Influence of Flaw Shapes on Stress Intensity Factors for Pressure Vessel Surface Flaws and Nozzle Corner Cracks

The influence of the flaw shape on the variation of stress-intensity factors along the crack front is examined for longitudinal inner surface flaws in pressure vessels, and for corner cracks present at the intersection of a pressure vessel and a nozzle. For the inner surface flaw problem, the geometry of the pressure vessel considered is that of a commercial pressure vessel. The analyzed flaw shapes are those recommended by the ASME Boiler and Pressure Vessel Code (Section III, App. G, 1977). In the case of corner cracks at nozzle-pressure vessel junctions, natural flaw shapes obtained through experiments are considered. A fully three-dimensional linear elastic hybrid displacement finite element procedure was used to analyze these problems of practical interest in pressure vessel analysis and design. The obtained solutions are compared with those in the literature using other numerical and/or experimental procedures, and a discussion of noted discrepancies is presented.

Introduction

Two of the most commonly encountered fracture problems in pressure vessels are longitudinal inner surface cracks (beltline flaws), which are generally semi-elliptical in geometry, and quarter-elliptical corner cracks, at the intersection of the nozzle and pressure vessel. Such flaws may be present in the pressure vessel due to manufacturing difficulties (e.g., tool marks) or material defects. The presence of such cracks elevates the stresses and strains considerably near the region of the cracks. This increases the possibility of catastrophic failure and reduces the safe operating lifetime of the pressure vessel considerably. Thus, a thorough and accurate analysis of structural integrity in the presence of flaws is necessary for the design of the pressure vessel.

A structural integrity analysis calls for the accurate estimation of stress intensity factors along the crack front. Stress intensity factor solutions can be used effectively in design strategies for brittle fracture and fatigue control of pressure vessels. Recently, several investigators have studied some of these problems using different analytical procedures and experimental methods. A series of papers were published by Kobayashi, et al. [1-3] concerning semi-elliptical outer or inner surface flaws in thick pressure vessels ($R_a/R_i = 1.1-2.0$) subjected to mechanical as well as thermal shock loading. In their procedure, Kobayashi, et al. [1-3] first obtain the solutions for surface flaws in flat plates using the Schwartz-Neumann alternating technique. From these they generate solutions for surface flaws of identical geometry in thick cylinders by applying the "curvature-correction factors" obtained from the two-dimensional analogues of edge cracks in strips. Atluri, et al. [4, 5] have also obtained solutions for inner or outer surface flaws of various aspect ratios in thick cylinders ($R_a/R_i = 1.5-2.0$) using the direct-solution technique for $K$-factors based on the hybrid-displacement finite element method [6, 7]. It was found that even though the solutions in [1-3] on the one hand and those in [4, 5] on the other, agreed well with each other for shallow (the ratio of maximum crack depth to cylinder thickness being small) semi-elliptical flaws, substantial differences existed in the case of deep, oblong semi-elliptical flaws. The possible reasons for these significant differences were discussed in [5]. On the other hand, the solutions in [4, 5] for deep oblong flaws agreed well with those obtained by Blackburn and Hellen [8] using the so-called "virtual crack-extension" or stiffness-derivative method.

More recently, geometries of commercial nuclear pressure vessels (with representative $R_a/R_i = 1.1$), with semi-elliptical inner surface flaws, were considered by McGowan and Raymund [9] for fracture analysis, using the stiffness derivative finite element technique with substructuring. Identical problems were also considered by Heliot, et al. [10] using the boundary integral equation method. In both [9, 10] only a sector of the cylinder ($\theta < \pi$) was considered in the case of a single semi-elliptical surface flaw at the inner surface of

Contributed by the Pressure Vessels and Piping Division and presented at the Pressure Vessels and Piping Conference, San Francisco, California, June 25-29, 1979, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received at ASME Headquarters, March 5, 1978; revised manuscript received January 2, 1980. Paper No. 79-PVP-65.

The authors wish to dedicate this paper to the memory of the late Mr. E. K. Lynn.

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the cylinder. The crack shapes investigated in [9 and 10] were those recommended by the ASME Boiler and Pressure Vessel Code [11].

