text{ MPa} \). The material is assumed to be elas-
tic-perfectly-plastic. It is noted that the above problem definition is identical to that used by 
Newman [5]. The plate is assumed to be in a state of 
plane stress.

Because of the symmetries of geometry, applied 
loading, and material homogeneity, only a quarter 
of the cracked plate is analysed. Fig. 2 shows the 
fine element breakdown that is used. A total of 
6 sector-shaped singularity elements near the crack-
tip, and 43 conventional quadric isoparametric 
elements are employed. Some of these isoparametric 
elements are 6 noded triangular elements. The majority 
are 8 noded quadrilaterals (See Fig. 2). It 
is seen that the total number of nodes in the infinite 
element mesh for the quarter-plate is 171, with a 
total number of degrees of freedom of 311.

The radius of the sector-shaped singularity 
elements is chosen as \( r = 2.14 \text{mm} \); i.e., \( r/a = 0.103. \) 
While the crack-extension per cycle of loading, \( \Delta a \), 
can be arbitrary (i.e., not related to the infinite 
element mesh size) in the present analysis pro-
cedure, it is chosen to be \( \Delta a = 0.15 \text{mm} \) in the present 
series of computations.

Techniques to Minimize Computational(CPU) Time

Firstly, we note that the near-tip elements 
used presently are of the order \( 10^3 \) times the semi-

crack length; and the total number of algebraic equa-
tions for the above problem are only 311.

In the present procedure, a tangent modulus 
(stiffness) approach is used in each increment and 
in each iteration in the respective increment. 

Thus a faster convergence is obtained in the pro-
cess of iterations of equilibrium correction, etc. 

However, it is noted that only the stiffness ma-
drices of the plastic portion of the structure need to 
be changed in the present tangent modulus ap-

proach, whereas, those of the elastic portion re-

main fixed.

For the presently considered levels of applied 
far-field tension on the specimen, only "small-

scale" yielding conditions prevail near the crack-
tip. A typical plastic-zone size near the crack-
tip for the presently considered class of Mode I 
problems is shown in Fig. 3, being superimposed on 
the finite element mesh. It is seen that the plas-
tic portion of the structure (designated "Region P") 
is considerably smaller than the elastic portion 
(designated "Region P'"). Thus only the stiffness ma-
drix of region P need to be changed in each load-
step and each iteration in the present procedure.

However, the total number of times, say \( N \), that 
the combined stiffness matrix (for Regions P + E) 
must be inverted in the course of analyzing a typ-
ical problem, say the case a growing crack under 
constant amplitude cyclic loading, is \( h = \left[ \text{no. of} \right. \)

Even though the general formulation presented in 
Eq. (1) is for finite deformations, the finite de-
formation effects in the present class of problems 
can be shown to be confined only to the tip of 
the crack-tip. Thus, changes to the stiffness of 
region E due to finite deformation effects, if any, are ignored henceforth.
iterations/cycle) \( X \) (number of load increments per cycle) \( X \) (no. or load cycles). For a typical problem, say \( P \) cycles of constant amplitude loading, a typical value for \( N \) can be \( N = 6 \times 28 \times 8 = 896 \), (ie., 4 iterations per cycle, with 28 load increments/cycle, etc.). This is a rather enormous amount of computing; and hence a more economical way of solving the stiffness equations is mandatory.

Since the plastic-zone is a small-size, by an appropriate node numbering scheme, the stiffness matrix of the plastic zone can be arranged, as in Fig. 3.b, so that it is a small sub-set of the global stiffness matrix of the structure (even though the plastic zone size keeps changing with load, an approximate preliminary analysis can be made to determine its size at maximum load). Then we use a static condensation procedure to first eliminate the equations corresponding to the nodes in the elastic portion, in the very first load increment. Thereafter only the equations for the nodes in Region \( P \) need to be operated upon, in all subsequent load steps and iterations. Thus, in the example cited above, \( (N = 896) \), in all but one of the 896 solutions, the number of equations being solved is rather very small, and correspond to the total number of unconstrained nodal displacements in Region \( P \). This enables the present computations to be economically feasible.

Also, the computer program is so arranged that the data obtained from computation up to the end of a given spectrum of loading can be used as input data at the beginning of a different spectrum of loading. For instance, at the end of a constant Hi-amplitude cyclic loading, the data is stored on a direct access permanent disc file and used as initial conditions for a low amplitude cyclic load spectrum; in this process not only the case of Hi-amplitude loading but also the case of Low to Hi block loading is solved.