The stress analysis of the pressure-vessel-nozzle juncture is a complex one. The situation becomes even more cumbersome in the presence of surface or corner cracks at the nozzle juncture. The problem has received widespread attention recently from experimental stress analysts. A wide variety of crack shapes at nozzle corners of thin BWR (Boiling Water Reactors), as well as thick ITV (Intermediate Test Vessels), vessels were considered for analysis by Smith, et al. [12], using frozen stress photoelasticity. Similar experimental work was also conducted by Broekhoven [13] and Miyazono [14], while limited numerical results, based primarily on the virtual crack extension method, were presented by Broekhoven [13] and Reynen [15]. One of the significant conclusions of the extensive experimental work by Smith, et al. [12] was that, if the crack shape inserted into an analytical (such as finite element) model is not a real one or if the inner fillet (for shallow flaws) or the outer boundary shape (for moderate to deep flaws) is improperly approximated, the obtained numerical stress-intensity factors may differ significantly from the physical behavior at the nozzle-vessel juncture. On the other hand, the limited numerical results published to date were based only on the consideration of mathematical shapes such as quarter-circle, for nozzle-corner cracks.

In the present paper, solutions are presented for flaws of various shapes at the inner surface of cylindrical pressure vessels, as recommended by the ASME Boiler and Pressure Vessel Code. In all of these, one half of cylinder \((\theta = \pi)\) is modeled by finite elements. Attention was also focussed on the effect of various boundary conditions at the ends of the cylinder on the stress-intensity factor solutions. The obtained solutions are compared with the limited set of results available in the literature. The second type of problem considered here is that of a "natural" shaped flaw at the nozzle corner in a pressure vessel. The flaw shape observed in the experiments of Smith, et al. [12] is used in the finite element model. The computed \(K\)-factor solutions are compared with those obtained from frozen-stress photoelastic technique by Smith, et al. [12]. However, since the material used in the experiments of [12] is necessarily incompressible in character, in order for the foregoing comparison to be meaningful, a near-incompressible material behavior (Poisson's ratio \(\nu = 0.45\)) was simulated in the present computations.

In the present set of results, the \(K\)-factors at various points along the crack front were calculated directly as unknowns, using the hybrid-displacement finite element method as originally developed by Atluri, et al. [6]. Finally, in order to demonstrate the level of confidence in the present solutions, the only problems of cracks in cylindrical geometries with known exact solutions (such as a through-thickness crack in a cylindrical shell [16], and an axisymmetric edge crack in a cylinder [17]) were also solved using the present procedure. The results were found to correlate excellently, and are discussed in the paper.

A Brief Outline of the Method of Approach

Special three-dimensional "crack front elements" developed through a hybrid-displacement finite element procedure are used for the analysis of the present problems. The hybrid-displacement finite element procedure uses the stationary condition of a modified total potential energy functional with a relaxed requirement of the continuity, a priori, of interelement boundary displacements. The developed finite element procedure is applicable to arbitrarily shaped crack fronts; thus enabling the application to observed natural flaw shapes in pressure vessel nozzle junctions. Three-dimensional asymptotic solutions for displacements and stresses near the crack front are embedded in this procedure. The variational principle governing the present finite element method is a three-field principle, with the element interior displacements, element boundary displacements, and the Lagrange multipliers (which are physically the interelement boundary tractions and which are used to match the element interior and boundary displacements at the interelement boundary), as the three field variables. The compatibility between the special crack front elements and neighboring regular elements at the interelement boundary is maintained through the variational principle, assuring the convergence of the finite element procedure. In this special finite element solution procedure the three (modes I, II and III) stress-intensity factors along the crack front are also treated as unknowns along with the generalized nodal displacements of the structure. It can be shown that the final set of algebraic equations, governing the structure's global nodal displacements and the three stress-intensity factors at various points along the crack front can be written in the following form:

\[
\begin{align*}
K_1 q^* + K_1^T k^* &= Q_1^1 \\
K_2 q^* + K_2^T k^* &= Q_2^1
\end{align*}
\]

where \(q^*\) are the structure's global nodal displacements, \(k^*\) are the mixed-mode stress intensity factors at various points along the crack front, \(K_1, K_2, \) and \(K_3\) are the corresponding stiffness matrices, and \(Q_1^1, Q_2^1\), are the corresponding nodal forces. It can be seen from equations (1) and (2) that the solution for the stress-intensity factors can be obtained directly from the finite element solution procedure instead of post-processing the displacement or stress solution to obtain the stress-intensity factors. A detailed description of the hybrid-displacement finite element procedure for 3D crack analysis can be found in references [6, 7].

The advantages of the hybrid-displacement finite element method may be briefly enumerated as follows. As mentioned before, the present formulation leads to a direct solution for stress-intensity factors along the crack front. The boundary displacements in this hybrid model are so chosen as to be compatible with the other neighboring regular elements developed through a conventional compatible displacement model. This not only establishes the convergence of the finite element model, but also enables one to develop "super-elements (combination of several singular elements)."


Fig. 1 Geometry of beltline surface flaw in a cylinder

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These can be used, in conjunction with the widely available finite element computer programs based on the more common displacement finite element model, to solve complex three-dimensional fracture mechanics problems such as pressure vessel surface flaws and nozzle corner cracks.

Since the boundary displacements as well as the interior displacements are assumed independently, it is an easy task to embed the correct asymptotic variation of displacements (i.e., correct $\sqrt{r}$ and $\theta$ variations for displacements), not only in the interior of the crack element, but also at its boundary. Since the asymptotic displacement solution near the crack front is self-equilibrated, it can be shown that the strain-energy integrals of the formulation can be converted into boundary integrals that are free from singularities by the use of the divergence theorem. In such cases, since the integrands are free of singularities, the integrals can be evaluated accurately using simple numerical integration procedures. In brief, it is felt by the authors that the hybrid-displacement finite element procedure offers an efficient approach, with no restrictive assumptions, for the solution of complex three-dimensional fracture problems.

**Description of Problems**

The problems considered here pertain to surface flaws of shapes as recommended for fracture analysis by the ASME Boiler and Pressure Vessel Code [11], and the considered pressure vessel geometries are representative of commercial pressure vessels. Several different ratios of crack depth to vessel wall thickness were considered. The geometric and material parameters for beltrine flaws are summarized in the following:

- Pressure vessel outer radius $R_o = 11$
- Pressure vessel inner radius $R_i = 10$
- Semi-minor axis of crack $a = 1$
- Semi-major axis of crack $c = 3$

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**Fig. 2** Geometry of corner crack at nozzle-pressure vessel junction

**Fig. 3** Geometry and finite element breakdown of a cylindrical shell with an axial crack

**Fig. 4** Geometry and finite element breakdown of a cylinder with an outer axisymmetric edge crack
The legend in Fig. 1 shows the geometry of the relatively thin \((R_o/R_i = 1.1)\) pressure vessel, with the semi-elliptical surface flaw, under consideration.

Next, the geometry of an Intermediate Test Vessel with a nozzle corner crack is also considered for analysis by the present method. The geometry of the ITV and the nozzle corner crack are depicted in Fig. 2. The pressure vessel-nozzle geometry given in Fig. 2 is identical to the problem studied by Smith, et al. \[12\] using the frozen stress photoelasticity technique. The flaw shape presented in Fig. 2 is that of a natural flaw shape obtained through experiments by Smith, et al. \[12\], and corresponds to the shape of the deepest flaw observed by Smith, et al. \[12\] in experiments.

Results

The accuracy and convergence of the present method have been tested and verified through the solution of several fracture problems of aerospace and pressure vessel structure applications \[4-7 and 18\]. Since both the thick \((R_o/t = 3.4)\) as well as relatively thin \((R_o/t = 11)\) pressure vessels are to be considered, an effort was first undertaken to study the accuracy of the present method in the few fracture problems with known exact solutions, in thick and thin pressure vessels.