Monitoring of Crack-Closure and Opening in the Finite Element Model

Let us assume, that at a given instant of time (at a given point in the loading history), the location of the crack-tip, the location of the 4 nodes on the radial line (which coincides with the crack surface) of the singular sector element, the locations of all other nodes on the crack axis, as well the current (deformed) profile of the crack surface, are all known. We denote the current nodes on the crack surface as "updated Lagrangean Nodes". Let us now assume that the crack tip is now further extended by an arbitrary amount \( (\Delta a) \) and the structure is then subjected to further loading. We first note that, in the present development, the boundary displacement (in the direction of the applied normal stress) along a radial line of a "singular" sector element is of the form given in Eq. (3). Using the above equation, and knowing, a priori, the radial coordinates (ie., \( r \), as measured from the current crack-tip), of the respective nodes on the radial line of the sector element in its immediately previous location, one can compute the values of \( \Delta v \) at the above mentioned "updated Lagrangean Nodes." By adding (or subtracting, as the case may be) these incremental displacements to the previously known values, \( \Delta u \), etc., the values of the current crack surface deformation profile is made.

During the unloading part of any cycle, of the present cyclic loading case, at the instant the displacement (in the direction of applied tension) at one or more nodes on the crack surface becomes negative, further unloading is stopped. The computational procedure is then switched to a displacement control type, and the above negative displacements are precisely enforced to be zero; thus finding the precise stress level at which the closure constraint on the respective node must be enforced.

After the crack-closure is detected, the respective node(s) are constrained thereafter, until the restraining force(s) at the node(s) just becomes zero and begins to be tensile in nature. The corresponding applied stress level defines the crack-opening stress. The details of the above process are discussed and graphically illustrated elsewhere [12,14].

Finally, some comments on the presently observed patterns of crack-closure are given, before proceeding to a discussion of specific cases. In general, closure was noticed to occur at the node closest to the current crack-tip, as indicated in the sequence of unloading steps in Figs. 4a-c. However, if the current crack-surface profile is irregular, as in the case of Hi-to-Low block loading to be discussed later, crack-closure may first occur at the node closest to the crack-tip; however, in the subsequent unloading step, closure may occur at a node far-removed from the crack-tip, as indicated in Fig. 4d. From the results to be discussed later, this pattern of crack-closure appears to contribute significantly to growth retardation and delay effects.

Criterion For Crack-Extension Stress Level

In the present work, a study is made to arrive at a criterion for the stress level, \( \sigma \), at which fatigue crack growth occurs. In prior literature, this crack-extension stress level was chosen arbitrarily. For instance, in [5] the crack is extended at the maximum applied stress in each cycle even in a general spectrum (for instance, high-to-low, low-to-high, etc.,) loading, where as in [7] the crack was extended at the applied stress level at which the restraining nodal force at the new crack-tip becomes zero. In the present study, for instance in a constant-amplitude (zero-to-tension) cyclic loading, it was found that the crack opening and closure stresses, \( \sigma_0 \) and \( \sigma_0 \), respectively, were sensitive to the chosen \( \sigma_{LS} \). In the present work, a criterion, \( \sigma_{LS} = \gamma \cdot p \cdot \sigma_{max} \), is chosen, where \( p \) is a constant of proportionality, is postulated; and \( p \) is obtained by calibration such that the calculated \( \sigma_{LS} \) correlated with that observed in experimental \( \sigma_{LS} \) studies such as in [2]. However, it is noted that this constant of proportionality \( p \) may, to some extent, be dependent on the numerical methodology employed in fatigue crack modeling itself. Thus the above described calibration may be considered as valid only in the context of the particular methodology employed in the present work. Three different test cases, each with a different magnitude of constant amplitude (zero-to-tension) cyclic loading, were studied with different \( \sigma_{LS} \) and \( p \) values, in each case for the above mentioned constant of proportionality, \( p \). The idea was to select a "\( p \)" that yields results, in each case, for \( \sigma_{LS} = \sigma_{max} \) that are in best agreement with the experimental results [2].
for 2024-T3 Aluminum alloy, which is the material simulated in analysis. The results, for instance, for the case \((\sigma_{\text{max}}/\sigma_{\text{ys}} = .40)\) and \((R = \sigma_{\text{min}}/\sigma_{\text{max}} = 0)\) are summarized as follows:

<table>
<thead>
<tr>
<th>p</th>
<th>Levelled-off (\sigma_{\text{op}})</th>
<th>([\sigma_{\text{op}}/\sigma_{\text{max}}]) at steady state</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>115 MPa</td>
<td>.82</td>
</tr>
<tr>
<td>.85</td>
<td>94 &quot;</td>
<td>.67</td>
</tr>
<tr>
<td>.62</td>
<td>79 &quot;</td>
<td>.56</td>
</tr>
<tr>
<td>.40</td>
<td>58 &quot;</td>
<td>.41</td>
</tr>
</tbody>
</table>

were obtained for the cases, \((\sigma_{\text{max}}/\sigma_{\text{ys}}) = .229\) and \((\sigma_{\text{max}}/\sigma_{\text{ys}}) = .314\). From these three sets of results, it was observed that \(p=0.62\) yields results for \((\sigma_{\text{op}}/\sigma_{\text{max}})\) that are in best agreement with experimental observations, which indicate that \([\sigma_{\text{op}}/\sigma_{\text{max}}]\) at steady state is about 0.56. It is hypothesized that the above constant \(p=0.62\) may be used throughout the rest of the analysis, i.e., for cases of general spectrum loading. We also note that when the load level \(\sigma_{\text{op}}\) during any cycle is first determined, the number (and size) of load steps between this \(\sigma_{\text{op}}\) and \(\sigma_{\text{max}}\) in the respective cycle, is so adjusted that the pre-chosen level of \(\sigma_{\text{op}}\) \(=\sigma_{\text{op}}+p(\sigma_{\text{max}}-\sigma_{\text{op}})\) coincides with one of the load-increments in the cycle. We now discuss the results of analysis of each of the loading cases.

**Constant Amplitude Zero-to-Tension Cyclic Loading**

(i) The results for \(\sigma_{\text{op}}\), for the case of \((\sigma_{\text{max}}/\sigma_{\text{ys}}) = 0.4\) and \(R = (\sigma_{\text{op}}/\sigma_{\text{max}}) = 0\), are shown in Fig. 5, for 8 cycles of loading. It is observed that \(\sigma_{\text{op}}\) reaches a "steady state" value of 0.56 after the 4th or 5th cycle. It is also noted that this value for \(\sigma_{\text{op}}/\sigma_{\text{max}} = 0.56\) is in reasonable accord with experimental results [2] for the same material, a 2024-T3 Aluminum alloy. Knowing \(\sigma_{\text{op}}\) in each cycle, we define the effective stress-intensity factor as:

\[
\Delta K_{\text{eff}} = C_1 \sqrt{\sigma_{\text{op}} + N \Delta a} \left(\sigma_{\text{max}} - \sigma_{\text{op}}\right)
\]

where \(C_1\) is the finite size correction factor for the present crack geometry (which was found to be \(C_1 = 1.017\) from a finite element linear analysis of the crack with \(a = a_0\); and thereafter assumed to be constant; \(N\) is the number of cycles and \(\Delta a\) is the crack growth per cycle. It is thus seen that \(\Delta K_{\text{eff}}\) levels off after a few cycles to a steady state value.

(ii) Fig. 6 shows the crack surface deformation profiles for instance, at various stages of unloading during the 8th cycle of the present constant-amplitude \((R = 0)\) cyclic loading. The large blunting at the initial crack-tip location \((a = a_0)\) is observed to remain permanently. The surface of the extended crack is observed to be fairly smooth. During the 8th cycle, it is observed that precise crack-closure occurs only over the area \(\Delta a\) (i.e., only at the previous location of the crack-tip node) even upon full unloading; however, it is also noted that at this stage, the opening of the crack faces between the points \(a\) and the precisely closed node is very small (See Fig. 6).