First, an axial through-the-thickness crack in a pressurized thin cylindrical \((R/t = 25)\) shell with \([a/(Rt) = 2] (2a being the crack length) was solved. The exact solution of the stress-intensity factor, at the middle surface of an infinitely long shell, normalized by a factor of \(\sigma_{mb} \sqrt{a} \), was given by Erdogan \[16\] to be 2.42. The problem geometry and the finite element breakdown, with 150 elements and 2766 (before imposition of boundary conditions) degrees of freedom, are shown in Fig. 3. The condition of zero axial displacement was imposed at the ends of the cylinder. The solution for the membrane component of the stress-intensity factor by the present procedure yielded a corresponding normalized value of 2.4521 which is 1.3 percent higher than the exact solution. However, it was found that the present solution for the bending component of the stress-intensity ratio \(\Delta K_{mb} \), as defined in equation (5.81) \[16\], was about 28 percent lower than that given in \[16\]. It is to be noted, however, that the analysis in \[16\] is based on an 8th order linearized shallow shell theory in which the well-known Kirchhoff-Love hypotheses regarding transverse shear and twisting moment are invoked. Thus, as stated in \[16\], in the use of such a theory, one may expect that the shell thickness will have only a slight effect on the membrane component, but a significant effect on the bending component of the stress-intensity factor.

Next, a thick cylinder \((R_o/t = 5)\) with an axisymmetric, part through, outer edge crack (crack depth/cylinder thickness = 0.5) subjected to an axial stress of \(a_o\) was considered. The problem geometry is given in Fig. 4. Since this is an axisymmetric problem, an angular sector of 10 deg was considered for modeling. The finite element breakdown is also presented in Fig. 4. This contains 36 elements and 945 (before imposition of boundary conditions) degrees of freedom. The solutions for stress-intensity factors, normalized with respect to \(\sigma_o \sqrt{a} \) (\(a\) being the crack depth), by Erdol and Erdogan \[17\] (which, however, is for an infinitely long cylinder) and by the present method (which is for a cylinder with \(2L/R_o = 1.0\) as shown in Fig. 4) were 1.6817 and 1.7967, respectively. The present solution is thus about 6.8 percent higher than the exact solution by Erdol and Erdogan \[17\].

The displacement and stress analysis of an uncracked shell \((R_o/t = 21)\) with clamped edges and subjected to internal pressure and uniform temperature increase was also considered by the present three-dimensional finite element model. The exact solution for this problem is given by Kraus \[19\]. The present solution for displacements as well as stresses agreed to 5 percent of the exact solution, the present solution being higher than the exact solution. The problem definition and the finite element breakdown, with 24 elements and 579 (before imposition of boundary conditions) degrees of freedom are given in Fig. 5.

The foregoing results are considered to lend sufficient engineering confidence in the present three-dimensional finite element procedure to accurately solve, not only problems of thin uncracked shells, but also problems of cracks in thin as well as thick cylindrical geometries.

Beltline Surface Flaws

After establishing the accuracy of the present procedure through problems whose exact solutions are known, the procedure was then applied to the problem of semi-elliptical surface flaws (beltline flaws) in relatively thin shells. The finite element breakdown of one quarter of the structure (which only was modeled due to the appropriate symmetry conditions) is shown in Fig. 6. This breakdown consists of 380 total number of elements and 6195 total degrees of freedom (before imposition of boundary conditions). We note that only a single crack was modeled.

The solutions for stress intensity factors along the crack

\[\frac{a}{R_o - R_i} = 0.25, 0.5 \text{ and } 0.8\]

Young’s modulus, \(E = 27.9 \times 10^6 \text{ psi}\)

Poisson’s ratio, \(\nu = 0.3\)

\[\frac{L}{R_i} = 1.06 \text{ for } a/t = 0.25 \text{ and } 0.5\]

\[= 1.48 \text{ for } a/t = 0.8.\]

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front for beltlime flaws in pressure vessels are presented in Figs. 7 through 9 for crack depth to cylinder thickness ratios of 0.25, 0.5 and 0.8, respectively. The solution by Heliot et al. [10], using the boundary integral equation method, and by McGowan and Raymund [9], using a stiffness derivative finite element method, are also presented in Figs. 7 through 9. From these it can be seen the present solutions for the K-factor at the surface (elliptical angle 90 deg) are lower by about 12, 10 and 0.0 percent for \((a/t)\) ratios of 0.25, 0.5 and 0.8, respectively, as compared with those of [10], where as these differences reduce to 7, 6 and 0 percent, respectively, when compared to those in [9]. Thus, it can be seen that the differences in the solutions are somewhat appreciable for the case of shallow cracks, but the differences vanish as the crack depth increases.

It is noted that the present solutions were obtained by directly applying the pressure on the inner surface of the cylinder (but not also on the crack face); while at the ends of the cylinder, axial displacements were suppressed. The problem of stress-intensity factor computation may be viewed, equivalently, within the assumption of linearity, as a local perturbation problem in which the only external loads are the tractions applied on the crack face that are equal and opposite to the ones that would exist in an otherwise uncracked cylinder. Viewed in this fashion, the crack face tractions that are applied in the present analysis are those computed directly, from the present finite element method, for the uncracked cylinder with the end conditions of suppressed axial displacements. It has been noted earlier that the present finite element procedure overestimates stresses in

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After the release of the preprint (ASME Paper No. 79-PVP-6) of this paper, certain possible errors in Figs. 7 and 10 were brought to our attention by R. C. Labbens. Upon checking, some errors in the computer input for the problems with \((a/t) = 0.25\) were found. The corrected results are being presented in this paper.
uncracked thin cylindrical shells, with clamped ends, by about 5 percent as compared to the exact solution [19].

However, the solutions for K-factors in [9, 10] were obtained directly from the local perturbation problem in which the only external loads are the crack face tractions that are equal and opposite to the ones that would exist in an otherwise uncracked cylinder. All the calculations in [9, 10], with the exception of one result for a deep crack \((a/t) = 0.8\) in [10], were obtained by imposing stress-free conditions at the ends of the cylinder. Also, in [9, 10], the stresses in the uncracked cylinder were taken to be those corresponding to the well-known, plane-strain, Lamé solution for cylinders. The Lamé solution may be considered as exact if the circular cross sections of the cylinder remain plane and when the axial tractions applied on the end cross sections of the cylinder are those corresponding to plane-strain (ie., \(\sigma_r = \sigma_\theta = \sigma_z = 0\)) where \(\sigma_r, \sigma_\theta, \) and \(\sigma_z\) are the axial, radial and circumferential stresses, respectively, in the cylinder). The Lamé solution was approximated in [9, 10] by algebraic polynomials in the thickness direction (constant, linear, and quadratic in the thickness coordinate). The stress-intensity factor solutions were then generated in [9, 10] in the form of influence functions corresponding to each mode of stress variation (constant, linear, etc.) in the thickness direction.

To understand further the previously mentioned discrepancies between the present solution and those in [9, 10], in the case of shallow \((a/t = 0.25\) and 0.5) flaws in internally pressurized cylinders, the present procedure was used to generate the influence functions analogous to those in [9, 10]. Particular emphasis was placed on the case of constant pressure distribution on the crack face when the crack is shallow \((a/t = 0.25\) and 0.5). The solutions for K-factors for uniform pressure applied on the crack face, for \((a/t)\) ratios of 0.25 and 0.5, are presented in Figs. 10 and 11, respectively, along with the comparison results from references [9, 10]. It is interesting to note from Figs. 10 and 11 that the differences between the present solutions and those in [9, 10] for the influence functions for the constant crack face pressure are of the same order as the differences between the solutions in the case of the actual crack in an internally pressurized cylinder (see Figs. 7 and 8). This may be due to the fact that in the case of shallow flaws, the Lamé solution stresses, which are used to pressurize the crack in the procedures of [9, 10], are nearly constant in the procedures over the crack face. Thus, it appears that the differences in the present solutions and those in [9, 10] for shallow cracks in internally pressurized cylinders arise, primarily, due to the differences in the influence functions for constant crack face pressure for such shallow flaws as shown in Figs. 10 and 11. It should be noted that in [10] only a segment of the shell, with the subtended circumferential angle \(\theta\) being less than 180 deg, was modeled. Specifically, for the case of \((a/t) = 0.25\) the value \(\theta = 60\) deg and for the case of \((a/t) = 0.5\), the value \(\theta = 80\) deg were used in [10]. However, for the case of \((a/t) = 0.8\) two models, with \(\theta = 80\) deg and \(\theta = 180\) deg, respectively, were analyzed. Moreover, in all the cases analyzed in [10], with the exception of the case with \((a/t) = 0.8\) and \(\theta = 180\) deg, stress-free conditions were used at the ends of the cylinder.