**Low-To-High Block Loading**

A two level block loading, from low to high, with \(\sigma_{\text{max}}\) in the higher level being 1.273 times the \(\sigma_{\text{max}}\) in the lower level, is considered. The maximum stress in the lower level is taken such that \((\sigma_{\text{max}})_{\text{Low}} = \sigma_{\text{op}}\) = 0.314. As mentioned earlier, the data at the end of 4 cycles of low level block loading (See Fig. 7) is recovered from a constant-amplitude test case, with the corresponding stress level. The following results were obtained:

(i) The variation of crack-opening stress, \(\sigma_{\text{op}}\), as the cyclic loading progresses, is shown in Fig. (7). It can be seen that immediately after the step up in the level of applied stress, \(\sigma_{\text{op}}\) decreases by about 33\% of its steady state value corresponding to the lower level of block loading. Subsequent to this, \(\sigma_{\text{op}}\) increases monotonically to a steady state value corresponding to the higher level of block loading, within about 5 cycles. Prior to this stabilization, \(\Delta K_{\text{eff}}\) (defined as before) in the higher level of block loading remains considerably higher than the steady state value corresponding to this stress level; thus indicating growth acceleration following the load step-up.

(ii) The representative crack surface profiles for instance, at various stages of unloading at the end of the 8th cycle (high stress) of the current low-to-high level block loading, are shown in Fig. (8). From this Figure, it can be seen that the step-up in the level of loading causes a blunting of the crack-tip (i.e., at the location \(x/a = 1.02\) in Fig. 8, when the step-up in loading occurs in the present finite element simulation). Even during the unloading at the end of the present two level block loading, as seen from Fig. (8), the crack-closure occurs only over the area \(\Delta a\) (i.e., only at the previous location of the crack-tip node).

**High-To-Low Block Loading**

After 8 consecutive cycles of a high level block loading (the data at which point is recovered from the corresponding constant amplitude test case), the \(\sigma_{\text{op}}\) is reduced by 21.4\% and 8 more cycles of this reduced level block loading were considered. The magnitude of the applied stress in the high-level block was such that \((\sigma_{\text{max}})_{\text{High}}/\sigma_{\text{ys}} = 0.40\). The following results were obtained:

(i) The variation of the crack-opening stress \(\sigma_{\text{op}}\), as the loading progresses, is shown in Fig. (9). It is seen that immediately after the step-down in the load level, no abrupt decrease in \(\sigma_{\text{op}}\) as was the case in Low-to-High loading, occurs in the present High-to-Low block loading case. After the load-level step down, \(\sigma_{\text{op}}\) stayed at about 0.70 \((\sigma_{\text{max}})_{\text{Low}}\) within the number of cycles of low-level load considered presently. It may be possible that, as further number of low-level load cycles are considered and the crack-tip grows further and eventually surpasses the plastic zone created during the high level block loading, the \(\sigma_{\text{op}}\) decreases to a base line value corresponding to the lower level block loading. However, limitations of computer funds precluded the possibility of a larger number of load cycles at the low level. It is also seen that after the load-level step down, \(\Delta K_{\text{eff}}\) remains remarkably lower than its base line value corresponding to the low-level block loading.
The presence of a considerable retardation of growth, but no delay.

(ii) Fig. (10) shows the crack surface profiles during various stages of unloading in one of the low level cycles of the present high to low block loading. It is seen that at the stage of unloading indicated by point 'B' in Fig. (10), the crack clos- es only at the previous crack-tip (closure area = Δa). Further unloading, represented by point C, causes another node away from the current crack-tip to close, as seen in Fig. 10. The area of crack- closure thus increases as the unloading progresses.

To understand the effects of the features of crack-closure as in the present case, the problem was reanalyzed with the constraint of closure being removed on the node (as discussed above) far away from the crack-tip, but leaving the closure constraint on the node closest to the crack-tip. The corresponding changes in σ are indicated by a broken line in Fig. 9. These results indicate the influence of properly imposing the closure constraints on nodes even far away from the crack-tip when the considered loading, as the present High to Low case, causes such a type of crack-closure.

**Single Over Load**

The case of a single overload after 4 cycles of a constant amplitude block loading, followed by further cycles of constant amplitude (equal in magnitude to that before overload) was considered. The following three cases were considered:

<table>
<thead>
<tr>
<th>Case</th>
<th>Base σ_max</th>
<th>σ_overload max - base</th>
<th>σ_op max - σ_op base</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>110MP</td>
<td>1.273</td>
<td>0.151</td>
</tr>
<tr>
<td>2</td>
<td>110MP</td>
<td>1.455</td>
<td>0.283</td>
</tr>
<tr>
<td>3</td>
<td>80MP</td>
<td>2.0</td>
<td>0.681</td>
</tr>
</tbody>
</table>