In contrast, in all the cases analyzed presently, one-half of the shell in the circumferential direction \((\theta = 180\) deg), as appropriate symmetry conditions due to the presence of a single-surface flaw would indicate, was modeled. Further, in all the cases analyzed presently, the condition of zero axial displacement was imposed at the ends. In [10], the stresses...
end was modeled, as compared to the case when a segment of these results, it was concluded in [10] that the results for solution at the free surface (\( \phi = 90 \text{ deg} \)) was more significant for higher-order variations of crack-face pressure. Based on these results, it was concluded in [10] that the results for stress-intensity-factor influence functions for constant crack-face pressure presented in [10] for \((a/t) = 0.25 \) and 0.5 (corresponding to the modeled segments \( \theta = 60 \text{ deg} \) and 80 deg, respectively; but with traction-free conditions at the ends in each case) may have been overestimated by about 5 percent. The present results confirm this conclusion qualitatively. However, as noted earlier, the present results for the \( K \)-factors at the surface of the crack (\( \phi = 90 \text{ deg} \), Fig. 1) are lower than those in [10] by about 12 and 10 percent for shallow flaws with \((a/t) = 0.25 \) and 0.5, respectively.

In conclusion, we feel that the presently found average discrepancies in stress-intensity factors all along the crack front, for surface flaws in internally pressurized cylinders, viz., about 7 percent for the case of \((a/t) = 0.25 \), (Fig. 7), about 8 percent for \((a/t) = 0.5 \) (Fig. 8), and almost zero percent for \((a/t) = 0.8 \) (Fig. 9), may be viewed as not being serious, especially when the results are used in the context of an engineering fracture theory to study the integrity of the flawed vessel.

Further investigations were carried out to study the effects of the simulation of different boundary conditions at the ends of the cylinders on the solution for stress-intensity factors. The results presented thus far correspond to a condition of suppressed axial displacements at the cylinder ends. Generally, in practical cases, the pressure vessels may consist of hemispherical end caps at the ends of the cylinder. The effects of the simulation of these hemispherical end caps were studied.

The effect of adding a hemispherical end cap to a cylindrical shell, and the combination being internally pressurized, is given by Flügge [20]. Fig. 12 illustrates the free body diagram of the cylindrical and hemispherical shell ends. At the juncture, the meridional stress resultants \( N_z \) and \( N_y \) are in equilibrium. But, there is a discrepancy in the hoop strains which causes the radial shear force \( X_1 \) and moment \( X_2 \). If one assumes that both the shells have the same thickness, then \( X_2 = 0 \), and \( X_1 \) is [20] equal to \( pa/8x \) where \( \kappa = 3(1-\nu^2) a^2/t^2 \). The radial shear forces have a parabolic distribution through the thickness of shell. The variation of this shear stress, \( \tau_{r2} \), is \( -3X_1 (1-4z^2/t^2)/2t \). The stresses due to the normal stress resultant \( N_z \) and the shear stress were then imposed as the boundary conditions at the ends of the cylinder in the finite element solution procedure.

First, only the stress corresponding to the axial stress resultant \( N_z \) was added. The solution for stress-intensity factors was practically unaffected by the normal pressure (i.e., they increased by less than 1 percent). Then, the shear stress was also incorporated in the solution procedure and stress-intensity factors dropped by about 3.5 percent all along the crack front. Finally, it is noted that, in some practical situations, the crack surface may also be pressurized in addition to the pressure on the inner surface of the vessel. In such cases, the solutions presented in Figs. 7 through 11 can be superposed appropriately to obtain the final solution for stress-intensity factors.

**Nozzle Corner Cracks**

Next, a corner crack at nozzle-pressure vessel juncture as given in Fig. 2 was analyzed by the present finite element procedure. For this finite element analysis, one half of the structure was modeled. The cross-sectional view of the finite element breakdown is given in Fig. 13 with an isometric view presented in Fig. 14 for clarity. The finite element breakdown consists of 452 total number of elements and 7677 total number (before imposition of boundary conditions) of degrees of freedom.

The geometry of the present problem is that of a typical intermediate test vessel. Experimental studies by Smith et al. [12] have revealed that the flaw shapes are significantly different from that of a quarter-ellipse or quarter-circle, which are the most common geometries assumed by the analytical investigators. Such assumptions in the analytical
The procedure may result in vast differences between the predicted stress-intensity factors and the true stress-intensity factors along the crack front. The main reasons for this discrepancy can be attributed to the fact that, near the nozzle-pressure vessel juncture, the stress gradients are rather severe and complex. The same conclusion was also arrived by Smith, et al. [12] in their experimental investigation.

The deepest natural flaw shape observed in a thick-walled pressure vessel nozzle (flaw shape no. 6 of reference [12]) was analyzed using the present finite element technique. Though Fig. 2 is not drawn to scale, the flaw shape depicted in Fig. 2 was drawn to scale and corresponds to the shape as found in the experiments of [12]. Initially, the Poisson ratio \( \nu \) of the material was taken as 0.3, which is typical for a material such as steel. The solution for the normalized stress-intensity factors along the crack front for the present nozzle-corner crack is shown in Fig. 15. Though good correlation was found with the experimental solution, the maximum value of stress intensity factor was about 10 percent lower than the experimental value. In the frozen-stress photoelasticity experiments by Smith, et al. [12], an incompressible material with Poisson ratio \( \nu = 0.5 \) was used. It was also estimated by Smith, et al. [21] that the experimental solutions may be higher than the solutions for materials with a lower Poisson ratio by a factor of \( ((1 - \nu)/(1 - 0.5\nu))^{1/2} \) when plane strain conditions exist at the crack front. Thus, for a material with \( \nu = 0.3 \), this experimental overestimation may be about 10 percent.

To understand this better, investigations were carried out by considering a nearly incompressible (\( \nu = 0.45 \)) material in the finite element model. Results for the nozzle-corner crack, with Poisson ratio \( \nu = 0.45 \), are presented in Fig. 15. This solution still exhibits a difference of about 5 percent as compared with the experimental solution. Since the present finite element method cannot be directly applied to a precisely incompressible material (\( \nu = 0.5 \)) without the rather complicated feature of introducing the hydrostatic pressure in the material as a variable in the formulation, such an analysis was not attempted. However, further studies were made on the effects of simulation of different boundary conditions on cylinder and nozzle ends. The normal stress and the shear stress, as described earlier, due to the introduction of hemispherical end caps, were imposed on the cylinder and nozzle ends. The calculated stress-intensity factors then increased by about 2.5 percent along the crack front, thus reducing the difference in the present solution and the experimental result to about 2.5 percent. Based on the foregoing considerations it appears that the present solutions can be applied in an engineering fracture mechanics analysis with confidence.

**Conclusion**

An accurate method for the solution of complex three-dimensionally curved cracks in pressure vessels with arbitrarily curved cracks is presented. The present solutions correlate excellently with the experimental solution for a complex three-dimensional problem of nozzle-pressure vessel juncture with a natural shaped crack. The present results also correlate excellently with available exact analytical solutions for through-thickness and axisymmetric part-through thickness cracks in thin as well as thick cylinders. Studies were made and presented on the effects of various Poisson ratios and on the effects of simulation of different boundary conditions at the cylinder and nozzle ends on the \( K \)-factor solutions. Also excellent correlation was found between the present solutions and those reported by Heliot, et al. [10] and McGowan, et al. [9] for deep oblong beltline surface flaws in cylindrical pressure vessels. However, the present results were found to be of the order of 7–8 percent lower, on the average, than those in [9, 10] for the case of shallow (\( a/t = 0.25 - 0.5 \)) beltline flaws. The possible reasons for these differences have been discussed. However, it is also felt that these differences may not be significant in the context of application of these results, within the limits of an engineering fracture theory.

Finally, it may be of interest to note that the present finite element program uses automatic mesh generation. Moreover, in the present hybrid singular element procedure, the \( K \)-factors at various points along the crack front are solved for directly as unknowns in the final finite element algebraic equations.

**Acknowledgments**

The results reported herein were obtained in the course of an investigation supported by the Union Carbide Corporation Subcontract No. 7565 (under Department of Energy Prime contract No. W-7405-ENG-26) with Georgia Tech. The authors express their appreciation to Mr. G. D. Whitman for his timely encouragement. The authors also thank Dr. R. C. Labbens of Creusot-Loire, Paris, France, for his constructive comments, on the preprint, 79-PVP-65, of this paper, which were very helpful in the preparation of this manuscript.

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