In the above, σ_max, base is the maximum applied stress prior to or after overload; σ_overload is the overload stress; σ_op max is the maximum calculated value for crack opening stress after overload; and σ_op base is the base line opening stress for an otherwise constant amplitude cyclic load at level σ_max, base and all these stresses are illustrated in Fig. (11). The obtained results are discussed below:

(i) The variations of σ during the load cycling, for the three different ratios of stress overload, are indicated in Figs. (11a,b,c) respectively. In all three overload cases an abrupt decrease in σ, (which relative decrease becomes more predominant as the overload stress ratio increases) is noticed immediately after the single overload application. After this, in all the three cases, σ increases again to reach a peak value σ_op max before levelling off to a steady-state value. This relative values of σ_op max increases as the overload stress ratio increases. Also as the overload stress ratio increases, the latter is the occurrence of this σ increases.

(ii) When the considered loading, as the present High to Low block constraint on nodes even far away from the current crack-tip, causes such a type of crack-closure.

(iii) The curve depicting the variation of the ratio (σ_op max - σ_op base)/(σ_max - σ_op base) with the overload stress ratio, which is drawn using the above discussed 3 data points, is shown in Fig. 12. By extrapolation, the threshold value of the overload ratio, at which retardation effects come into play, is seen to be about 1.10. Bernard et al [16] report a threshold overload ratio of 1.3 - 1.4 based on a series of experiments on the material Ducol W30B whose yield strength is 366 MP (comparable to the presently considered σ = 350 MP). It is noted however, that the present analysis is based on a plane-stress assumption, while Bernard et al [16] note the dependence of the experimentally determined threshold value on the specimen thickness.

(iv) The crack-line deformation profiles at various stages of unloading at the end of the considered number of cycles are shown in Fig. 13 for the case of overload ratio of 2, while similar results were noted for the other two overload-ratio cases also. It is seen that the application of the single overload to the specimen (at the instant when a/a0 = 1.02 in Fig. 13) causes a large (plastic) blunting which is retained in the crack-surface profile even as the crack advances in further cyclic loading. When the specimen is fully unloaded, at the end of the cycle illustrated in Fig. 13, almost the whole surface area ahead of the previously mentioned location of blunting is noticed to close, while the crack surface area behind this blunting location is seen never to close.

**Analysis of a Center-Cracked Specimen under Pure Mode II Cyclic Loading**

A center cracked square plate under a constant amplitude cyclic loading of pure shear, which is uniformly distributed at the edges of the plate, is analyzed. Plane stress conditions are assumed. The material is considered to be 2024-T3 Aluminum alloy (same as in the pure Mode I case discussed earlier). The dimensions of the plate are: L = 140mm; a0 = 40mm. The maximum amplitude of the uniformly distributed shear is taken to be 80 MP (σ_max/σ_y = .23). In the present problem, the geometry of the plate with the crack is symmetrical about both the x and y axis (See Fig. 14), and the external loading is anti-symmetric with respect to both x and y axes.

As earlier, the present material is modeled as an elastic-perfect-plastic material. We note also that the presently considered material has the...
same properties in tension as in compression. Thus, the displacement field has the following antisymmetric properties:

\[ u(x, y) = -u(x, -y) = u(-x, y) = -u(-x, -y) \]

\[ v(x, y) = -v(x, -y) = v(-x, y) = -v(-x, -y) \]

where \( u \) is the displacement in \( x \) direction, etc. Further, it is noted that these displacements may be discontinuous at the crack surface, \( -\alpha < x < \alpha \). Thus, in the finite element modeling, only a quarter of the plate is modeled, as shown in Fig. 14, with the displacement boundary conditions: \( u = 0 \) along \( y = 0 \), in the uncracked ligament only; and \( u = 0 \) at nodes along \( x = 0 \). The linear elastic results, for the first load increment, from the present finite element analysis indicate:

\[ K_I = 0.075; \quad K_{II} = 3.777 \]

which compare favorably with the following results (obtained by using the finite-size correction factors) of Bowles and Heal [17] for an identical problem:

\[ K_I = 0.0; \quad K_{II} = 3.899 \]

The fact that \( K_{II} \neq 0 \) in the present finite element analysis is the result of inherent numerical errors, such as round-off and truncation, in the finite element analysis.

In Fig. (15a) the results for the displacements \( u \) at the upper and lower crack surfaces, \( u^U \) and \( u^L \), respectively, are plotted for the linear elastic case. These numerical results for \( u^U \) and \( u^L \) are identical in the linear elastic case. This equality of \( u^U \) and \( u^L \) is noticed as the loading continues in the first cycle (crack being stationary) and when the plastic zone size is significant at \( \tau = 70 \) MPa (see Fig. 16) for the shape of the plasticity zone at \( \tau = 70 \) MPa. For lack of any other criteria, the crack was extended, in the present procedure, at 70MPa, (the maximum applied stress being 80 MPa) in the direction of the initial crack axis, i.e., in the \( x \)-direction. It is seen from Fig. (15a) that significant increase in \( u \) is brought about by the process of crack extension and hence the attendant plastic unloading; however, again, \( u^U \) and \( u^L \) are almost identical (to the 4th significant digit). Thus it is seen that through the all stages of loading, crack extension and plastic-unloading, and further loading (to 8 MPa in this case) after crack extension, the upper and lower crack faces experience identical displacements in the \( y \) direction, i.e., perpendicular to the initial crack axis. On the other hand, the displacements \( u \) at the upper and lower surfaces of the crack, \( u^U \) and \( u^L \), are plotted in Fig. (15b) for the complete unloading when linear-elastic conditions prevail (\( \tau = 31.1 \) MPa); when appreciable plasticity develops at the crack-tip (\( \tau = 70 \) MPa), when the crack is extended (and hence there is plastic loading) at \( \tau = 70 \) MPa, and when the load is further increased to 80 MPa after crack extension. It is seen that the magnitudes of \( u^U \) and \( u^L \) are nearly identical (to the 4th significant digit), but with opposite sign, in all the above cases. However it is interesting to observe that the change (as a ratio of the respective value prior to crack extension, at the same load) brought about by the process of crack extension (and hence plastic unloading) in \( u \) is much more pronounced than in \( u^U \) and \( u^L \).

It is interesting to note that in the elastic case the crack-surface displacements \( u^U \) and \( u^L \) (see Fig. 15a) exhibit almost a linear variation from the crack-tip; thus indicating lack of any \( \sigma \) (in particular, \( \sqrt{r} \) type for linear elasticity) component in \( u \) for the linear elastic case. On the other hand, for the linear elastic case, the tangential displacements \( u_t^U \) and \( u_t^L \) (see Fig. 15b) do exhibit a \( \sqrt{r} \) behavior near the crack-tip, for the present Mode II problem. Also, it is seen from Fig. (15) that, as plasticity develops, the tangential displacements \( u_t^U \) and \( u_t^L \) exhibit a \( \sqrt{r} (\sigma < 0) \) variation near the crack-tip. However, for the pure Mode II case, even in the presence of plasticity, the analyses of Hutchinson [8] and Rice and Rosengren [9], indicate that there may not be a \( \sqrt{r} (\sigma < 0) \) type variation in \( u_t \) near the crack-tip.

For the present results for \( u \), in the presence of plasticity, Fig. 16, are seen to contain such a \( \sqrt{r} (\sigma < 1/2) \) type variation near the crack-tip. However, it should be noted that the angular variation of the singularity functions \( u_t (\theta) \), as embedded in the present sector elements, are being approximated as quadratic polynomials in each sector element. The fact that \( \sqrt{r} (\sigma < 1/2) \) type variations in \( u_t \) are numerically obtained along the radial line of the sector element lying on the crack surface, as in Fig. 15a, suggests that the above angular variations are not being solved highly accurately in the present numerical method. However, it appears that these numerical errors are identical at \( \gamma = 0 \) as at \( \gamma = \pi \), so that \( u_t^U = u_t^L \) as in Fig. 15a. Finally the crack-surface displacements \( u_t^U \) and \( u_t^L \) at the end of the first cycle of loading (i.e., when the applied stress is brought back to zero) are also indicated in Figs. (15a) and (15b) respectively. Once again it is seen that even after complete unloading, \( u_t^U \) and \( u_t^L \) are identical in magnitude and direction, whereas \( u_t^U \) are identical in magnitude but opposite in direction. Thus for the present material, with identical properties in tension as in compression, it is seen that in all cases of pure-shear type external loading, the upper and lower surfaces of the crack move together in the direction perpendicular to the initial crack-axis, whereas they slide past one another in the direction of the crack-axis. Thus, it appears for these types of materials the phenomenon of crack-closure, as observed experimentally and as analysed presently in Mode I type loading conditions, does not occur in pure Mode II type cyclic loading. However, the present experience indicates that crack-closure may occur even in pure Mode II cyclic loading if the material has different properties in uniaxial tension and compression. Consideration of such materials is not pursued in the present work. Finally, the computed shapes and sizes of the plastic zone near the crack-tip at various stages of pure shear loading are indicated in Fig. 16.

**Summary and Conclusions:**

If the crack-growth rate, \( d \Delta \), in Mode I fatigue loading, can be assumed to be related to the effective stress-intensity range, \( \Delta K_{eff} \), the present results indicate that: (i) growth retardation occurs in high to low and single overload cases, and (ii) significant delay effects prior to retardation occur in the case of a single-overload and linear-elastic retardation effect, etc., for such dominant as the overload ratio increases. From the crack-surface deformation profiles shown in Figs. 8 and 13, it is seen that a considerable crack-surface blunting occurs at the instant when the applied load is stepped up. This blunting at the instant...
of load-step up alters the possible pattern of crack closure during subsequent load cycles, and is seen to be responsible for the "delay" effects such as the delayed retardation in the single overload case, and the delayed transition of opening stress levels from the base line values for lower amplitude block loading to the higher base-line value for the higher amplitude block loading, in the low-to-high case. The physical mechanism behind these effects is further detailed in [12,14]. Finally, the phenomenon of crack-closure was not observed in the present numerical modeling of a thin center-cracked sheet (of an elastic-plastic material with identical properties in uniaxial tension as in compression), subject a pure shear (Mode II) cyclic loading of constant amplitude. Thus, it appears that in the study of the more general problem of fatigue crack-growth under mixed-mode loading, that crack-closure effects need be considered in the case mode I component only. However, the present experience indicates that crack-closure may occur even in pure Mode II if the material has different properties in uniaxial tension and compression.

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References:
FIG 1. SCHEMATIC REPRESENTATION OF TRANSLATION OF SINGULAR ELEMENTS.

FIG 2. FINITE ELEMENT MODEL OF A CENTER CRACKED SPECIMEN UNDER UNIAXIAL CYCLIC LOADING (SINGULAR SECTOR ELEMENTS SHOWN WITHIN DETAIL 'A').

FIG 3a. REPRESENTATIVE SIZE OF YIELD ZONE AT \( \sigma_{\text{max}} \).

FIG 3b. SCHEMATIC REPRESENTATION OF INCREMENTAL EQUATIONS IN THE PRESENCE OF YIELDING.

FIG 4. REPRESENTATIVE PATTERNS OF CRACK CLOSURE: (a) CRACK CLOSURES ONLY AT NODAL NODES CLOSEST TO CRACK TIP, \( \Delta \gamma \) CRACK CLOSURE OCCURS ONLY AT NODAL NODES AWAY FROM CRACK TIP.
Fig 5. Crack closure and crack opening stresses in constant amplitude (R=0) cyclic loading.

Fig 8. Crack line profile during unloading in low-to-high block loading.

Fig 9. Crack closure and crack opening stresses in high-to-low block loading.

Fig 10: a: Crack line profile during unloading in high-to-low block loading; b: Inset 'A' in Fig 10a is magnified and shown.

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FIG 11. CRACK CLOSURE AND CRACK OPENING STRESS FOR THREE DIFFERENT CASES OF A SINGLE OVERLOAD.

FIG 12. EFFECT OF OVERLOAD-STRESS RATIO ON \( \frac{\sigma_{op, max} - \sigma_{op, base}}{\sigma_{max, base} - \sigma_{op, base}} \).

FIG 13. CRACK LINE PROFILE DURING UNLOADING AFTER A SINGLE OVERLOAD.

FIG 14. GEOMETRY AND FINITE ELEMENT MODEL OF A CENTER CRACKED PANEL UNDER PURE SHEAR CYCLIC LOADING.

FIG 15. a: NORMAL DISPLACEMENT PROFILES OF THE UPPER AND LOWER SURFACES OF THE CRACK; b: TANGENTIAL DISPLACEMENT PROFILES OF UPPER AND LOWER SURFACES OF THE CRACK.

FIG 16. PLASTIC ZONE NEAR CRACK TIP IN A CENTER CRACKED PANEL UNDER PURE SHEAR CYCLIC LOADING.